Coupled Mode Analysis

Chapter 4

Physics 208, Electro-optics

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Class Outline

- The dielectric tensor
- Plane wave propagation in anisotropic media
- The index ellipsoid
- Phase velocity, group velocity and energy
- Crystal types
- Propagation in uniaxial crystals
- Propagation in biaxial crystals
- Optical activity and Faraday rotation
- Coupled mode analysis of wave propagation

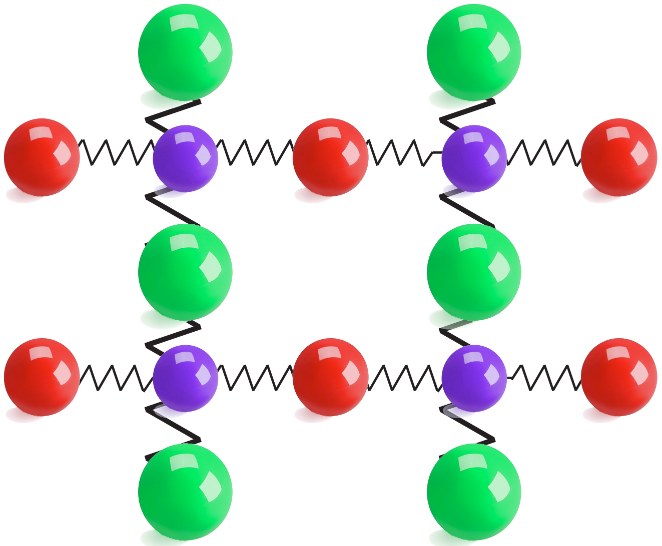
The Dielectric Tensor

- ϵ relates the electric field to the electric displacement by

$$\vec{D} = \epsilon \vec{E} = \epsilon_0 \vec{E} + \vec{P}$$

- In anisotropic materials the polarization may not be in the same direction as the driving electric field. Why?

Anisotropic Media



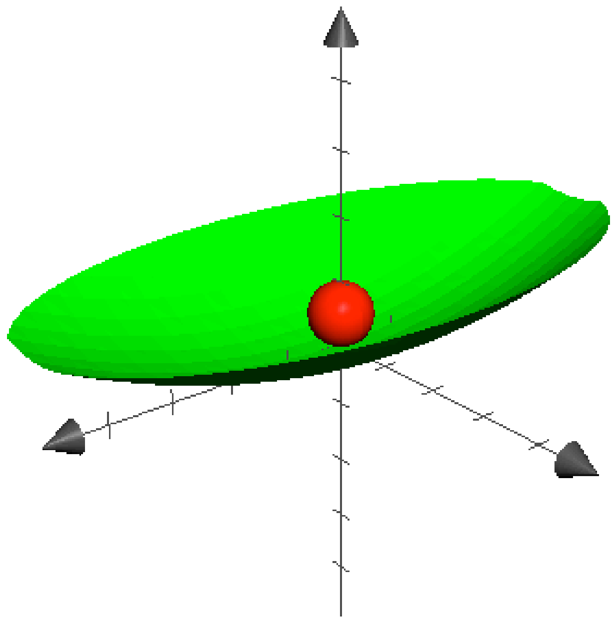
Charges in a material are the source of polarization. They are bound to neighboring nuclei like masses on springs.

In an anisotropic material the stiffness of the springs is different depending on the orientation.

The wells of the electrostatic potential that the charges sit in are not symmetric and therefore the material response (material polarization) is not necessarily in the direction of the driving field.

Analogies

What every-day phenomena can have a response in a direction different than the driving force?



The Dielectric Tensor

The susceptibility tensor χ relates the polarization of the material to the driving electric field by

$$\vec{P} = \epsilon_0 \bar{\chi} \vec{E}$$

or

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \epsilon_0 \begin{bmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

Thus the material permittivity ϵ in $\vec{D} = \epsilon \vec{E} = \epsilon_0 \vec{E} + \vec{P}$

is also a tensor $\bar{\epsilon} = \epsilon_0 \left(\bar{I} + \bar{\chi} \right)$

Tensor Notation

For brevity, we often express tensor equations such as

$$\vec{D} = \epsilon_0 \left(\bar{I} + \bar{\chi}\right) \vec{E}$$

in the form

$$D_i = \epsilon_0(\delta_{ij} + \chi_{ij})E_j$$

where summation over repeated indices is assumed, so that this is equivalent to

$$D_i = \sum_j \epsilon_0(\delta_{ij} + \chi_{ij})E_j$$

Dielectric Tensor Properties

The dielectric tensor is Hermetian such that

$$\epsilon_{ij} = \epsilon_{ji}^*$$

In a lossless material all elements of ϵ are real so the tensor is symmetric $\epsilon_{ij} = \epsilon_{ji}$ and can be described by 6 (rather than 9) elements

$$\begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix}$$

Plane Wave Propagation in Anisotropic Media

For a given propagation direction in a crystal (or other anisotropic material) the potential wells for the charges will be ellipsoidal, and so there will be two directions for the driving field for which the material response is in the same direction.

These two polarization states each have a (different) phase velocity associated with them. Light polarized at either of these angles will propagate through the material with the polarization unchanged.

Wave Equation in Anisotropic Materials

From Maxwell's equations in a dielectric (ρ=J=0), using d/dt→iω and ∇→$i\vec{k}$ we have

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}$$

$$i\vec{k} \times \vec{E} + i\omega \vec{B} = 0 \qquad i\vec{k} \times \vec{H} - i\omega \vec{D} = \vec{J}$$

$$\vec{k} \times \vec{E} = -\omega\mu\vec{H} \qquad \vec{k} \times \vec{H} = \omega\epsilon\vec{E}$$

$$\vec{k} \times \vec{k} \times \vec{E} + \omega^2\mu\epsilon\vec{E} = 0$$

Which is the wave equation for a plane wave, most easily analyzed in the **principle coordinate system** where ϵ is diagonal (i.e. a coordinate system aligned to the crystal axes).

