

# Transport Phenomena in Biomedical Engineering (196 C)

**DATES:** January 28 to May 20, 2008

**TIMES:** 6:00-8:45 PM

**ROOM:** 333

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# OUTLINE

- Solutions to Midterm
- Solutions to Assignment#3
- Review of Significant Figures
- Review of Extraction (Problem 2.22 )
- Blood Rheology Review (Problems 4.22,4.24 and 4.29)
- Introduction to Mass Transfer

# USEFUL REFERENCES

- *Basic Transport Phenomena in Biomedical Engineering* by R.L. Fournier
- *Transport Processes and Unit Operations, Second Edition* by C.J. Geankoplis
- *Transport Phenomena, Revised Second Edition* by Bird , Stewart and Lightfoot

<http://bcs.wiley.com/he-bcs/Books?action=index&bcsId=3406&itemId=0470115394>

- *Elementary Differential Equations and Boundary Value Problems* by Boyce and Diprima or any other Differential Equations Book.

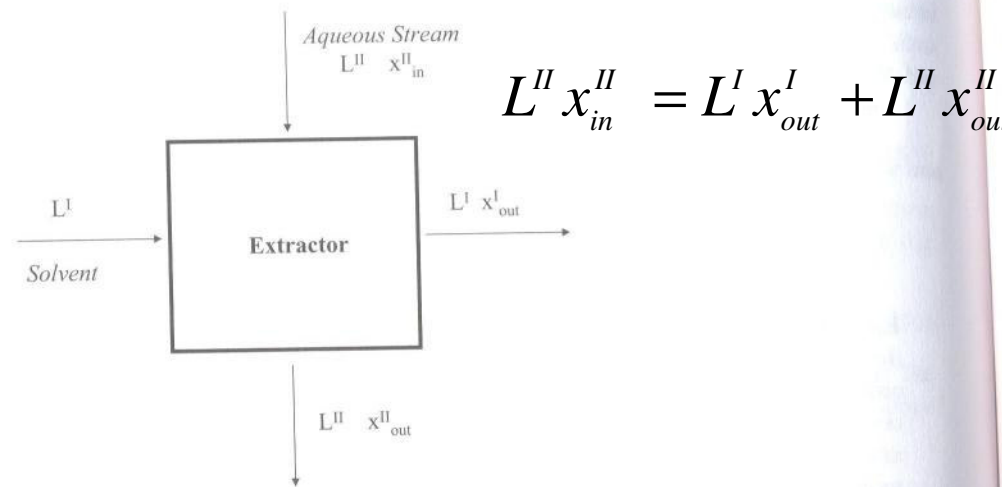
# REVIEW OF SIGNIFICANT DIGITS

- The number of significant digits carried varies from discipline to discipline. However, the rules remain the same.

- This is a helpful link that summarizes the rules:

<http://www.fordhamprep.com/gcurran/sho/sho/lessons/lesson23.htm>

# SINGLE STAGE EXTRACTION (REVIEW)



- The streams leaving the extractor are in equilibrium:

$$x_{out}^I = K x_{out}^{II}$$

- The extraction factor  $E$ , and the % extraction are not the same.

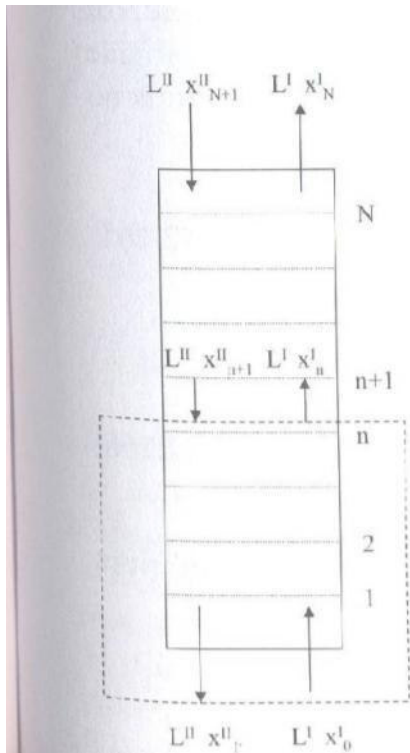
Figure 2.6 A single stage liquid-liquid extractor.

