Chapter 3: Matrix Determinants

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Outline

Sections 3.1-3.2 Matrix determinants

Sections 3.3 Applications

Introduction

Briefly speaking, determinant is a mathematical rule to evaluate a square matrix to a real number

$$\det : \mathbf{A} \in \mathbb{R}^{n\times n} \longrightarrow \det(\mathbf{A}) \in \mathbb{R}$$

in order to determine whether the matrix is invertible or not.

For example, we have seen the following formula for computing the inverse of a 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad (\text{assuming } ad - bc \neq 0)$$

The denominator $ad - bc$ is a number computed from the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, which can indicate whether the 2×2 matrix is invertible.

This would be called the determinant of the 2×2 matrix, and denoted as

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

For a 1×1 matrix $\mathbf{A} = [a]$, it is invertible if and only if a is nonzero, so we can just define the determinant of a 1×1 matrix as the number in it:

$$\det[a] = a$$

What about 3×3 or even larger matrices?

Matrix minor and cofactor

We will define the notion of determinants for general square matrices using a recursive approach, and for that goal, we first need to define the matrix minors and cofactors.

Def 0.1 (Matrix minor and cofactor). Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a square matrix. For any pair of indices $1 \le i, j \le n$, let $\mathbf{A}_{ij}$ denote the submatrix formed by deleting the ith row and jth column of $\mathbf{A}$. We define

- the (i, j) **minor** of $\mathbf{A}$ as $M_{ij} = \det(\mathbf{A}_{ij})$, and
- the (i, j) **cofactor** of $\mathbf{A}$ as $C_{ij} = (-1)^{i+j} \det(\mathbf{A}_{ij}) = (-1)^{i+j} M_{ij}$ (thus cofactor is just signed minor).

Example 0.1 ($n = 3$)**.** Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Then

- the $(1,1)$-**submatrix** is $\mathbf{A}_{11} = \begin{bmatrix} & & \\ & 5 & 6 \\ & 8 & 9 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$
- the $(1,1)$-**minor** is $M_{11} = \det(\mathbf{A}_{11}) = 5 \cdot 9 - 6 \cdot 8 = -3$, and
- the $(1,1)$-**co-factor** is $C_{11} = (-1)^{1+1} M_{11} = -3$

Similarly, the $(1, 2)$-

- Submatrix: $\mathbf{A}_{12} = \begin{bmatrix} & & \\ 4 & & 6 \\ 7 & & 9 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$
- Minor: $M_{12} = \det(\mathbf{A}_{12}) = 4 \cdot 9 - 6 \cdot 7 = -6$, and
- Co-factor: $C_{12} = (-1)^{1+2} M_{12} = 6$

What about the $(1, 3)$-submatrix, minor, and cofactor?

And how many minors and cofactors are there in total?

We can arrange all the minors and cofactors of a square matrix separately into two matrices of the same size:

$$\underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}}_{\text{cofactors matrix } \mathbf{C}} = \underbrace{\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}}_{\text{signs matrix } (-1)^{i+j}} \underbrace{\circ}_{\substack{\text{entrywise} \\ \text{multiplication}}} \underbrace{\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}}_{\text{minors matrix } \mathbf{M}}$$

In the previous example, we have obtained that

$$\begin{bmatrix} -3 & 6 & -3 \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \circ \begin{bmatrix} -3 & -6 & -3 \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

Determinants of square matrices

We define the determinants of all square $\mathbf{A} \in \mathbb{R}^{n\times n}$ recursively as follows:

- If $n = 1$: Define $\det(\mathbf{A}) = \det[a_{11}] = a_{11}$, which is the trivial case.
- For any larger n, define

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \underbrace{a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + \cdots + a_{1n} \cdot C_{1n}}_{\text{cofactor expansion along first row}}$$

where $C_{1j} = (-1)^{1+j} \det(\mathbf{A}_{1j})$ are the cofactors along the first row, which are signed determinants of $(n-1) \times (n-1)$ matrices.