Wave Equation in Anisotropic Materials

$$\vec{k} \times \vec{k} \times \vec{E} + \omega^2 \mu \epsilon \vec{E} = 0$$

or

$$\begin{bmatrix} \omega^2\mu\epsilon_x - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & \omega^2\mu\epsilon_y - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & \omega^2\mu\epsilon_z - k_x^2 - k_y^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0$$

for which the determinant of the matrix must be zero for solutions (other than k=ω=0) to exist

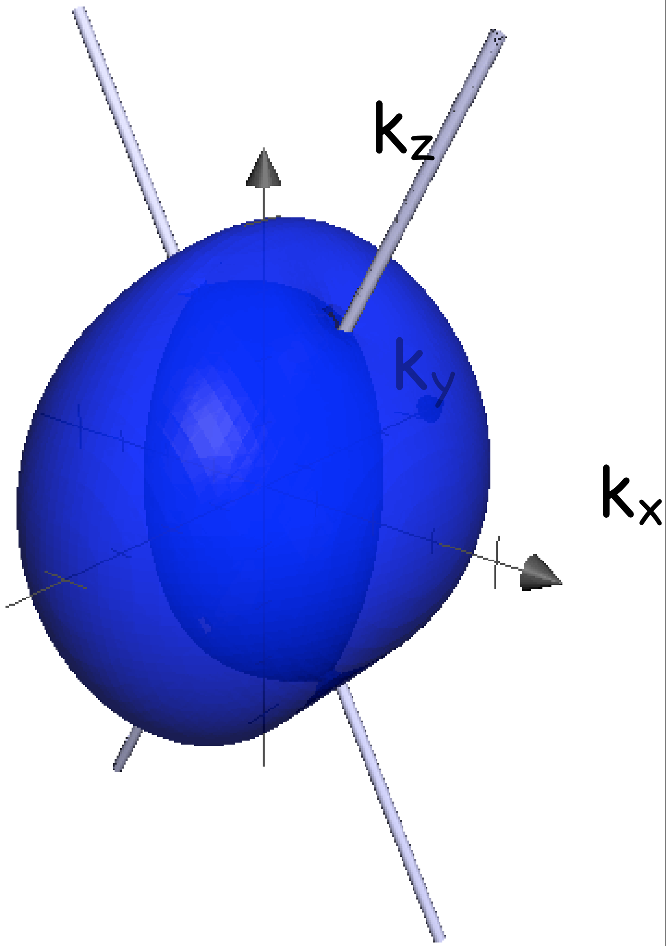
$$\begin{vmatrix} \omega^2\mu\epsilon_x - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & \omega^2\mu\epsilon_y - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & \omega^2\mu\epsilon_z - k_x^2 - k_y^2 \end{vmatrix} = 0$$

This defines two surfaces in k-space called the normal shells.

Normal Shells

In a given direction, going out from the origin, a line intersect this surface at two points, corresponding to the magnitude of the two k-vectors (and hence the two values of the phase velocity) wave in this direction can have.

The two directions where the surfaces meet are called the optical axes. Waves propagating in these directions will have only one possible phase velocity.



Wave Equation in Anisotropic Materials

$$\vec{k} \times \vec{k} \times \vec{E} + \omega^2 \mu \epsilon \vec{E} = 0$$

or

$$\begin{bmatrix} \omega^2 \mu \epsilon_x - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & \omega^2 \mu \epsilon_y - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & \omega^2 \mu \epsilon_z - k_x^2 - k_y^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0$$

$$\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = E_0 \begin{bmatrix} \frac{k_x}{k^2 - \omega^2 \mu \epsilon_x} \\ \frac{k_y}{k^2 - \omega^2 \mu \epsilon_y} \\ \frac{k_z}{k^2 - \omega^2 \mu \epsilon_z} \end{bmatrix}$$

where $E_0^2 = E_x^2 + E_y^2 + E_z^2$

Example

Find the possible phase velocities ($v_p=\omega/k$) for a wave propagating along the x-axis in a crystal, and the associated polarization directions.

L'Hopital's Rule

$$v_p = \frac{1}{\sqrt{\mu\epsilon_y}} \quad \text{or} \quad v_p = \frac{1}{\sqrt{\mu\epsilon_z}}$$

L'Hopital's rule can be applied to limit problems if the following conditions are met:

(1) The limit is written as a quotient AND
(2) The quotient is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

polarized in the y or z direction respectively

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

L'Hopital's rule can be applied several times, as long as the quotient is $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Wave Propagation in Anisotropic Materials

Gauss' law $\vec{\nabla} \cdot \vec{D} = \rho$
in a dielectric (ρ=0) can be written as

$$i\vec{k} \cdot \vec{D} = 0$$

implying the propagation direction $\hat{k}$ is orthogonal to the displacement vector $\vec{D}$.

But the Poynting vector $\vec{S} = \vec{E} \times \vec{H}$ which gives the direction of energy flow is orthogonal to the electric field $\vec{E}$, not the displacement vector. Thus energy does not flow in the direction of the wave's propagation if the polarization of the wave ($\hat{E}$) is not an eigen-vector of the material's dielectric tensor!

Spatial Walkoff

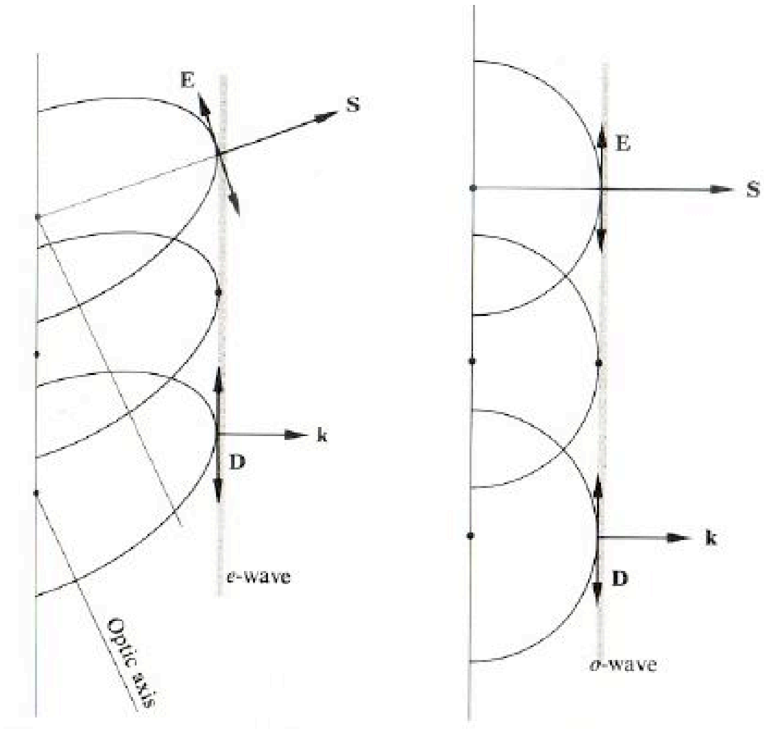


Image 8.26 from Hecht

Eigenstates of a material

For any direction of propagation there exists two polarization directions in which the displacement vector is parallel to the transverse component of the electric field. These are the eigenstates of the material.