$$\frac{x_{out}^{II}}{x_{in}^{II}} = \frac{1}{1 + E} \quad (2.157)$$

where  $E$  is defined as the extraction factor,  $\frac{L^I K}{L^{II}}$ . The % extraction of the solute from phase II is given by the next equation:

$$\% \text{ extraction} = \left( 1 - \frac{x_{out}^{II}}{x_{in}^{II}} \right) \times 100 \quad (2.158)$$

# MULTI-STAGE EXTRACTION



$$L^{II} x_{N+1}^{II} + L^I x_0^I = L^{II} x_1^{II} + L^I x_N^I$$

$$\% \text{ extraction} = \left[ 1 - \frac{x_1^{II}}{x_{N+1}^{II}} \right] * 100$$

$$N = \frac{\ln \frac{x_{N+1}^{II} - x_e^{II}}{x_1^{II} - x_e^{II}}}{\ln E}$$

$$\eta = \frac{N}{N_{\text{actual}}}$$

By *Ana log y*

$$x_{N+1}^{II} \approx x_{in}^{II} \quad x_1^{II} \approx x_{out}^{II} \quad x_N^I \approx x_{out}^I$$

$$x_N^I = K x_N^{II}$$

$$x_0^I = K x_e^{II}$$

*SINGLE STAGE*

$$L^{II} x_{in}^{II} = L^I x_{out}^I + L^{II} x_{out}^{II}$$

# MULTI-STAGE EXTRACTION REFERENCE MATERIAL

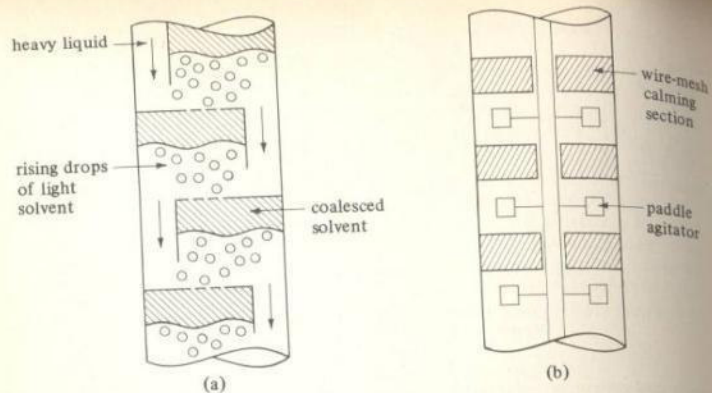


FIGURE 12.6-2. Extraction towers: (a) perforated-plate or sieve-tray tower, (b) agitated extraction tower.

be repeated by contacting the exit  $L_1$  stream with fresh solvent  $V_2$  in Fig. 12.5-6. In this way a greater percentage removal of the solute  $A$  is obtained. However, this is wasteful of the solvent stream and also gives a dilute product of  $A$  in the outlet solvent extract streams. In order to use less solvent and to obtain a more concentrated exit extract stream, countercurrent multistage contacting is often used.

Many of the fundamental equations of countercurrent gas absorption and of rectification are the same or similar to those used in countercurrent extraction. Because of the frequently high solubility of the two liquid phases in each other, the equilibrium relationships in extraction are more complicated than in absorption and distillation.

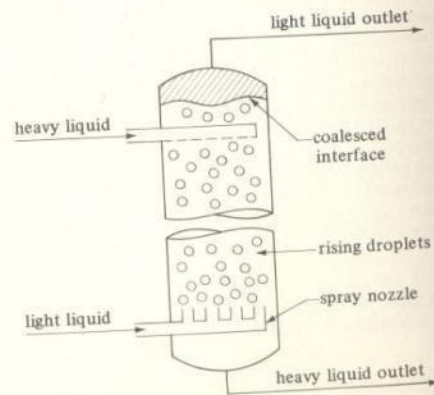


FIGURE 12.6-3. Spray-type extraction tower.

