Experiment 13

Numerical Analysis

Objective

The objective of this experiment is to examine the accuracy of numerical solutions to analytical expressions and to determine the effect of a velocity-dependent drag force on a moving object by using computer-assisted numerical methods to solve Newtons equation of motion.

Concepts

Projectile kinematics homework problems instruct you to neglect air resistance because the velocity-dependent drag force invalidates the simple constant-acceleration solution. Fortunately nature ignores this hint; otherwise raindrops would hit with lethal impact. This week we learn to compute numerical solutions to Newton's equation of motion that include the effect of air resistance. Many real-world engineering and science calculations are set up this way.

For a moving spherical object, such as a baseball, the forces acting on the object are gravity and a drag force due to air resistance. The drag force has two main characteristics; it opposes the direction of motion and it increases in magnitude with increasing velocity. We shall assume the drag force is proportional to the velocity squared.

Figure 13.1: Free body diagram for a falling baseball

Newtons 2nd law in the vertical direction is written as:

$$F_d - mg = ma$$

with $F_d = 0.22d^2v^2$. As the ball falls, the velocity increases until the drag force balances the gravitational force so that the net force is zero.

To predict the motion of the ball, we need to calculate the resultant force and velocity produced by the drag using simple step functions over a small time interval in which the acceleration is assumed constant. New values of position and velocity are calculated from previous values of velocity and acceleration. The repetitive arithmetic in these step equations is just what computers were developed for. These equations use the simplest estimate for the velocity and acceleration in each interval. The larger the interval the less accurate the estimate over this interval is. Our text book develops slightly more detailed expressions, but the core concept for this and all the more sophisticated treatments is the same: if we know the forces, Newtons second law lets us predict the motion.

Calculations

1. **Air Resistance on a Baseball**. A thrown ball will encounter an air resistance force proportional to the square of its diameter d and its speed v. This drag force is given by

$$F_d(v) = \left(0.22\frac{\text{kg}}{\text{m}^3}\right) d^2v^2$$

for air at STP. The constant (0.22) has units of kg/m^3 so that when MKS units are used for diameter and velocity the resulting force is in Newtons. When the sphere reaches its terminal velocity v_T, the net force on the sphere is zero, so from Newtons second law we have

$$\left(0.22\frac{\text{kg}}{\text{m}^3}\right) d^2v^2 - mg = 0.$$

If a baseball is at hand, measure the diameter and mass of the baseball and compute its terminal velocity. If a baseball is not available, assume its mass is 0.14 kg and its diameter is 0.072 m.

2. **Free Fall from Rest**. A baseball is dropped from rest. Use the spreadsheet program Excel to evaluate the motion over a large number of small time steps. Enter the n=0 row of initial values. Define the n=1 elements as equations using previous values and constants. Then use the fill down command (in the Edit menu) to repeat the calculations for as long and as far as you want the baseball to fall. Program each of the following equations into the following columns, using references to cells containing the value for Δt, v_T, g, and m. You can use a time step of $\Delta t = 0.1v_T/g$ which is one tenth of a characteristic time v_T/g describing how long before the effects of drag become significant. In step 3 we will investigate the significance of this value. The first couple of rows in your spreadsheet should look similar to this

		No Drag				Drag			
index	time	velocity	position	net force	acceleration	velocity	position	net force	acceleration
0	0	0	0	-1.372	-9.8	0	0	-1.372	-9.8

-mg

-g

Figure 13.2: Example of the first row of the spreadsheet

where the values mg and g reference the cells where these constants have been defined. The value of successive cells in the n^{th} row of data should be calculated from the following formulas:

(a) Index: n

(b) Time: $t_n = t_{n-1} + \Delta t$

(c) Exact solution to velocity without a drag force acting: $v_n = -gt_n$

(d) Exact solution to distance without a drag force acting: $y_n = -\frac{1}{2}gt_n^2$

(e) Net force for numerical solution without drag force acting: $F_n = -mg$

(f) Acceleration for numerical solution without drag force acting $a_n = F_n/m$

(g) Numerical solution to velocity with drag force acting: $v_n = v_{n-1} + a_{n-1}\Delta t$

(h) Numerical solution to distance with drag force acting: $y_n = y_{n-1} + v_{n-1}\Delta t$ (the $\frac{1}{2}a\Delta t^2$ term can be neglected for sufficiently small timesteps)

(i) Net force for numerical solution with drag force acting: $F_n = 0.22d^2v_n^2 - mg$

(j) Acceleration for numerical solution with drag force acting $a_n = F_n/m$

Plot velocity and position as functions of time for at least 32 points. Highlight the column of time values. Holding the ctrl key to skip columns, also highlight the two columns of velocity values. Then select Chart from the Insert menu and Excel will guide you through the plotting options. Show on this graph the terminal velocity you calculated. Also plot distance as a function of time, again comparing the drag-free case to the realistic solution.

3. **Effect of Timestep**. Set the diameter d to zero in your spreadsheet. This has the effect of eliminating air resistance form the numerical solutions (if this causes division by zero errors in your cells, don't worry, the issue will be resolved once you manually set the timestep $Deltat$). We will observe how changing the time step affects the accuracy of these solutions. Set the time step to 2 seconds. Record the exact position of the ball at 2 seconds and the numerical solution to the position at 2 seconds. Reduce your time step by a factor of 2 (i.e. set it to 1 second) and again record the exact and numerical solution to the position at 2 seconds (because the time step has changed, these values will be in a different row than before). Continue decreasing the time step by factors of 2 down to 1/16th of a second, recording the position of the ball at 2 seconds for the exact and numerical solutions for each time step. Determine the percent error in the numerical solution at each time step by comparing it to the exact value. Below what value for the time step does the error decrease to less than 10%? How does this compare to the value of Δt used in step 2? Explain why smaller time steps give less error and why there is diminishing value to having time steps less than $\delta t = 0.1v_T/g$

Data Table

Timestep	Exact value for y(2 s)	Numerical solution to y(2 s)	% difference
2 s			
1 s			
$\frac{1}{2}$ s			
$\frac{1}{4}$ s			
$\frac{1}{8}$ s			
$\frac{1}{16}$ s			

Report

In addition to the standard elements of a well written lab report described in the introduction to this manual, your report must include:

1. A combined graph including curve of exact velocity without drag vs. time, and the curve of numerical solution to the velocity with drag vs. time. Indicate on the graph what the terminal velocity is and what the mass and diameter of the ball are for the results appearing in the graphs.
2. A combined graph including the distance fallen as a function of time for the exact solution without drag and the numerical solution with drag.
3. Printout of the spreadsheet data table.
4. A conclusion discussing what you have learned regarding how the time scale used in numerical calculations affect the accuracy of the results.