Requiring $\vec{D}$ and $\vec{E}$ be parallel in the matrix equation

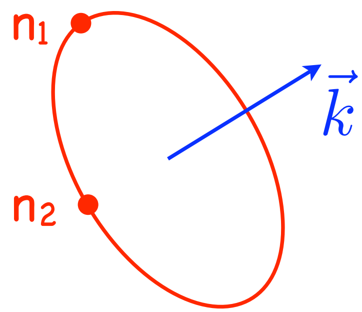
$$\vec{D} = \bar{\epsilon}\vec{E}$$

is equivalent to finding the eigenvalues and eigenvectors of ϵ satisfying

$$\lambda_i \vec{u_i} = \bar{\epsilon}\vec{u_i}$$

Index Ellipsoid

Index of refraction as a function of polarization angle

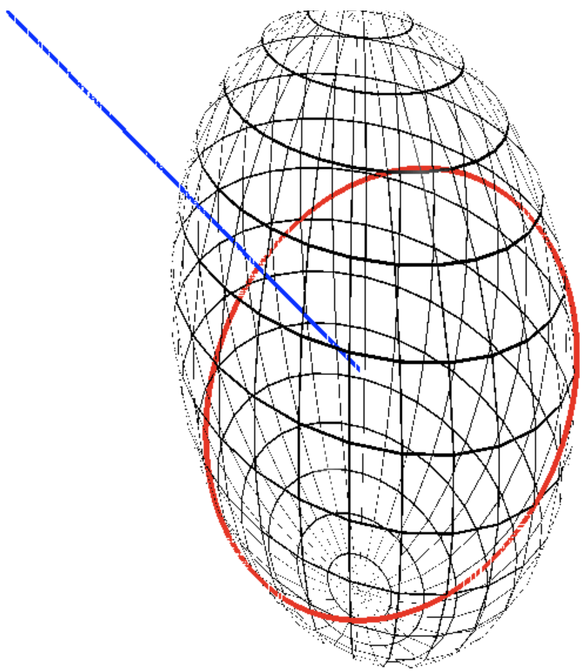


The polarization directions that have a max and min index of refraction form the major and minor axes of an ellipse defining n(θ) the index fora wave with the electric displacement vector at an angle of θ in the transverse plane.

Index Ellipsoid

The sum of all index ellipses plotted in three dimensions is the index ellipsoid, defined by

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1$$



Example

Find the angle of the direction of the optical axis for Lithium Niobate

Example

Find the angle of the direction of the optical axis for Topaz

Crystal Types

The index ellipsoid is determined by the three principle indices of refraction n_x, n_y and n_z.

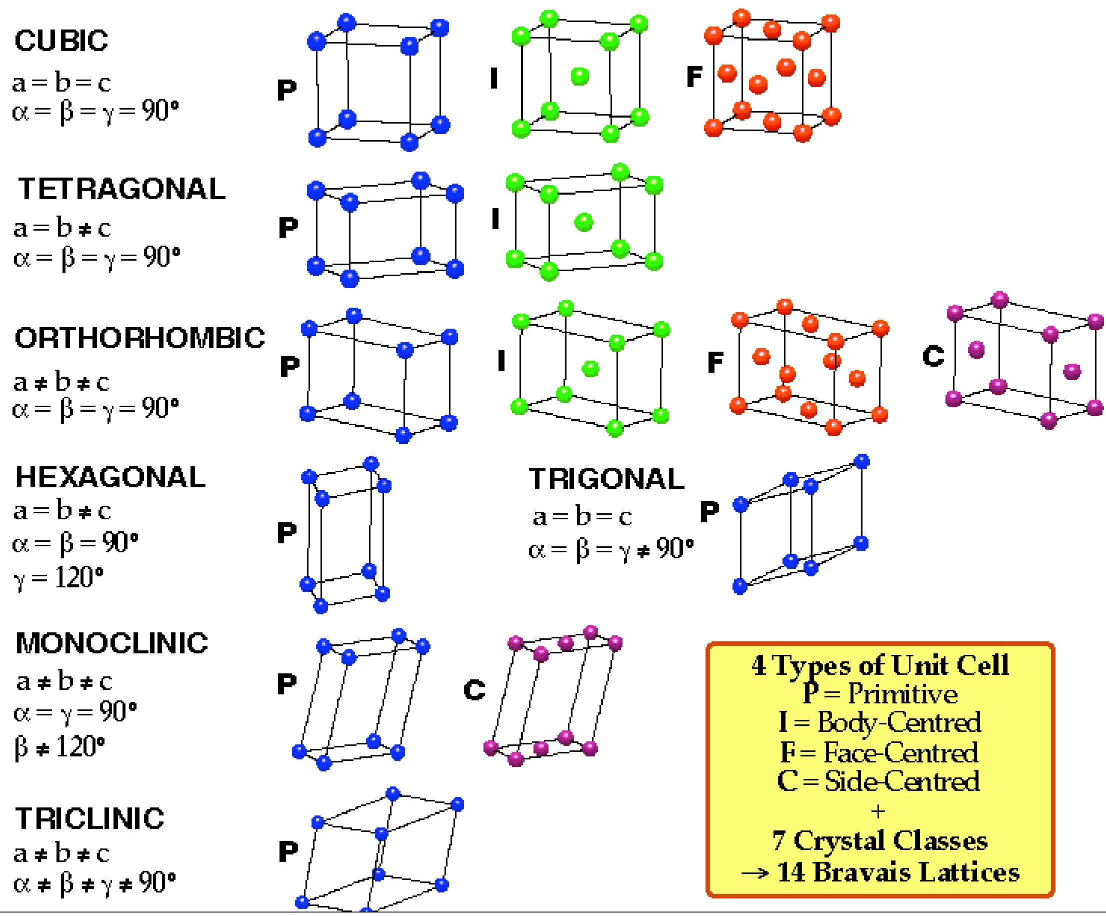
A crystal with $n_x \neq n_y \neq n_z$ will have _____ optical axes

A crystal with $n_x = n_y \neq n_z$ will have _____ optical axes

A crystal with $n_x = n_y = n_z$ will have ______ optical axes

Crystal Properties

The geometry of a crystal determines the form of the dielectric tensor



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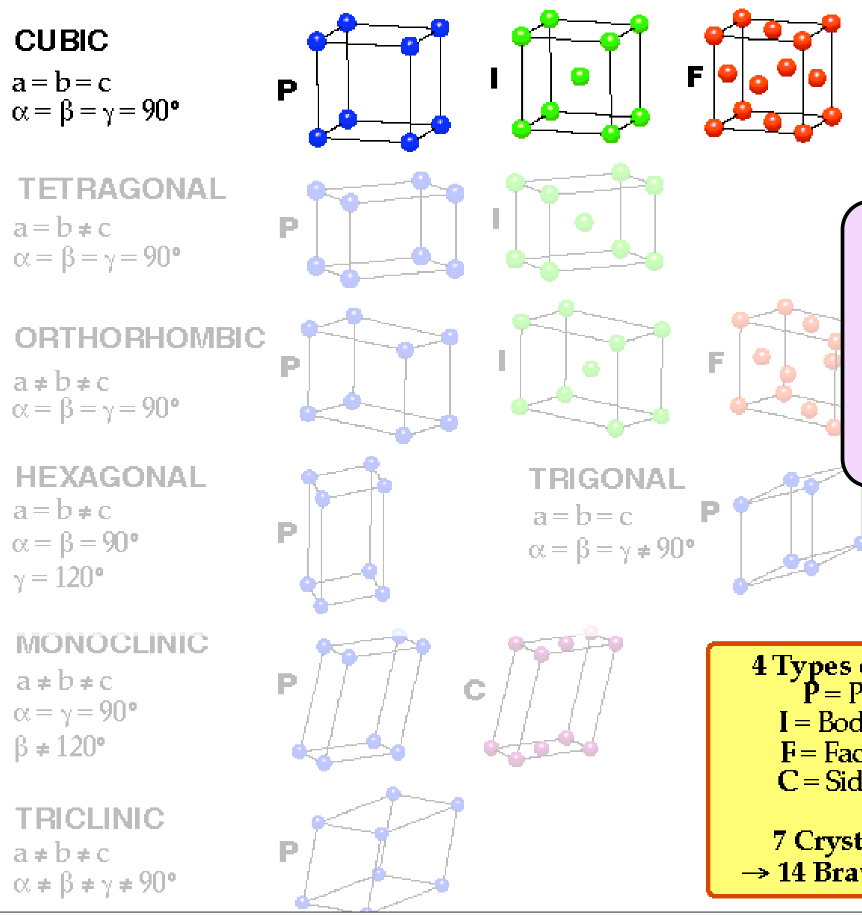


Isotropic

$$\epsilon = \epsilon_0 \begin{bmatrix} n^2 & 0 & 0 \\ 0 & n^2 & 0 \\ 0 & 0 & n^2 \end{bmatrix}$$

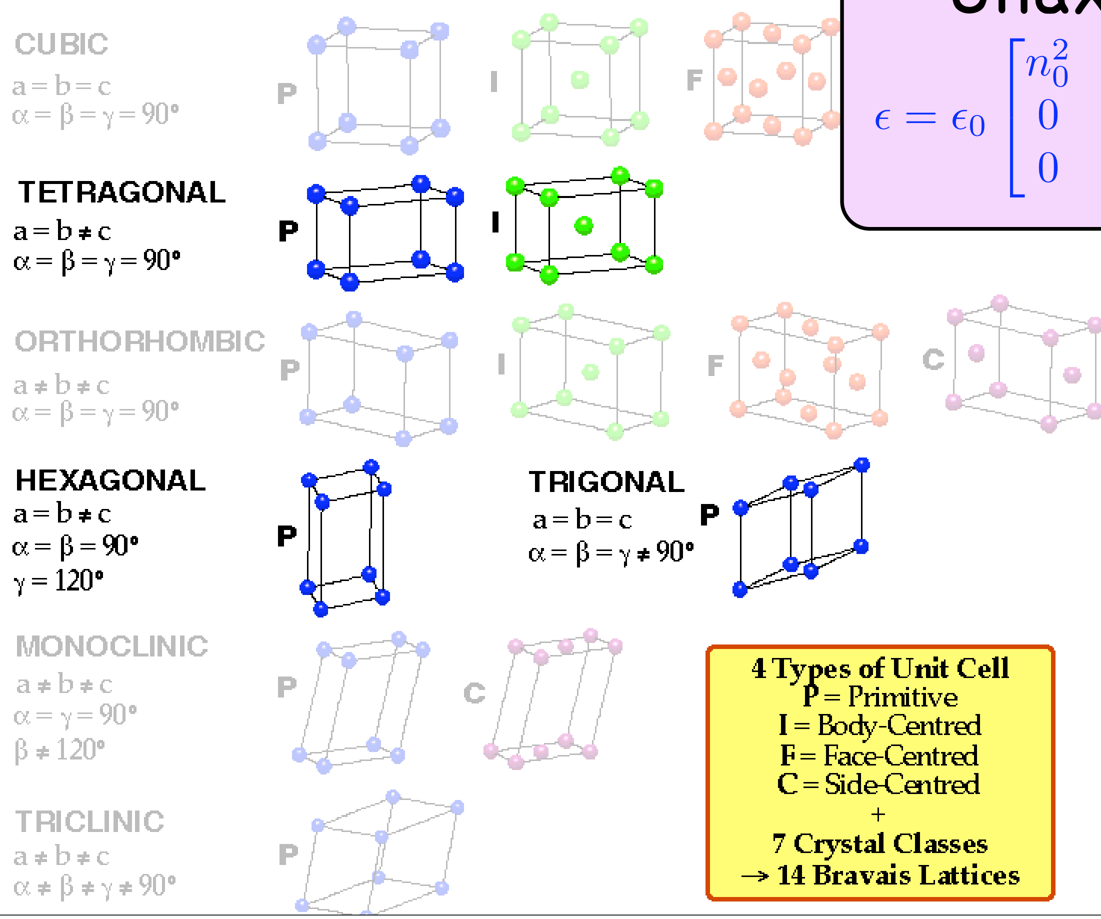
4 Types of Unit Cell
P = Primitive
I = Body-Centred
F = Face-Centred
C = Side-Centred
+
7 Crystal Classes
→ 14 Bravais Lattices

Crystal Properties

The geometry of a crystal determines the form of the dielectric tensor

Unaxial

$$\epsilon = \epsilon_0 \begin{bmatrix} n_0^2 & 0 & 0 \\ 0 & n_0^2 & 0 \\ 0 & 0 & n_e^2 \end{bmatrix}$$



4 Types of Unit Cell
P = Primitive
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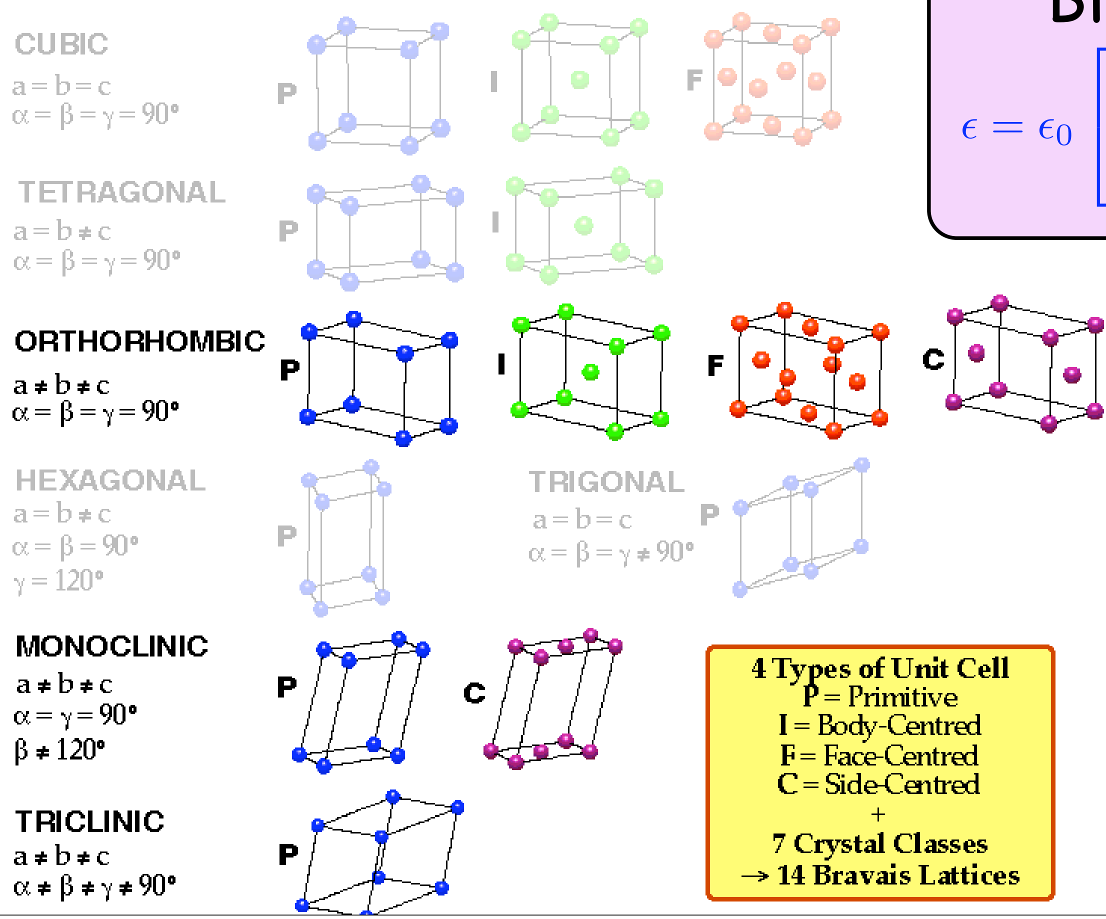
+
7 Crystal Classes
→ 14 Bravais Lattices

Crystal Properties

The geometry of a crystal determines the form of the dielectric tensor

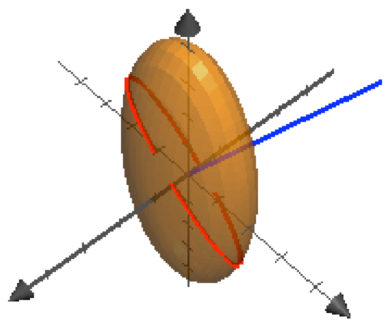
Biaxial

$$\epsilon = \epsilon_0 \begin{bmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{bmatrix}$$



Biaxial Crystals

By convention $n_x < n_y < n_z$ so the optical axis is always in the xz plane

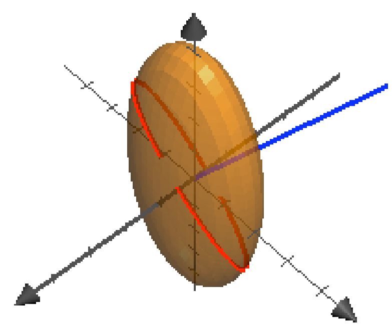


Uniaxial Crystals

By convention $n_o \equiv n_x = n_y$ and $n_e \equiv n_z$

The optical axis in a uniaxial crystal is always in the z-direction

A negative uniaxial crystal is one in which $n_e < n_o$ while a positive uniaxial crystal has $n_e > n_o$.



Example

A wave is propagating in the [1,1,1] direction in Mica, what are the two principle indices of refraction and in which direction are the eigenpolarizations?

Example

A wave is propagating in the [1,1,1] direction in Mica, what are the two principle indices of refraction and in which polarization directions do these correspond to?

plane of polarization is given by $\vec{k} \cdot \vec{r} = 0$ so x+y+z=0.
The intersection of this plane and the index ellipsoid, given by

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1$$

is,

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{(x+y)^2}{n_z^2} = 1$$

Find extremes of $r^2=x^2+y^2+z^2$ subject to the preceding two constraints to find directions of transverse eigenpolarizations. Plug directions back into $r=(x^2+y^2+z^2)^{1/2}$ to find index for each polarization

Example Solution in Mathematica

```
nx = 1.552;
ny = 1.582;
nz = 1.588;

In[148]:= eq1 = (x / nx) ^2 + (y / ny) ^2 + (z / nz) ^2 == 1;
          eq2 = x + y + z == 0;
          eq3 = r == Sqrt[x^2 + y^2 + z^2];

In[151]:= rsol = r /. Solve[{eq1, eq2, eq3}, {x, y, r}][[1]][[1]]
```

Out[151]= $\sqrt{\left(2.45481 + 0.\ z + 0.0269076\ z^2 - 0.0328601\ \sqrt{1.29082 - 1.\ z}\ z\ \sqrt{1.29082 + z}\right)}$

```
In[208]:= rsolmax = Maximize[rsol, z]
          zmax = z /. rsolmax[[2]][[1]];
          rmax = rsolmax[[1]];
          rsolmin = Minimize[rsol, z]
          zmin = z /. rsolmin[[2]][[1]];
          rmin = rsolmin[[1]];

Out[208]= {1.58295, {z → -0.936293}}

Out[211]= {1.56375, {z → 0.280416}}

In[220]:= solmax = Solve[{eq1, eq2, eq3}, {x, y, r}] /. z → zmax
          xmax = x /. solmax[[1]];
          ymax = y /. solmax[[1]];
          solmin = Solve[{eq1, eq2, eq3}, {x, y, r}] /. z → zmin
          xmin = x /. solmin[[1]];
          ymin = y /. solmin[[1]];

Out[220]= {{r → 1.58295, x → -0.303467, y → 1.23976},
           {r → 1.56559, x → 1.22184, y → -0.285544}}

Out[223]= {{r → 1.56375, x → -1.21895, y → 0.938534},
           {r → 1.57116, x → 0.943902, y → -1.22432}}

In[228]:= p1 =
           {rmax, {xmax, ymax, zmax} /
             Sqrt[xmax^2 + ymax^2 + zmax^2]}
          p2 =
           {rmin, {xmin, ymin, zmin} /
             Sqrt[xmin^2 + ymin^2 + zmin^2]}

Out[228]= {1.58295, {-0.19171, 0.783194, -0.591485}}

Out[229]= {1.56375, {-0.779504, 0.600181, 0.179323}}
```

Index of extraordinary ray

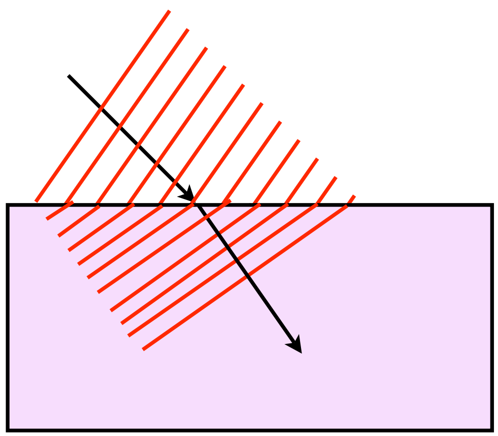
The index of refraction of the ordinary ray is always equal to the index seen when the ray propagates along the optical axis

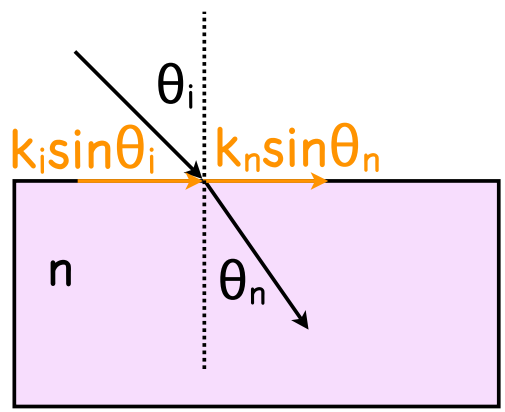
The extra-ordinary ray sees an index of refraction that is a function of the ray propagation angle from the optical axis. In a uniaxial crystal

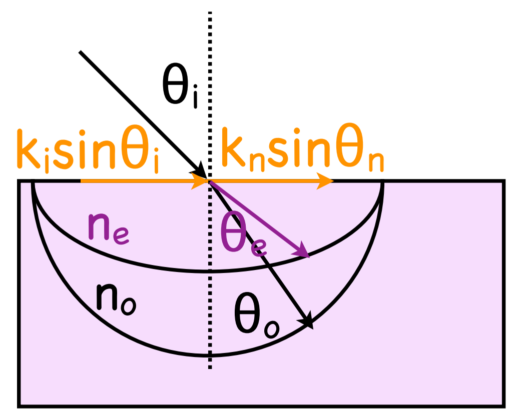
$$\frac{1}{n_e^2(\theta)} = \frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2}$$

Double Refraction

Snell's law requires the tangential component of the k-vector be continuos across a boundary



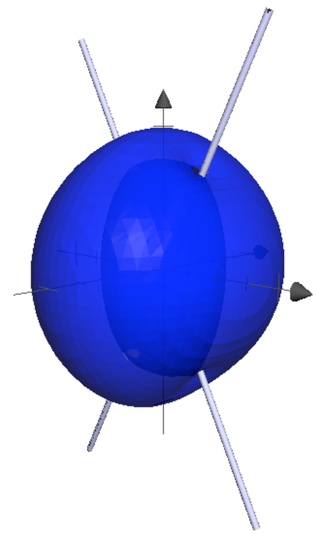




Conical Refraction

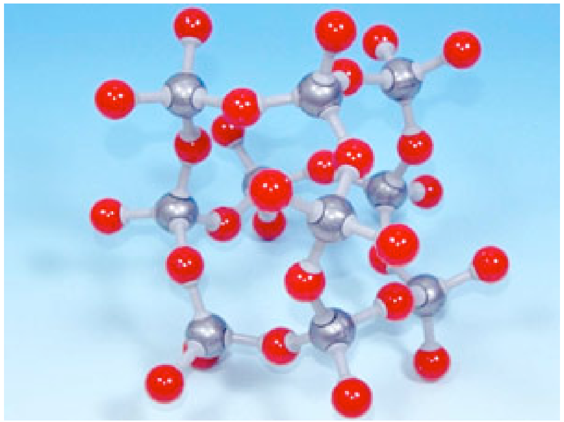
The normal surface is a surface of constant ω in k-space. Group velocity $v_g = \nabla_k \omega$ is perpendicular to the normal surfaces, but in a biaxial crystal the normal surfaces are singular along the optical axis.

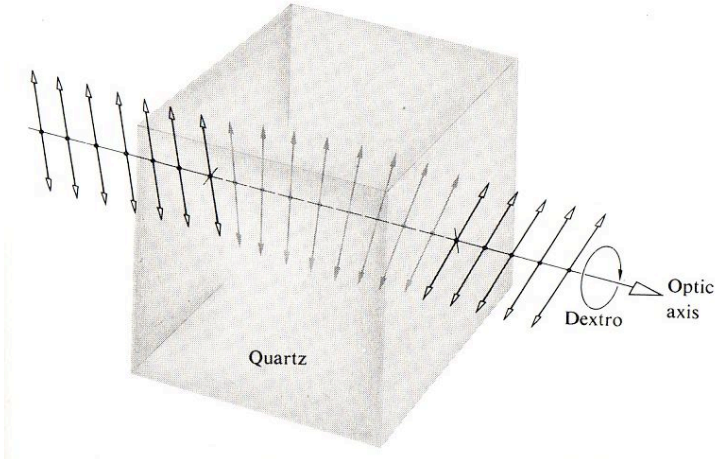
The surrounding region is conical, thus a wave along the optical axis will spread out conically



Optical Activity

Quartz and other materials with a helical molecular structure exhibit "Optical Activity", a rotation of the plane of polarization when passing through the crystal.





Optical Activity

Optically active materials can be thought of as birefringent, having a different index of refraction for right-circular polarization and for left-circular polarization.

The specific rotary power describes how much rotation there is per unit length

$$\rho = \frac{\pi}{\lambda}(n_l - n_r)$$

The source of this optical activity is the induced dipole moment of the molecule formed by changing magnetic flux through the helical molecule

Optical Activity

The displacement vector thus has an additional contribution in a direction perpendicular to the electric field

$$\vec{D} = \epsilon \vec{E} + i\epsilon_0 \vec{G} \times \vec{E}$$

where $\vec{G} = (g_{ij} k_i k_j / k_0^2)\hat{k}$ is called the gyration vector and

$$\vec{G} \times \vec{E} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ G_x & G_y & G_z \\ E_x & E_y & E_z \end{bmatrix} = \begin{bmatrix} 0 & -G_z & G_y \\ G_z & 0 & -G_x \\ -G_y & G_x & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = [G]\vec{E}$$

so we can define an effective dielectric tensor

$\epsilon' = \epsilon + i\epsilon_0 [G]$ so that $\vec{D} = \epsilon' \vec{E}$

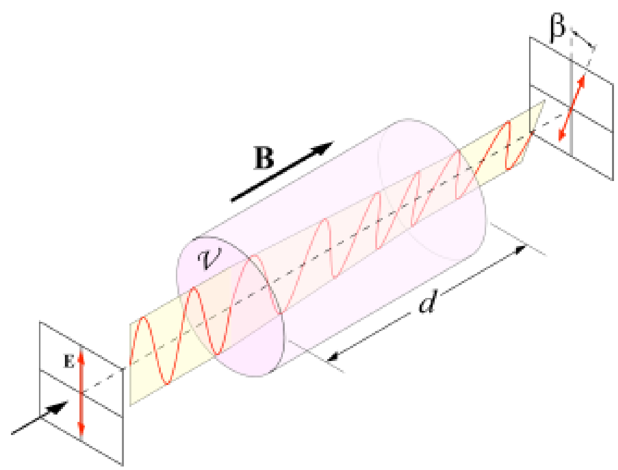
allowing the wave equation to be solved for eigenpolarizations of propagation and two corresponding indices of refraction that depend on [G]

Faraday Rotation

A form of optical activity induced by an external magnetic field.

$$\rho = VB$$

where V is called the Verdet constant. The sense of rotation depends on the direction of propagation relative to the direction of the magnetic field



Faraday Rotation

The motion of the charges in a material driven by the electric field feel a Lorentz force $q\vec{v} \times \vec{B}$ resulting in an induced dipole with a term proportional to $\vec{B} \times \vec{E}$ such that

$$\vec{D} = \epsilon \vec{E} + i\epsilon_0 \gamma \vec{B} \times \vec{E}$$

where $\gamma = -V n_0 \lambda_0 / \pi$.

For Faraday rotation the imaginary term is proportional to B, while for optical activity it is proportional to k.

Thus a wave double passing (in opposite directions) a material will net zero polarization rotation from optical activity, but twice the one-way rotation due to Faraday rotation.

Exercise

Given an isotropic material that is perturbed such that

$$\vec{D} = \epsilon\vec{E} + i\Delta\epsilon\vec{E}$$

where $\Delta\epsilon$ is

$$\Delta\epsilon = \epsilon_o \begin{bmatrix} 0 & -G_z & G_y \\ G_z & 0 & -G_x \\ -G_y & G_x & 0 \end{bmatrix}$$

find the eigenpolarizations for the electric field and associated indices of refraction. If a linearly polarized plane wave were to propagate along the direction of G through a thickness d of such material, what would the output wave look like?

Exercise

In[1]:=
```
M = ({{ε, 0, 0}, {0, ε, 0}, {0, 0, ε}} -
     I ε0 {{0 , -Gz, Gy}, {Gz, 0, -Gx}, {-Gy, Gx, 0}})
MatrixForm[M]
```

Out[2]//MatrixForm=

$$\begin{pmatrix} \epsilon & i\,Gz\,\epsilon 0 & -i\,Gy\,\epsilon 0 \\ -i\,Gz\,\epsilon 0 & \epsilon & i\,Gx\,\epsilon 0 \\ i\,Gy\,\epsilon 0 & -i\,Gx\,\epsilon 0 & \epsilon \end{pmatrix}$$

In[38]:=
```
NumericalValues =
 {ε → 1.5^2, Gx → 0, Gy → 0 , Gz → 0.001,  ε0 → 1}
```

Out[38]= {ε → 2.25, Gx → 0, Gy → 0, Gz → 0.001, ε0 → 1}

In[5]:=
```
Simplify[Eigensystem[M]]
```

Out[5]=

$$\Bigg\{\Big\{\epsilon,\ \epsilon-\sqrt{(Gx^2+Gy^2+Gz^2)\,\epsilon 0^2},\ \epsilon+\sqrt{(Gx^2+Gy^2+Gz^2)\,\epsilon 0^2}\Big\},$$

$$\Big\{\Big\{\frac{Gx}{Gz},\ \frac{Gy}{Gz},\ 1\Big\},\ \Big\{\frac{Gx\,Gy\,\epsilon 0+i\,Gz\sqrt{(Gx^2+Gy^2+Gz^2)\,\epsilon 0^2}}{Gy\,Gz\,\epsilon 0-i\,Gx\sqrt{(Gx^2+Gy^2+Gz^2)\,\epsilon 0^2}},$$

$$-\frac{i\,(Gx^2+Gz^2)\,\epsilon 0}{i\,Gy\,Gz\,\epsilon 0+Gx\sqrt{(Gx^2+Gy^2+Gz^2)\,\epsilon 0^2}},\ 1\Big\},$$

$$\Big\{\frac{Gx\,Gy\,\epsilon 0-i\,Gz\sqrt{(Gx^2+Gy^2+Gz^2)\,\epsilon 0^2}}{Gy\,Gz\,\epsilon 0+i\,Gx\sqrt{(Gx^2+Gy^2+Gz^2)\,\epsilon 0^2}},$$

$$\frac{i\,(Gx^2+Gz^2)\,\epsilon 0}{-i\,Gy\,Gz\,\epsilon 0+Gx\sqrt{(Gx^2+Gy^2+Gz^2)\,\epsilon 0^2}},\ 1\Big\}\Big\}\Bigg\}$$

In[39]:=
```
IndexofRefraction =
 Sqrt[Eigenvalues[M /. NumericalValues]]
val1 =
  (Eigensystem[M /. NumericalValues])[[2]][[1]][[1]];
```

Out[39]= {1.50033, 1.5, 1.49967}

In[41]:=
```
EigenPolarization =
 Chop[Eigenvectors[M /. NumericalValues] / val1]
```

Out[41]= {{1., -1. i, 0}, {0, 0, 1.41421 i}, {1., 1. i, 0}}

Coupled Mode Analysis

When the eigenstates of an unperturbed system are known, a small perturbation can be treated as a coupling between these states.

Rather than rediagonalizing the matrix for ϵ one could solve the wave equation with the perturbed matrix $\epsilon+\Delta\epsilon$

$$E = A_1 e^{i(k_1 z - \omega t)}\hat{e}_1 + A_2 e^{i(k_2 z - \omega t)}\hat{e}_2$$

with ϵ

$$\downarrow$$

$$E = A_3 e^{i(k_3 z - \omega t)}\hat{e}_3 + A_4 e^{i(k_4 z - \omega t)}\hat{e}_4$$

or

with $\epsilon+\Delta\epsilon$

$$E = A_1(z) e^{i(k_1 z - \omega t)}\hat{e}_1 + A_2(z) e^{i(k_2 z - \omega t)}\hat{e}_2$$

Exercise

Given an isotropic material that is perturbed such that

$$\vec{D} = \epsilon\vec{E} + i\Delta\epsilon\vec{E}$$

where $\Delta\epsilon$ is

$$\Delta\epsilon = \begin{bmatrix} 0 & -G_z & G_y \\ G_z & 0 & -G_x \\ -G_y & G_x & 0 \end{bmatrix}$$

find the coupled mode expression for the electric field for a linearly polarized plane wave propagating along the direction of G.

From

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} - \omega^2 \mu (\varepsilon + \Delta\varepsilon) \vec{E} = 0$$

Find solutions of the form

$$E = E_x(z)\, e^{i(kz - \omega t)}\, \hat{i} + E_y(z)\, e^{i(kz-\omega t)}\, \hat{j}$$

where $\frac{\omega}{k} = \frac{1}{\sqrt{\mu \varepsilon}}$ i.e. we are finding slowly varying amplitudes $E_x(z)$ and $E_y(z)$ to the unperturbed solutions.

Since E varies only along z $\vec{\nabla} \rightarrow \hat{k} \frac{d}{dz}$

use $\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\nabla^2 E + \vec{\nabla}(\vec{\nabla} \cdot \vec{E})$

$$-\frac{d^2}{dz^2} \vec{E} + \hat{k} \frac{d}{dz} \left(\hat{k} \frac{d}{dz} \cdot \vec{E} \right) - \omega^2 \mu (\varepsilon + \Delta\varepsilon) \vec{E} = 0$$

$$-\frac{d^2}{dz^2} \underbrace{\left[\vec{E} - \hat{k}(\hat{k} \cdot E) \right]}_{(E_x \hat{i} + E_y \hat{j} + E_z \hat{k}) - \hat{k} E_z} - \omega^2 \mu (\varepsilon + \Delta\varepsilon) \vec{E} = 0$$

$(E_x \hat{i} + E_y \hat{j} + E_z \hat{k}) - \hat{k} E_z$ is transverse component of $\vec{E}$, $\vec{E}_T$

for $\vec{E} \cdot \vec{k} = 0$ (assume isotropic medium) $\vec{E}_T = \vec{E}$

$$-\frac{d^2}{dz^2} \vec{E} + \omega^2 \mu (\varepsilon + \Delta\varepsilon) \vec{E} = 0$$

← The 1-dimensional wave equation in isotropic media

$$-\frac{d^2}{dz^2}\left(E_x(z)\, e^{i(kz-\omega t)}\right) + \omega^2\mu(\epsilon+\Delta\epsilon)\, E_x(z)\, e^{i(kz-\omega t)} \;\hat{i}$$

$$-\frac{d^2}{dz^2}\left(E_y(z)\, e^{i(kz-\omega t)}\right) + \omega^2\mu(\epsilon+\Delta\epsilon)\, E_y(z)\, e^{i(kz-\omega t)} \;\hat{j} \quad = 0$$

using chain rule to evaluate terms like

$$-\frac{d^2}{dz^2}\left(E_x(z)\, e^{i(kz-\omega t)}\right) = -\frac{d}{dz}\left[E_x(z)\, ik\, e^{i(kz-\omega t)} + e^{i(kz-\omega t)}\frac{d}{dz}E_x(z)\right]$$

$$= +E_x(z)\, k^2 e^{i(kz-\omega t)} - ik\, e^{i(kz-\omega t)}\frac{d}{dz}E_x(z)$$

$$- ik\, e^{i(kz-\omega t)}\frac{d}{dz}E_x(z) - e^{i(kz-\omega t)}\frac{d^2}{dz^2}E_x(z)$$

$$= \left[-\frac{d^2}{dz^2}E_x(z) - 2ik\frac{d}{dz}E_x(z) + k^2 E_x(z)\right] e^{i(kz-\omega t)}$$

using $k^2 = \omega^2\mu\epsilon$ we get

This is from wave equation solutions to the unperturbed material

$$\left[\frac{d^2}{dz^2}E_x(z) + 2ik\frac{d}{dz}E_x(z) + \omega^2\mu\,\Delta\epsilon\, E_x(z)\right]\hat{i}$$

$$+ \left[\frac{d^2}{dz^2}E_y(z) + 2ik\frac{d}{dz}E_y(z) + \omega^2\mu\,\Delta\epsilon\, E_y(z)\right]\hat{j} = 0$$

for slow variations $\frac{d^2}{dz^2} E << k \frac{d}{dz} E$ we can neglect $\frac{d^2}{dz^2}$ terms

Envelope varies slowly compared to optical frequency

$$\left[2ik \frac{d}{dz} E_x(z) + \omega^2 \mu \Delta\varepsilon\, E_x(z)\right] \hat{i}$$
$$+ \left[2ik \frac{d}{dz} E_y(z) + \omega^2 \mu \Delta\varepsilon\, E_y(z)\right] \hat{j} \approx 0$$

for $G_x = G_y = 0;\ G_z = G$ we have $\Delta\varepsilon = i\varepsilon_0 \begin{bmatrix} 0 & -G & 0 \\ G & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

giving

$$\left[2ik \frac{d}{dz} E_x(z) - \omega^2 \mu (i\varepsilon_0 G) E_y(z)\right] \hat{i}$$
$$+ \left[2ik \frac{d}{dz} E_y(z) + \omega^2 \mu (i\varepsilon_0 G) E_x(z)\right] \hat{j} = 0$$

These coupled differential equations reduce to coupled equations that can be solved using linear algebra when we use phasors to represent the fields

let $E_x(z) = \mathrm{Re}\left[\tilde{E}_x(z)\right]$ where $\tilde{E}_x(z) = \tilde{E}_x e^{i(\kappa z + \alpha_x)}$

$E_y(z) = \mathrm{Re}\left[\tilde{E}_y(z)\right]$ $\quad \tilde{E}_y(z) = \tilde{E}_y e^{i(\kappa z + \alpha_y)}$

then with $\frac{d}{dz} \tilde{E}(z) = i\kappa \tilde{E}(z)$ we have

$$-2k\kappa \tilde{E}_x - i\omega^2 \mu \varepsilon_0 G \tilde{E}_y = 0$$

and $-2k\kappa \tilde{E}_y + i\omega^2 \mu \varepsilon_0 G \tilde{E}_x = 0$

or with $k^2 = \omega^2 \mu \varepsilon$

$$\begin{bmatrix} -2k\kappa & -ik^2 G \\ ik^2 G & -2k\kappa \end{bmatrix} \begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \end{bmatrix} = 0$$

$$\begin{bmatrix} -2k\chi & -ik^2G \\ ik^2G & -2k\chi \end{bmatrix}\begin{bmatrix}\tilde{E}_x \\ \tilde{E}_y\end{bmatrix} = 0$$

requiring $\begin{vmatrix} -2k\chi & -ik^2G \\ ik^2G & -2k\chi \end{vmatrix} = 0$ for non trivial solutions to $\tilde{E}_x$ and $\tilde{E}_y$ to exist.

This gives $\chi = \pm \frac{1}{2} k G$

Assuming input field is polarized along x and has amplitude E_0

$$Re[\tilde{E}_x(0)] = E_0 \quad \text{and} \quad Re[\tilde{E}_y(0)] = 0$$

thus $\alpha_x = 0$

$$E_x(z) = E_0 \cos(\tfrac{1}{2} k G z)$$

Solving for $\tilde{E}_y$ gives $\tilde{E}_y(z) = i\frac{2\chi}{k}\frac{E_0}{G} e^{i\chi z}$

So $E_y(z) = -E_0 \sin(\tfrac{1}{2} k G z)$

Diagonalization

Consider the same problem where

$$\epsilon = \epsilon_0 \begin{bmatrix} n^2 & -iG & 0 \\ i\,G & n^2 & 0 \\ 0 & 0 & n^2 \end{bmatrix}$$

Find the normal modes of propagation by finding the eigenvectors of the dielectric tensor

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$$\epsilon = \begin{bmatrix} n^2 & -i\epsilon_0 G & 0 \\ i\epsilon_0 G & n^2 & 0 \\ 0 & 0 & n^2 \end{bmatrix}$$

Find the normal modes of propagation by finding the eigenvectors of the dielectric tensor

$\lambda = n^2,\ n^2 - \epsilon G, n^2 + \epsilon G$

$\vec{u} = [0,0,1],\ [i,1,0],\ [-i,1,0]$

Summary

- Material polarization is not necessarily parallel to electric field in anisotropic materials
  - permittivity of material (ϵ) is a tensor relating E to D
  - Direction of wave vector and energy flow can be different
- Normal shells and index ellipsoid are geometrical constructions to visualize the propagation parameters of a wave

References

- Yariv & Yeh “Optical Waves in Crystals” chapter 4
- Hecht “Optics” section 8.4
- Fowles “Modern Optics” section 6.7