The **cofactor expansion** formula, when applied recursively, reduces determinants calculation to smaller and smaller matrices toward the 1×1 case.

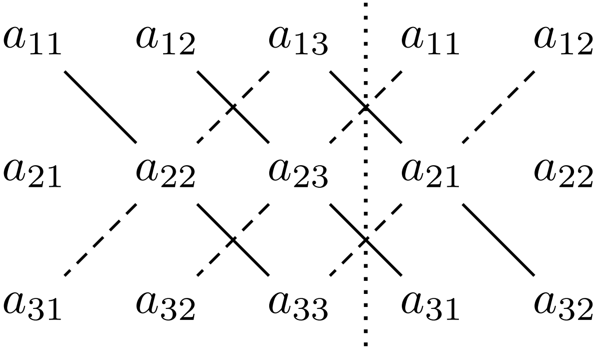
Let's apply the recursive formula to the cases of $n = 2, 3$:

- $n = 2$ (to verify the previous formula):

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot (-1)^{1+1} a_{22} + a_{12} \cdot (-1)^{1+2} a_{21} = a_{11}a_{22} - a_{12}a_{21}$$

- $n = 3$ (this leads to a new formula):

$$
\begin{aligned}
&\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\
&\qquad\qquad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}
\end{aligned}
$$


a_{11} a_{12} a_{13} a_{11} a_{12}
a_{21} a_{22} a_{23} a_{21} a_{22}
a_{31} a_{32} a_{33} a_{31} a_{32}

Example 0.2. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

From previous calculations, we have

$$C_{11} = -3, \quad C_{12} = 6, \quad C_{13} = -3$$

Thus, by the cofactor expansion formula,

$$\det(\mathbf{A}) = 1(-3) + 2(6) + 3(-3) = 0.$$

(We will see later that this implies that $\mathbf{A}$ is not invertible)

It turns out that the determinant of an $n \times n$ matrix $\mathbf{A}$ can be computed by a cofactor expansion along any row or column.

Theorem 0.1. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. Then for any $1 \le i \le n$,

$$\det(\mathbf{A}) = a_{i1} \cdot C_{i1} + a_{i2} \cdot C_{i2} + \cdots + a_{in} \cdot C_{in} \longleftarrow \text{expansion along row } i$$

and for any $1 \le j \le n$,

$$\det(\mathbf{A}) = a_{1j} \cdot C_{1j} + a_{2j} \cdot C_{2j} + \cdots + a_{nj} \cdot C_{nj} \longleftarrow \text{expansion along column } j$$

We omit the proof but verify this result using an example.

Example 0.3. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Compute the determinant of this matrix by using a cofactor expansion along

(a) the 2nd row, or

(b) the 3rd column.

Example 0.4. Compute the determinant of

$$\mathbf{A} = \begin{bmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{bmatrix}$$

Remark. We should perform the co-factor expansion along a row or column that has the most zeros (for fast computing).

The previous theorem also implies the following result.

Corollary 0.2. The determinant of a (square) diagonal, or lower/upper triangular matrix is equal to the product of its diagonal entries.

We use the following example to illustrate the corollary.

Example 0.5. Find the determinants of

$$\mathbf{A} = \begin{bmatrix} 5 & & & \\ & 2 & & \\ & & 1 & \\ & & & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}$$

In fact, for block diagonal and block lower/upper triangular matrices with square main blocks, similar results hold true:

$$\det \begin{bmatrix} \mathbf{A} & \\ & \mathbf{B} \end{bmatrix} = \det(\mathbf{A}) \det(\mathbf{B})$$

$$\det \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ & \mathbf{B} \end{bmatrix} = \det(\mathbf{A}) \det(\mathbf{B})$$

$$det \begin{bmatrix} \mathbf{A} & \\ \mathbf{C} & \mathbf{B} \end{bmatrix} = \det(\mathbf{A}) \det(\mathbf{B})$$