## 12.7B Continuous Multistage Countercurrent Extraction

1. *Countercurrent process and overall balance.* The process flow for this extraction process is the same as given previously in Fig. 10.3-2 and is shown in Fig. 12.7-1. The feed stream containing the solute  $A$  to be extracted enters at one end of the process and the solvent stream enters at the other end. The extract and raffinate streams flow countercurrently from stage to stage, and the final products are the extract stream  $V_1$  leaving stage 1 and the raffinate stream  $L_N$  leaving stage  $N$ .

Making an overall balance on all the  $N$  stages,

$$L_0 + V_{N+1} = L_N + V_1 = M \quad (12.7-1)$$

where  $M$  represents total kg/h (lb<sub>m</sub>/h) and is a constant,  $L_0$  the inlet feed flow rate in kg/h,  $V_{N+1}$  the inlet solvent flow rate in kg/h,  $V_1$  the exit extract stream, and  $L_N$  the exit raffinate stream. Making an overall component balance on component  $C$ ,

$$L_0 x_{C0} + V_{N+1} y_{CN+1} = L_N x_{CN} + V_1 y_{C1} = M x_{CM} \quad (12.7-2)$$

Combining Eqs. (12.7-1) and (12.7-2) and rearranging,

$$x_{CM} = \frac{L_0 x_{C0} + V_{N+1} y_{CN+1}}{L_0 + V_{N+1}} = \frac{L_N x_{CN} + V_1 y_{C1}}{L_N + V_1} \quad (12.7-3)$$

A similar balance on component  $A$  gives

$$x_{AM} = \frac{L_0 x_{A0} + V_{N+1} y_{AN+1}}{L_0 + V_{N+1}} = \frac{L_N x_{AN} + V_1 y_{A1}}{L_N + V_1} \quad (12.7-4)$$

Equations (12.7-3) and (12.7-4) can be used to calculate the coordinates of point  $M$  on the phase diagram that ties together the two entering streams  $L_0$  and  $V_{N+1}$  and the two exit streams  $V_1$  and  $L_N$ . Usually, the flows and compositions of  $L_0$  and  $V_{N+1}$  are known and the desired exit composition  $x_{AN}$  is set. If we plot points  $L_0$ ,  $V_{N+1}$ , and  $M$  as in Fig. 12.7-2, a straight line must connect these three points. Then  $L_N$ ,  $M$ , and  $V_1$  must lie on one line. Also,  $L_N$  and  $V_1$  must also lie on the phase envelope, as shown. These balances also hold for lb<sub>m</sub> and mass fraction, kg mol and mol fractions, and so on.

### EXAMPLE 12.7-1. Material Balance for Countercurrent Stage Process

Pure solvent isopropyl ether at the rate of  $V_{N+1} = 600$  kg/h is being used to extract an aqueous solution of  $L_0 = 200$  kg/h containing 30 wt % acetic acid ( $A$ ) by countercurrent multistage extraction. The desired exit acetic acid concentration in the aqueous phase is 4%. Calculate the compositions and amounts of the ether extract  $V_1$  and the aqueous raffinate  $L_N$ . Use equilibrium data from Appendix A.3.

**Solution:** The given values are  $V_{N+1} = 600$ ,  $y_{AN+1} = 0$ ,  $y_{CN+1} = 1.0$ ,  $L_0 = 200$ ,  $x_{A0} = 0.30$ ,  $x_{B0} = 0.70$ ,  $x_{C0} = 0$ , and  $x_{AN} = 0.04$ . In Fig. 10.3-3,  $V_{N+1}$

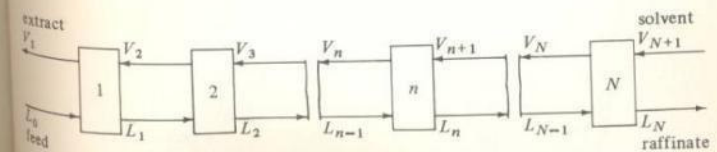
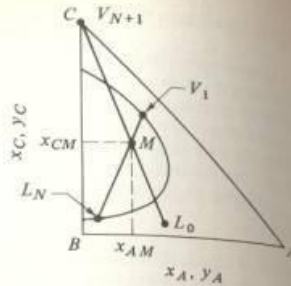


FIGURE 12.7-1. Countercurrent-multistage-extraction-process flow diagram.

FIGURE 12.7-2. Use of the mixture point  $M$  for overall material balance in countercurrent solvent extraction.



and  $L_0$  are plotted. Also, since  $L_N$  is on the phase boundary, it can be plotted at  $x_{AN} = 0.04$ . For the mixture point  $M$ , substituting into Eqs. (12.7-3) and (12.7-4),

$$x_{CM} = \frac{L_0 x_{C0} + V_{N+1} y_{CN+1}}{L_0 + V_{N+1}} = \frac{200(0) + 600(1.0)}{200 + 600} = 0.75 \quad (12.7-3)$$

$$x_{AM} = \frac{L_0 x_{A0} + V_{N+1} y_{AN+1}}{L_0 + V_{N+1}} = \frac{200(0.30) + 600(0)}{200 + 600} = 0.075 \quad (12.7-4)$$

Using these coordinates, the point  $M$  is plotted in Fig. 12.7-3. We locate  $V_1$  by drawing a line from  $L_N$  through  $M$  and extending it until it intersects the phase boundary. This gives  $y_{A1} = 0.08$  and  $y_{C1} = 0.90$ . For  $L_N$  a value of  $x_{CN} = 0.017$  is obtained. By substituting into Eqs. (12.7-1) and (12.7-2) and solving,  $L_N = 136$  kg/h and  $V_1 = 664$  kg/h.

2. Stage-to-stage calculations for countercurrent extraction. The next step after an overall balance has been made is to go stage by stage to determine the concentrations at each stage and the total number of stages  $N$  needed to reach  $L_N$  in Fig. 12.7-1.

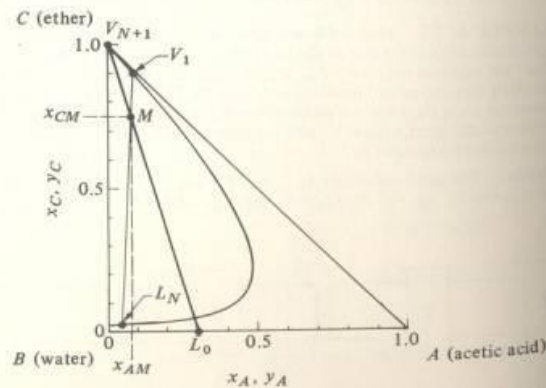


FIGURE 12.7-3. Method to perform overall material balance for Example 12.7-1.

Making a total balance on stage 1,

$$L_0 + V_2 = L_1 + V_1 \quad (12.7-5)$$

Making a similar balance on stage  $n$ ,

$$L_{n-1} + V_{n+1} = L_n + V_n \quad (12.7-6)$$

Rearranging Eq. (12.7-5) to obtain the difference  $\Delta$  in flows,

$$L_0 - V_1 = L_1 - V_2 = \Delta \quad (12.7-7)$$

This value of  $\Delta$  in kg/h is constant and for all stages,

$$\Delta = L_0 - V_1 = L_n - V_{n+1} = L_N - V_{N+1} = \dots \quad (12.7-8)$$

This also holds for a balance on component A, B, or C.

$$\Delta x_A = L_0 x_0 - V_1 y_1 = L_n x_n - V_{n+1} y_{n+1} = L_N x_N - V_{N+1} y_{N+1} = \dots \quad (12.7-9)$$

Combining Eqs. (12.7-8) and (12.7-9) and solving for  $x_\Delta$ ,

$$x_\Delta = \frac{L_0 x_0 - V_1 y_1}{L_0 - V_1} = \frac{L_n x_n - V_{n+1} y_{n+1}}{L_n - V_{n+1}} = \frac{L_N x_N - V_{N+1} y_{N+1}}{L_N - V_{N+1}} \quad (12.7-10)$$

where  $x_\Delta$  is the x coordinate of point  $\Delta$ .

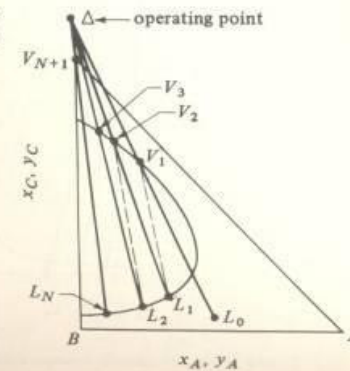
Equations (12.7-7) and (12.7-8) can be written as

$$L_0 = \Delta + V_1 \quad L_n = \Delta + V_{n+1} \quad L_N = \Delta + V_{N+1} \quad (12.7-11)$$

From Eq. (12.7-11), we see that  $L_0$  is on a line through  $\Delta$  and  $V_1$ ,  $L_n$  is on a line through  $\Delta$  and  $V_{n+1}$ , and so on. This means  $\Delta$  is a point common to all streams passing each other, such as  $L_0$  and  $V_1$ ,  $L_n$  and  $V_{n+1}$ ,  $L_N$  and  $V_{N+1}$ , and so on. The coordinates to locate this  $\Delta$  operating point are given for  $x_{CA}$  and  $x_{AA}$  in Eq. (12.7-10). Since the end points  $V_{N+1}$ ,  $L_N$  or  $V_1$ , and  $L_0$  are known,  $x_\Delta$  can be calculated and point  $\Delta$  located. Alternatively, the  $\Delta$  point is located graphically in Fig. 12.7-4 as the intersection of lines  $L_0 V_1$  and  $L_N V_{N+1}$ .

In order to step off the number of stages using Eq. (12.7-11) we start at  $L_0$  and draw the line  $L_0 \Delta$ , which locates  $V_1$  on the phase boundary. Next a tie line through  $V_1$  locates

FIGURE 12.7-4. Operating point  $\Delta$  and number of theoretical stages needed for countercurrent extraction.



$L_1$ , which is in equilibrium with  $V_1$ . Then line  $L_1\Delta$  is drawn giving  $V_2$ . The tie line  $V_2L_2$  is drawn. This stepwise procedure is repeated until the desired raffinate composition  $L_N$  is reached. The number of stages  $N$  is obtained to perform the extraction.

**EXAMPLE 12.7-2. Number of Stages in Countercurrent Extraction**

Pure isopropyl ether of 450 kg/h is being used to extract an aqueous solution of 150 kg/h with 30 wt % acetic acid ( $A$ ) by countercurrent multistage extraction. The exit acid concentration in the aqueous phase is 10 wt %. Calculate the number of stages required.

**Solution:** The known values are  $V_{N+1} = 450$ ,  $y_{AN+1} = 0$ ,  $y_{CN+1} = 1.0$ ,  $L_0 = 150$ ,  $x_{A0} = 0.30$ ,  $x_{B0} = 0.70$ ,  $x_{C0} = 0$ , and  $x_{AN} = 0.10$ . The points  $V_{N+1}$ ,  $L_0$ , and  $L_N$  are plotted in Fig. 12.7-5. For the mixture point  $M$ , substituting into Eqs. (12.7-3) and (12.7-4),  $x_{CM} = 0.75$  and  $x_{AM} = 0.075$ . The point  $M$  is plotted and  $V_1$  is located at the intersection of line  $L_N M$  with the phase boundary to give  $y_{A1} = 0.072$  and  $y_{C1} = 0.895$ . This construction is not shown. (See Example 12.7-1 for construction of lines.)

The lines  $L_0 V_1$  and  $L_N V_{N+1}$  are drawn and the intersection is the operating point  $\Delta$  as shown. Alternatively, the coordinates of  $\Delta$  can be calculated from Eq. (12.7-10) to locate point  $\Delta$ . Starting at  $L_0$  we draw line  $L_0 \Delta$ , which locates  $V_1$ . Then a tie line through  $V_1$  locates  $L_1$  in equilibrium with  $V_1$ . (The tie-line data are obtained from an enlarged plot such as the bottom of Fig. 12.5-3.) Line  $L_1 \Delta$  is next drawn locating  $V_2$ . A tie line through  $V_2$  gives  $L_2$ . A line  $L_2 \Delta$  gives  $V_3$ . A final tie line gives  $L_3$ , which has gone beyond the desired  $L_N$ . Hence, about 2.5 theoretical stages are needed.

3. **Minimum solvent rate.** If a solvent rate  $V_{N+1}$  is selected at too low a value a limiting

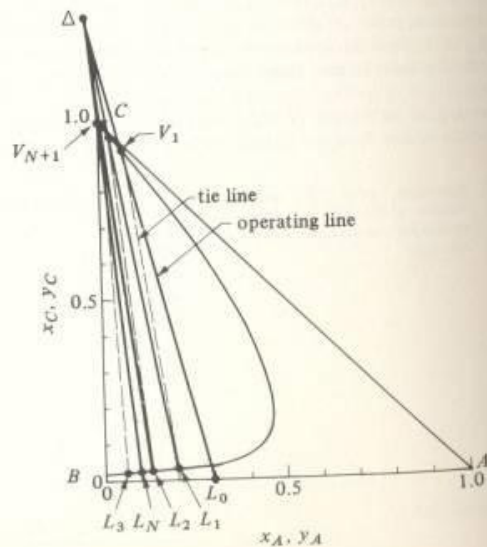


FIGURE 12.7-5. Graphical solution for countercurrent extraction in Example 12.7-2.

case will be reached with a line through  $\Delta$  and a tie line being the same. Then an infinite number of stages will be needed to reach the desired separation. The minimum amount of solvent is reached. For actual operation a greater amount of solvent must be used.

The procedure to obtain this minimum is as follows. A tie line is drawn through point  $L_0$  (Fig. 12.7-4) to intersect the extension of line  $L_N V_{N+1}$ . Other tie lines to the left of this tie line are drawn including one through  $L_N$  to intersect the line  $L_N V_{N+1}$ . The intersection of a tie line on line  $L_N V_{N+1}$  which is nearest to  $V_{N+1}$  represents the  $\Delta_{min}$  point for minimum solvent. The actual position of  $\Delta$  used must be closer to  $V_{N+1}$  than  $\Delta_{min}$  for a finite number of stages. This means that more solvent must be used. Usually, the tie line through  $L_0$  represents the  $\Delta_{min}$ .

**12.7C Countercurrent-Stage Extraction with Immiscible Liquids**

If the solvent stream  $V_{N+1}$  contains components  $A$  and  $C$  and the feed stream  $L_0$  contains  $A$  and  $B$  and components  $B$  and  $C$  are relatively immiscible in each other, the stage calculations are made more easily. The solute  $A$  is relatively dilute and is being transferred from  $L_0$  to  $V_{N+1}$ .

Referring to Fig. 12.7-1 and making an overall balance for  $A$  over the whole system and then over the first  $n$  stages,

$$L \left( \frac{x_0}{1-x_0} \right) + V' \left( \frac{y_{N+1}}{1-y_{N+1}} \right) = L \left( \frac{x_n}{1-x_n} \right) + V' \left( \frac{y_1}{1-y_1} \right) \quad (12.7-12)$$

$$L \left( \frac{x_0}{1-x_0} \right) + V' \left( \frac{y_{n+1}}{1-y_{n+1}} \right) = L \left( \frac{x_n}{1-x_n} \right) + V' \left( \frac{y_1}{1-y_1} \right) \quad (12.7-13)$$

where  $L = \text{kg inert } B/h$ ,  $V' = \text{kg inert } C/h$ ,  $y = \text{mass fraction } A \text{ in } V \text{ stream}$ , and  $x = \text{mass fraction } A \text{ in } L \text{ stream}$ . This Eq. (12.7-13) is an operating-line equation whose slope  $\cong L/V'$ . If  $y$  and  $x$  are quite dilute, the line will be straight when plotted on an  $xy$  diagram.

The number of stages are stepped off as shown previously in cases in distillation and absorption.

If the equilibrium line is relatively dilute, then since the operating line is essentially straight, the analytical Eqs. (10.3-21)-(10.3-26) given in Section 10.3D can be used to calculate the number of stages.

**EXAMPLE 12.7-3. Extraction of Nicotine with Immiscible Liquids.**

An inlet water solution of 100 kg/h containing 0.010 wt fraction nicotine ( $A$ ) in water is stripped with a kerosene stream of 200 kg/h containing 0.0005 wt fraction nicotine in a countercurrent stage tower. The water and kerosene are essentially immiscible in each other. It is desired to reduce the concentration of the exit water to 0.0010 wt fraction nicotine. Determine the theoretical number of stages needed. The equilibrium data are as follows (C5), with  $x$  the weight fraction of nicotine in the water solution and  $y$  in the kerosene.

$x$	$y$	$x$	$y$
0.001010	0.000806	0.00746	0.00682
0.00246	0.001959	0.00988	0.00904
0.00500	0.00454	0.0202	0.0185

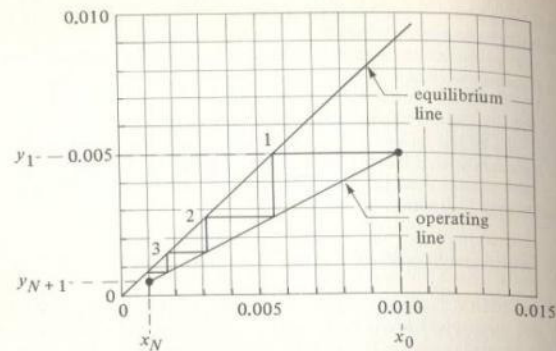


FIGURE 12.7-6. Solution for extraction with immiscible liquids in Example 12.7-3.

**Solution:** The given values are  $L_0 = 100$  kg/h,  $x_0 = 0.010$ ,  $V_{N+1} = 200$  kg/h,  $y_{N+1} = 0.0005$ ,  $x_N = 0.0010$ . The inert streams are

$$L = L(1 - x) = L_0(1 - x_0) = 100(1 - 0.010) = 99.0 \text{ kg water/hr}$$

$$V' = V(1 - y) = V_{N+1}(1 - y_{N+1}) = 200(1 - 0.0005) = 199.9 \text{ kg kerosene/hr}$$

Making an overall balance on  $A$  using Eq. (12.7-12) and solving,  $y_1 = 0.00497$ . These end points on the operating line are plotted in Fig. 12.7-6. Since the solutions are quite dilute, the line is straight. The equilibrium line is also shown. The number of stages are stepped off, giving  $N = 3.8$  theoretical stages.

## 12.8 INTRODUCTION AND EQUIPMENT FOR LIQUID-SOLID LEACHING

### 12.8A Leaching Processes

**1. Introduction.** Many biological and inorganic and organic substances occur in a mixture of different components in a solid. In order to separate the desired solute constituent or remove an undesirable solute component from the solid phase, the solid is contacted with a liquid phase. The two phases are in intimate contact and the solute or solutes can diffuse from the solid to the liquid phase, which causes a separation of the components originally in the solid. This process is called *liquid-solid leaching* or simply *leaching*. The term *extraction* is also used to describe this unit operation, although it also refers to liquid-liquid extraction. In leaching when an undesirable component is removed from a solid with water, the process is called *washing*.

**2. Leaching processes for biological substances.** In the biological and food processing industries, many products are separated from their original natural structure by liquid-solid leaching. An important process is the leaching of sugar from sugar beets with hot water. In the production of vegetable oils, organic solvents such as hexane, acetone, and

other are used to extract the oil from peanuts, soybeans, flax seeds, castor beans, sunflower seeds, cotton seeds, tung meal, and halibut livers. In the pharmaceutical industry, many different pharmaceutical products are obtained by leaching plant roots, leaves, and stems. For the production of soluble "instant" coffee, ground roasted coffee is leached with fresh water. Soluble tea is produced by water leaching of tea leaves. Tannin is removed from tree barks by leaching with water.

**3. Leaching processes for inorganic and organic materials.** Large uses of leaching processes occur in the metals processing industries. The useful metals usually occur in mixtures with very large amounts of undesirable constituents, and leaching is used to remove the metals as soluble salts. