The Wave Equation and the Speed of Light

Chapter 1

Physics 208, Electro-optics

Peter Beyersdorf

Class Outline

- Maxwell's equations
- Boundary conditions
- Poynting's theorem and conservation laws
- Complex function formalism
    - time average of sinusoidal products
- Wave equation

Maxwell's Equations

Electric field ($\vec{E}$) and magnetic field ($\vec{H}$) in free-space can be generalized to the electric displacement ($\vec{D}$) and the magnetic induction ($\vec{B}$) that include the effects of matter. Maxwell's equations relate these vectors

$$
\begin{aligned}
\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 && \text{Faraday's law} \\
\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} &= \vec{J} && \text{Ampere's law} \\
\vec{\nabla} \cdot \vec{D} &= \rho && \text{Gauss' law (for magnetism)} \\
\vec{\nabla} \cdot \vec{B} &= 0 && \text{Gauss' law (for electricity)}
\end{aligned}
$$

What do each of these mean?

Faraday's Law

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

- The Curl of the electric field is caused by changing magnetic fields
- A changing magnetic field can produce electric fields with field lines that close on themselves

Ampere's Law

$$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}$$

- The Curl of the magnetic field is caused by current of charged particles (J) or of the field they produce (dD/dt)
- A changing electric field can produce magnetic fields (with field lines that close on themselves)
- For all cases considered in this class, J=0

Gauss' Law

$$\vec{\nabla} \cdot \vec{D} \quad = \quad \rho$$

- Electrical charges are the source of the electric field
- $\vec{D} = \epsilon \vec{E} = \epsilon_0 \vec{E} + \vec{P}$
- For all cases considered in this class, $\rho=0$
- ϵ is a 3x3 tensor not a scalar (unless the material is isotropic)!
- ϵ may be a function of E and H! (giving rise to non-linear optics)
- ϵ can be determined via measurements on a parallel plate capacitor filled with a given material using the equation $C=\epsilon A/d$

Gauss' Law for Magnetism

$$\vec{\nabla} \cdot \vec{B} = 0$$

- There are no source of magnetic fields
  - No magnetic monopoles
  - Magnetic field lines can only circulate
- $$\vec{B} = \mu \vec{H} = \mu_0 \vec{H} + \vec{M}$$
- μ is a 3x3 tensor not a scalar (unless the material is isotropic)!
- μ may be a function of E and H! (giving rise to non-linear optics)
- μ can be measured using the Biot-Savart law $d\vec{B} = \dfrac{\mu I d\vec{L} \times \hat{r}}{4\pi r^2}$

Waves and Maxwell's Equations

- A charged particle is a source of an electric field
- When that particle moves it changes the (spatial distribution of) the electric field
- When the electric field changes it produces a circulating magnetic field
- If the particle accelerates this circulating magnetic field will change
- A changing magnetic field produces a circulating electric field
- The circulating electric field becomes the source of a circulating magnetic field

Boundary Conditions

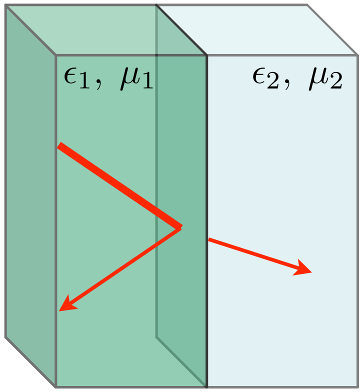
When an EM wave propagates across an interface, Maxwell's equations must be satisfied at the interface as well as in the bulk materials. The constraints necessary for this to occur are called the "boundary conditions"

$$\oint \vec{D} \cdot d\vec{A} = \int \sigma dA$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$$

$$\oint \vec{B} \cdot d\vec{A} = 0$$

$$\oint \vec{H} \cdot d\vec{s} = \int \vec{J} \cdot d\vec{A} + \frac{d}{dt} \int \vec{D} \cdot d\vec{A}$$



Boundary Conditions

Gauss' law can be used to find the boundary conditions on the component of the electric field that is perpendicular to the interface.

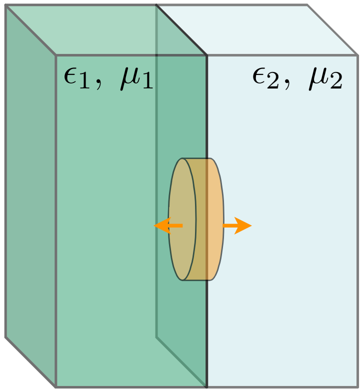
If the materials are dielectrics there will be no free charge on the surface (σ=0)

$$\oint \vec{D} \cdot d\vec{A} = \int \sigma dA$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{A}$$

$$\oint \vec{B} \cdot d\vec{A} = 0$$

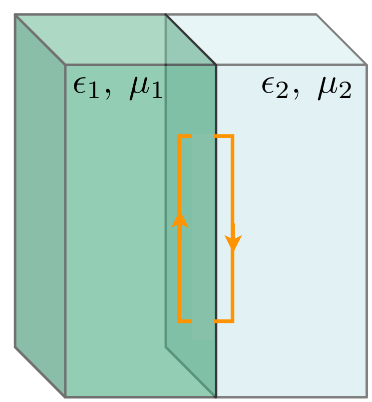
$$\oint \vec{H} \cdot d\vec{s} = \int \vec{J} \cdot d\vec{A} + \frac{d}{dt}\int \vec{D} \cdot d\vec{A}$$



$$D_{1\perp} - D_{2\perp} = \int \cancelto{0}{\sigma} dA \qquad \therefore \qquad D_{1\perp} = D_{2\perp}$$

Boundary Conditions

Faraday's law can be applied at the interface. If the loop around which the electric field is computed is made to have an infintesimal area the right side will go to zero giving a relationship between the parallel components of the electric field



$$\oint \vec{D} \cdot d\vec{A} = \int \sigma dA$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$$

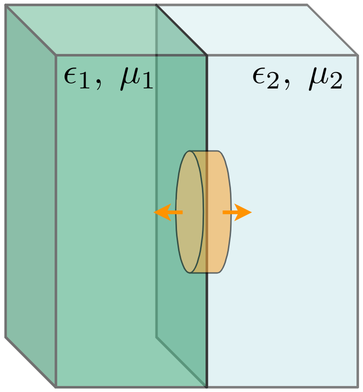
$$\oint \vec{B} \cdot d\vec{A} = 0$$

$$\oint \vec{H} \cdot d\vec{s} = \int \vec{J} \cdot d\vec{A} + \frac{d}{dt} \int \vec{D} \cdot d\vec{A}$$

$$E_{2\parallel} - E_{1\parallel} = -\frac{d}{dt} \int B \cdot \cancelto{0}{dA} \qquad \therefore \qquad E_{1\parallel} = E_{2\parallel}$$

Boundary Conditions

Gauss' law for magnetism gives a relationship between the perpendicular components of the magnetic field at the interface



$$\oint \vec{D} \cdot d\vec{A} = \int \sigma dA$$

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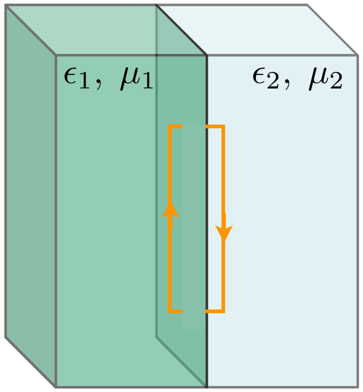
$$\oint \vec{B} \cdot d\vec{A} = 0$$

$$\oint \vec{H} \cdot d\vec{s} = \int \vec{J} \cdot d\vec{A} + \frac{d}{dt} \int \vec{D} \cdot d\vec{A}$$

$$B_{1\perp} A - B_{2\perp} A = 0 \qquad \therefore \qquad B_{1\perp} = B_{2\perp}$$