Example 0.6. Show that the determinants of the three kinds of elementary matrices are

$$\det(\mathbf{M}_i(r)) = r$$
$$\det(\mathbf{R}_{i\leftarrow j}(k)) = 1$$
$$\det(\mathbf{P}_{ij}) = -1.$$

Proof. We verify these three statements for the case of 3×3. □

Properties of matrix determinants

Theorem 0.3. Let $\mathbf{A}, \mathbf{B}$ be square matrices of the same size. Then

- $\det(\mathbf{A}^T) = \det(\mathbf{A})$
- $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$

Example 0.7. Verify the above results using the following matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$$

The second statement on the preceding slide implies the following result.

Corollary 0.4. Let $\mathbf{E}$ be an elementary matrix and $\mathbf{A}$ a square matrix of the same size. Then

$$\det(\mathbf{EA}) = \det(\mathbf{E})\det(\mathbf{A}) = \begin{cases} r \cdot \det(\mathbf{A}) & \text{if } \mathbf{E} = \mathbf{M}_i(r); \\ \det(\mathbf{A}) & \text{if } \mathbf{E} = \mathbf{R}_{i \leftarrow j}(k); \\ -\det(\mathbf{A}) & \text{if } \mathbf{E} = \mathbf{P}_{ij}; \end{cases}$$

Remark. This shows that row replacements do not change matrix determinants while interchanging two rows would flip the sign of the determinant (but the absolute value is still the same).

Example 0.8. Compute

$$\begin{vmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{vmatrix}$$

by performing only row replacement operations.

Remark. Because $\det(\mathbf{A}^T) = \det(\mathbf{A})$, performing elementary column operations on a square matrix have the same effects on the determinant as the elementary row operations.

Example 0.9. Compute

$$\begin{vmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{vmatrix}$$

by performing only column replacement operations.

Remark. It is certainly fine to use both kinds of replacement operations together to generate as many zeros in the matrix as possible.

Remark. Another consequence of the corollary is that a scalar multiple of a row (or column) corresponds to the same scalar multiple of the determinant.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ ra_{31} & ra_{32} & ra_{33} \end{vmatrix} = r \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & ra_{12} & a_{13} \\ a_{21} & ra_{22} & a_{23} \\ a_{31} & ra_{32} & a_{33} \end{vmatrix} = r \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Example 0.10. Compute $\begin{vmatrix} 0 & 2 & 6 \\ 1 & -3 & -3 \\ -1 & 0 & 9 \end{vmatrix}$

Remark. Be sure to distinguish from the case of scalar multiple of a matrix:

$$\begin{bmatrix} ra_{11} & ra_{12} & ra_{13} \\ ra_{21} & ra_{22} & ra_{23} \\ ra_{31} & ra_{32} & ra_{33} \end{bmatrix} = r \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Applying the formula on the previous slide three times (for a 3×3 matrix $\mathbf{A}$) yields that

$$\det(r\mathbf{A}) = r^3 \det(\mathbf{A}).$$

This could also be obtained as follows

$$\det(r\mathbf{A}) = \det(r\mathbf{I} \cdot \mathbf{A}) = \det(r\mathbf{I}) \det(\mathbf{A}) = r^3 \det(\mathbf{A}).$$

More generally, for $n \times n$ matrices $\mathbf{A}$,

$$\det(r\mathbf{A}) = r^n \det(\mathbf{A}).$$

Finally, we are ready to present the following important result.

Theorem 0.5. A square matrix $\mathbf{A}$ is invertible if and only if $\det(\mathbf{A}) \neq 0$.

Proof. For any square matrix $\mathbf{A}$, there exist a sequence of elementary matrices $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$ (which are either row permutation or row replacements) such that

$$\mathbf{E}_k \cdots \mathbf{E}_2 \cdot \mathbf{E}_1 \cdot \mathbf{A} = \underbrace{\mathbf{U}}_{\text{REF}} \longleftarrow \text{square matrix}$$

Taking determinants of both sides give that

$$(-1)^r \det(\mathbf{A}) = \det(\mathbf{U})$$

where r is the number of row permutations used. It follows that $\det(\mathbf{A}) \neq 0$ if and only if $\det(\mathbf{U}) \neq 0$, which is if and only if $\mathbf{U}$ contains n nonzero pivots, which is if and only if $\mathbf{A}$ is invertible. □

Corollary 0.6. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a square matrix. Then the following statements are all equivalent to "$\det(\mathbf{A}) \neq 0$".

1. $\mathbf{A}$ is invertible.
2. $\mathbf{A}$ has n pivot positions.
3. The equation $\mathbf{Ax} = \mathbf{0}$ only has the trivial solution.
4. The equation $\mathbf{Ax} = \mathbf{b}$ (for any $\mathbf{b}$) always has a unique solution.
5. The columns of $\mathbf{A}$ form a linearly independent set.
6. The columns of $\mathbf{A}$ span $\mathbb{R}^n$.
7. The linear transformation $f(\mathbf{x}) = \mathbf{Ax}$ is one-to-one.
8. The linear transformation $f(\mathbf{x}) = \mathbf{Ax}$ (from $\mathbb{R}^n$ to $\mathbb{R}^n$) is onto.

Example 0.11. Find the determinant of

$$\mathbf{A} = \begin{bmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ 2 & 7 & -2 \end{bmatrix}$$

and use it to determine

(a) if the columns of $\mathbf{A}$ are linearly independent;

(b) if the linear transformation $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ is one-to-one, or onto, or both.

Other applications of matrix determinants

- Solving systems of linear equations $\mathbf{A}\mathbf{x} = \mathbf{b}$
- Finding matrix inverse (if $\det(\mathbf{A}) \neq 0$)
- Computing the area/volume of parallelogram/parallelepiped

We will teach these based on three examples.

Theorem 0.7 (Cramer's Rule). Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be an invertible matrix. Then for any vector $\mathbf{b} \in \mathbb{R}^n$, the unique solution $\mathbf{x}$ of $\mathbf{A}\mathbf{x} = \mathbf{b}$ has the following entries:

$$x_i = \frac{\det[\mathbf{a}_1 \ldots \mathbf{b} \ldots \mathbf{a}_n]}{\det \underbrace{[\mathbf{a}_1 \ldots \mathbf{a}_i \ldots \mathbf{a}_n]}_{\mathbf{A}}}, \quad \text{for all } i = 1, \ldots, n$$

Remark. This formula is inefficient unless the matrix $\mathbf{A}$ is 2×2 or perhaps also 3×3.

Example 0.12. Use Cramer's rule to solve the following equation

$$5x_1 + 7x_2 = 3$$
$$2x_1 + 4x_2 = 1$$

Theorem 0.8 (Adjoint matrix). Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be an invertible matrix. Then

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \underbrace{\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}}_{\text{transposed cofactors matrix, called } \textbf{adjoint}} = \frac{1}{\det(\mathbf{A})} \mathrm{adj}\mathbf{A}.$$

Remark. This formula is not efficient either.

Example 0.13. Find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$$

Theorem 0.9. For n vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ in $\mathbb{R}^n$, the volume of the n-dimensional parallelpiped spanned by them is given by $\det[\mathbf{a}_1, \ldots, \mathbf{a}_n]$.

Example 0.14. Find

(a) the area of the parallelogram spanned by $\mathbf{a}_1 = (1, 2)^T, \mathbf{a}_2 = (3, 4)^T$

(b) the volume of the parallelepiped spanned by $\mathbf{a}_1 = (1, 0, 0)^T$, $\mathbf{a}_2 = (0, 1, 0)$, and $\mathbf{a}_3 = (1, 1, 1)$.