Boundary Conditions

Ampere's law applied to a loop at the interface that has an infintesimal area gives a relationship between the parallel components of the magnetic field. (Note that in most common materials $\mu=\mu_o$) In the absence of currents J=0 so



$$\oint \vec{D}\cdot d\vec{A} = \int \sigma dA$$

$$\oint \vec{E}\cdot d\vec{s} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A}$$

$$\oint \vec{B}\cdot d\vec{A} = 0$$

$$\oint \vec{H}\cdot d\vec{s} = \int \vec{J}\cdot d\vec{A} + \frac{d}{dt}\int \vec{D}\cdot d\vec{A}$$

$$H_{1\parallel}L - H_{2\parallel}L = \int \cancel{\vec{J}}^{\,0}\cdot d\vec{A} + \frac{d}{dt}\int \vec{D}\cdot \cancel{d\vec{A}}^{\,0} \quad \therefore \qquad H_{1\parallel} = H_{2\parallel}$$

Poynting's Theorem

The flow of electromagnetic energy is given by the Poynting vector

$$\vec{S} = \vec{E} \times \vec{H}$$

which has a magnitude that is the power per unit area carried by an electromagnetic wave in the direction of S.

S [W/m^2]$\Leftrightarrow$E [v/m] H [A/m]

Complex-Function Formalism

Steady-state sinusoidal functions of the form

$$a(t) = A\cos(\omega t + \alpha)$$

can be treated as having a complex amplitude

$$\widetilde{A} = Ae^{i\alpha}$$

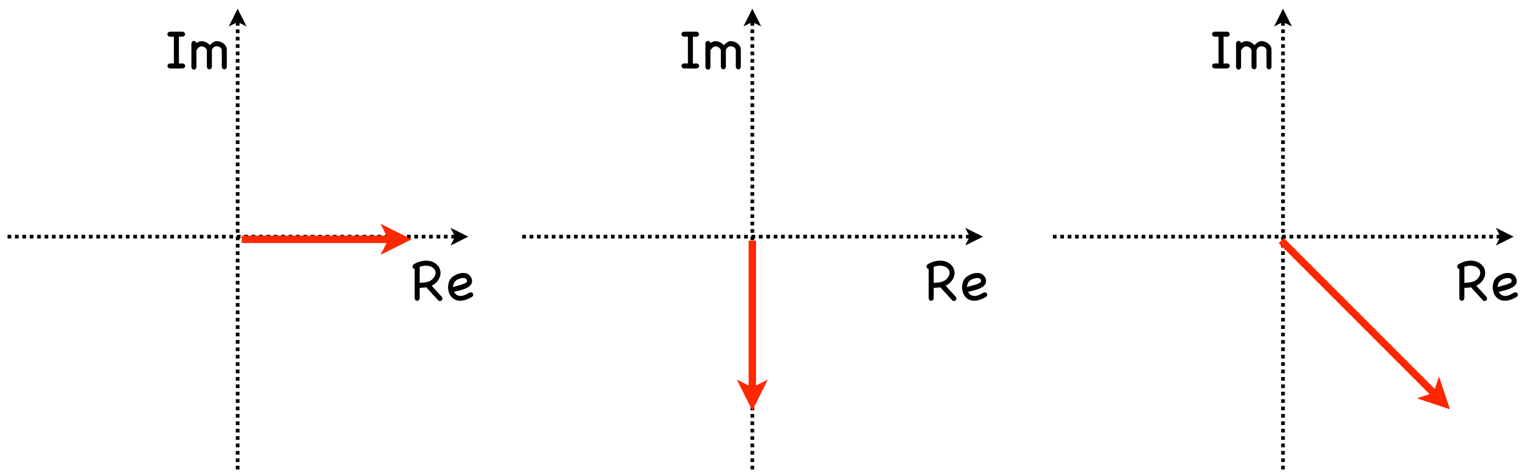
such that the function can be written as

$$a(t) = \mathrm{Re}\left[\widetilde{A}e^{i\omega t}\right]$$

or in shorthand $\widetilde{A}e^{i\omega t}$ where it is understood that the real part of this complex expression represents the original sinusoidal function

Phasors

The complex amplitude of a sinusoidal function can be represented graphically by a point (often an arrow from the origin to a point) in the complex plane



$$a(t) = \cos \omega t \qquad a(t) = \sin \omega t \qquad a(t) = \cos(\omega t) + \sin(\omega t)$$

$$\widetilde{A} = 1 \qquad \widetilde{A} = -i \qquad \widetilde{A} = \sqrt{2} e^{-i\pi/4}$$

Phasors

Addition of same-frequency sinusoidal functions involves factoring out the time dependance and simply adding the phasor amplitudes.

Addition of difference frequency sinusoidal function is often simplified by factoring out a sinusoidal component at the average frequency.

Multiplication of sinusoidal functions can not be done by multiplying phasors since

$$\mathrm{Re}\left[x\right]\mathrm{Re}\left[y\right] \neq \mathrm{Re}\left[xy\right]$$

Phasor Example

For electric field amplitudes described by

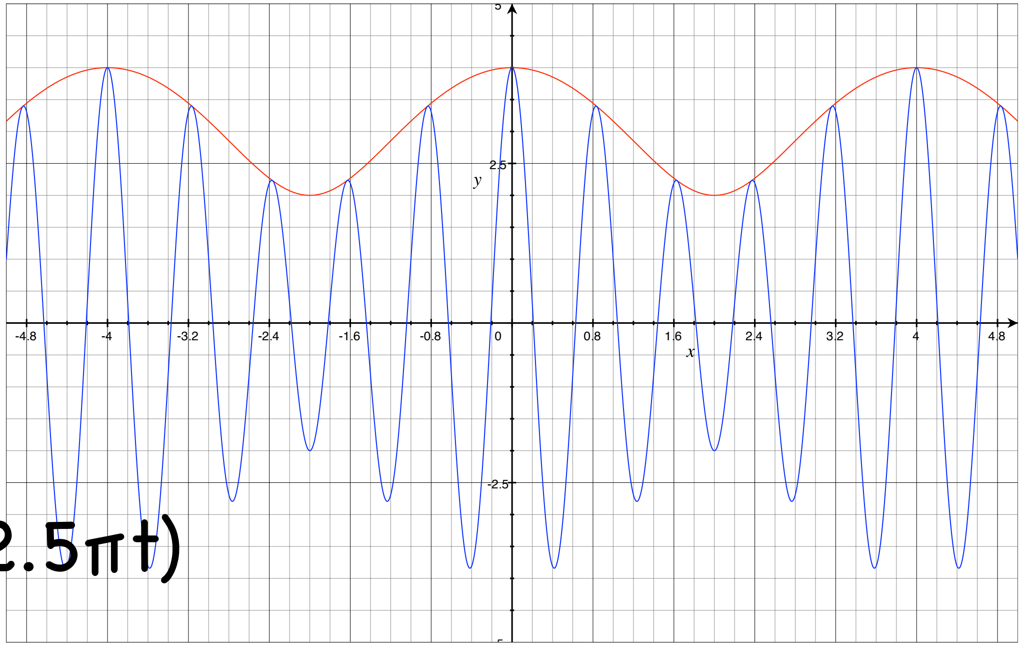
$$E_1 = E_{10} \cos\left(\omega_1 t\right)$$

and

$$E_2 = E_{20} \cos\left(\omega_2 t\right)$$

Use the phasor representation to find a representation of E_1+E_2 as a slow modulation of a field at the average frequency $\bar{\omega}=(\omega_1+\omega_2)/2$

Phasor Example



Example

E=cos(2πt)+3cos(2.5πt)

$$= \left[4E_{avg}^2 \cos^2\left(\frac{\Delta\omega t}{2}\right) + (\Delta E)^2 \sin^2\left(\frac{\Delta\omega t}{2}\right)\right]^{\frac{1}{2}} e^{i\bar{\omega}t + i\alpha}$$

$$E_{avg} \equiv \frac{E_{10} + E_{20}}{2} \qquad \bar{\omega} \equiv \frac{\omega_1 + \omega_2}{2} \qquad \alpha \equiv \arctan\left[\frac{\Delta E}{2E_{avg}} \tan\left(\frac{\Delta\omega t}{2}\right)\right]$$

$$\Delta E \equiv E_{10} - E_{20} \qquad \Delta\omega \equiv \omega_1 - \omega_2$$

Time Averages

Optical fields vary too fast to be directly detected, instead it is the irradiance averaged over many cycles that is detected as light.

$$\langle a(t)b(t)\rangle = \frac{1}{T}\int_0^T A\cos(\omega t + \alpha)B\cos(\omega t + \beta)dt$$

In terms of the phasor amplitudes this is

$$\langle a(t)b(t)\rangle = \frac{1}{2}\mathrm{Re}\left[\widetilde{A}\widetilde{B}^*\right]$$

Poynting Vector Example

For electric and magnetic fields given by

$$E = E_0 e^{i\omega t+\phi}$$

$$H = \frac{E_0}{Z_0} e^{i\omega t+\phi}$$

where

$$Z_0 = \eta_0 \equiv \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\Omega$$

is the impedance of free space, what is the irradiance of the wave? How much power is measured by a detector of area A?

Poynting Vector Example

For electric and magnetic fields given by

$$E = E_0 e^{i\omega t + \phi}$$

$$H = \frac{E_0}{\eta_0} e^{i\omega t + \phi}$$

where

$$\eta_0 \equiv \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\Omega$$

is the impedance of free space, what is the irradiance of the wave?

$$S_{avg} = \left\langle \vec{E} \times \vec{H} \right\rangle = \frac{1}{2}\mathrm{Re}\left[\widetilde{E}\widetilde{H}^*\right] = \frac{1}{2}\mathrm{Re}\left[E_0 e^{i\phi}\frac{E_0}{\eta_0}e^{-i\phi}\right] = \frac{E_0^2}{2\eta_0}$$

This is analogous to $P_{avg}=V^2/2R$ for AC circuits

$$P_{avg} = \vec{S}_{avg} \cdot \vec{A} \leq A\frac{E_0^2}{2\eta_0}$$

Wave Equation in Isotropic Materials

Starting with Faraday's law

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

take the curl of both sides

$$\begin{aligned} \vec{\nabla} \times \vec{\nabla} \times \vec{E} &= \vec{\nabla} \times (-\partial \vec{B}/\partial t) \\ &= -\partial(\vec{\nabla} \times \vec{B})/\partial t \end{aligned}$$

use vector calculus relationship to get

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \left(\vec{\nabla} \cdot \vec{\nabla}\right)\vec{E} = -\frac{\partial(\vec{\nabla} \times \vec{B})}{\partial t}$$

Use Ampere's law (in free space where J=0) $\vec{\nabla} \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t}$

and Gauss' law (in free space where ρ=0) $\vec{\nabla} \cdot \vec{D} = 0$

in an isotropic medium

$$\boxed{\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}}$$

Wave Equation in Crystals

In an anisotropic medium

$$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right)-\left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{E}=-\frac{\partial(\vec{\nabla}\times\vec{B})}{\partial t}$$

does not simplify as much since

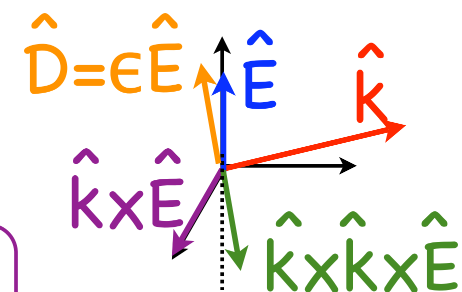
$\vec{\nabla}\cdot\vec{D}=0$ does not imply $\vec{\nabla}\cdot\vec{E}=0$

but rather

$$\vec{\nabla}\cdot\vec{D}=\vec{\nabla}\cdot\epsilon\vec{E}=\epsilon\vec{\nabla}\cdot\vec{E}+\vec{E}\cdot\nabla\epsilon$$

where $\nabla\epsilon\neq 0$. In this case it is usually easiest to write the wave equation as

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E}+\mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2}=0$$



or

$$\boxed{\vec{k}\times\vec{k}\times\widetilde{E}+\mu\epsilon\omega^2\widetilde{E}=0}$$

Spherical Solutions to the Wave Equation

$$\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

Consider solutions for E such that $\nabla^2 E$ and $\delta^2 E/dt^2$ are both proportional to E - allowing the two sides to differ only by a constant term.

$$\vec{E}(\vec{r},t) = \frac{\vec{E}_0}{r} e^{i\left(\vec{k}\cdot\vec{r} \pm \omega t\right)}$$

is one such solution in spherical coordinates. Using the relationship for the Laplacian of a spherically symmetric function:

$$\nabla^2 \psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)$$

Show that $\vec{E}(\vec{r},t)$ given above is a solution to the wave equation

Solutions to the Wave Equation

$$\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Downarrow$$

$$k^2 = \mu\epsilon\omega^2$$

Recall the meaning of k and ω ($k=2\pi/\lambda$, $\omega=2\pi/T$) we can express this as

$$\frac{\lambda}{T} = \frac{1}{\sqrt{\mu\epsilon}}$$

Since λ is the distance travelled by the wave in one cycle, and T is the time to travel one cycle, λ/T is the velocity of the wave, which can be determined from electrostatics and magnetostatics!

$$v = \frac{1}{\sqrt{\mu\epsilon}}$$

Solutions to the Wave Equation

From our solution in free space

$$\vec{E}(\vec{r},t) = \frac{\vec{E}_0}{r} e^{i\left(\vec{k}\cdot\vec{r} \pm \omega t\right)}$$

and Gauss' law in free space (ρ=0)

$$\vec{\nabla} \cdot \vec{E} = 0$$

We find that since $\vec{E}(\vec{r},t)$ only has a spatial dependance on r its divergence, given by

$$\vec{\nabla} \cdot \vec{\psi} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \psi_r) + \frac{1}{r\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi\ \psi_\phi) + \frac{1}{r\sin\phi}\frac{\partial \psi_\theta}{\partial \theta}$$

must be $$\vec{\nabla} \cdot \vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r(\vec{r},t)) = 0$$

implying $E_r(\vec{r},t) = 0$ meaning this must be a transverse wave (and isn't a solution in anisotropic media)

Speed of Light

In free-space where $\epsilon=\epsilon_0$ and $\mu=\mu_0$ the speed of light is **defined** to be $c\equiv 299792458$ m/s. In this sense any measurement of the speed of light in a vacuum is really a measurement of the length of a meter (the unit of time is also a defined quantity)

In material where $\epsilon=\kappa\epsilon_0$ and $\mu=\mu_r\mu_0$ the speed of light is $v=c/(\kappa\mu_r)^{1/2}$. We let $n\equiv(\kappa\mu_r)^{1/2}$ and call n the **index of refraction** for a material.

What is the physical interpretation of n?

If it is complex, what do the real and imaginary parts represent?

Index of Refraction

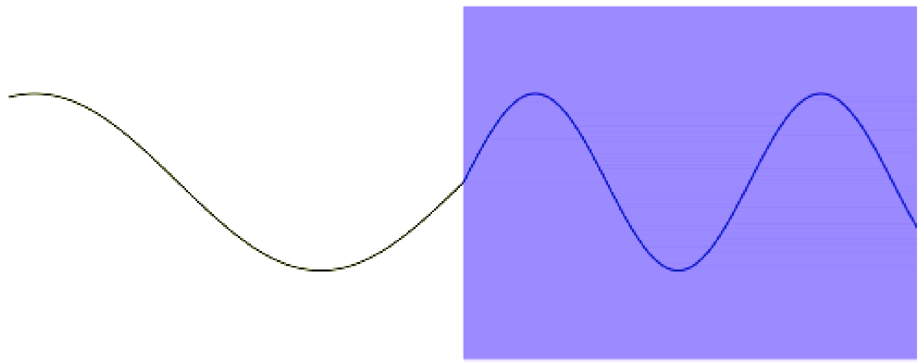
From our expression for the velocity of the wave $v=c/(\kappa\mu_r)^{1/2}$ we can substitute $n=(\kappa\mu_r)^{1/2}$ to get $v=c/n$

Thus n represents how much slower light travels in a material compared to free space.

Given the relation $c=nv=\omega/k$. If a wave travels from free space into a material causing it to slow down, does ω change, or does k change (or both)?

Index of Refraction

Consider a wave in free space entering a material. Doe the wavelength change, does the frequency change or both?



The frequency cannot change (or else the boundary would be discontinuous) so the wavelength (and hence k) must change so that $\lambda=\lambda_0/n$ and $k=nk_0$

Index of Refraction

Going back to the solution to the wave equation, we can express it explicitly for propagation in a material with index of refraction n

$$\vec{E}(\vec{r},t) = \frac{\vec{E}_0}{r} e^{i\left(n\vec{k}_0\cdot\vec{r}\pm\omega t\right)}$$

If n is complex such that $n=n'+i\,n''$

we have $$\vec{E}(\vec{r},t) = \frac{\vec{E}_0}{r} e^{i\left(n'\vec{k}_0\cdot\vec{r}\pm\omega t\right)} e^{-n''\vec{k}_0\cdot\vec{r}}$$

and we see n″ is related to the absorption coefficient α used in **Beer's law:** $I(x) = I_0 e^{-\alpha x}$ by $\alpha=2n''k_0$.

Phase Velocity

For a sinusoidal wave, or a waveform comprised of many sinusoidal components that all propagate at the same velocity, the waveform will move at the **phase velocity** of the sinusoidal components

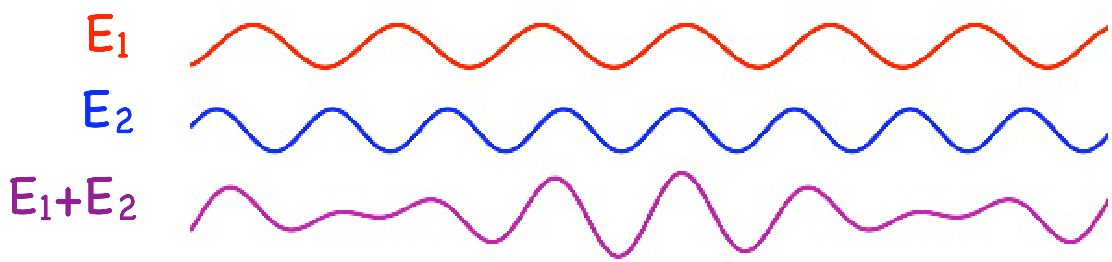
We've seen already that the phase velocity is

$$v_p=\omega/k$$

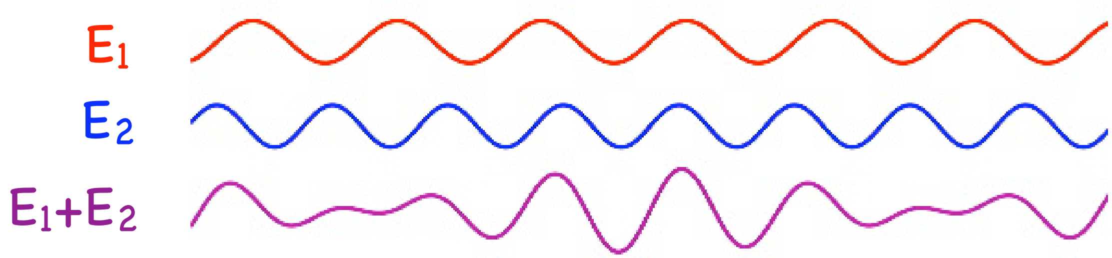
What happens if the different components of the wave have different phase velocities (i.e. because of dispersion)?

Phase and Group Velocity

No dispersion ($v_p = v_g$)



Dispersion ($v_p \neq v_g$)



Group Velocity

When the various frequency components of a waveform have different phase velocities, the phase velocity of the waveform is an average of these velocities (the phase velocity of the **carrier wave**), but the waveform itself moves at a different speed than the underlying carrier wave called the **group velocity**.

Group vs Phase velocity

An analogy that may be useful for understanding the difference comes from velodrome cycling:

Riders race as a team and take turns as leader with the old leader peeling away and going to the back of the pack

As riders make their way from the rear of the pack to the front they are moving faster than the group that they are in

Group Velocity

The phase velocity of a wave is

$$v = \frac{\omega}{k}$$

and comes from the change in the position of the wavefronts as a function of time

The waveform moves at a rate that depends on the relative position of the component wavefronts as a function of time. This is the group velocity and is

$$v_g = \frac{d\omega}{dk}$$

which can be found if you have

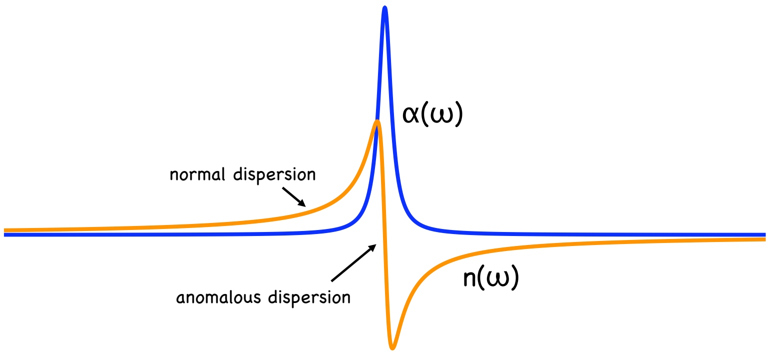
$$\omega = vk = \frac{c}{n(k)}k \qquad \text{giving} \qquad v_g = v\left(1 - \frac{k}{n}\frac{dn}{dk}\right)$$

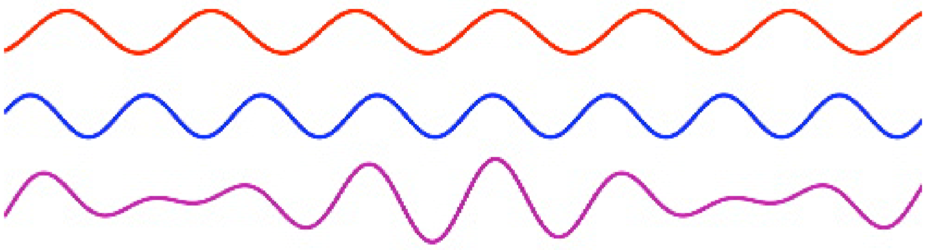
Slow Light

How slow can light be made to go?

In a Bose-Einstein Condensate light tuned to the atomic resonance tremendous dispersion and has been slowed to a speed of...

See Hau, et al. "Light speed reduction to 17 metres per second in an ultracold atomic gas", Nature 397, 594 - 598 (18 February 1999)





Example

Given the dispersion equation

$$n^2(\omega) = 1 + \frac{Ne^2}{\epsilon_0 m_e} \sum_j \left(\frac{f_j}{\omega_{0j}^2 - \omega^2} \right)$$

where f_j is the fraction of electrons that have a resonant frequency of ω_{0j}, find the phase velocity and group velocity of high frequency electromagnetic waves ($\omega >> \omega_{oj}$)

Example

$$n^2(\omega) = 1 + \frac{Ne^2}{\epsilon_0 m_e} \sum_j \left(\frac{f_j}{\omega_{0j}^2 - \omega^2} \right)$$

The phase velocity is v=c/n so

$$v = \frac{c}{\sqrt{1 + \frac{Ne^2}{\epsilon_0 m_e} \sum_j \left(\frac{f_j}{\omega_{0j}^2 - \omega^2} \right)}} \approx c \left(1 + \frac{Ne^2}{2\epsilon_0 m_e \omega^2} \right)$$

The group velocity can be found from

$$v_g = \frac{d\omega}{dk}$$

Example

$$n^2(\omega) = 1 + \frac{Ne^2}{\epsilon_0 m_e} \sum_j \left(\frac{f_j}{\omega_{0j}^2 - \omega^2} \right) \qquad v_g = \frac{d\omega}{dk}$$

using $\sum_j f_j = 1$ and $k = \frac{n\omega}{c}$

$$k = \frac{n\omega}{c} \approx \frac{\omega}{c} \left(1 - \frac{Ne^2}{2\epsilon_0 m_e \omega^2} \right)$$

$$\frac{dk}{d\omega} = \frac{1}{c} \frac{d(n\omega)}{d\omega} \approx \left(\frac{1}{c} + \frac{Ne^2}{2\epsilon_0 m_e \omega^2} \right)$$

$$v_g = \frac{d\omega}{dk} = \frac{c}{1 + Ne^2/2m_e\epsilon_0\omega^2}$$

Warning

For the analysis so far we have treated μ and ϵ as being scalars meaning the waves are propagating through isotropic materials.

What changes if the materials are not isotropic?

References

- Yariv & Yeh "Optical Waves in Crystals" chapter 1.1-1.5
- Hecht "Optics" chapter 2.3-2.9, 7.2, 7.3, 7.6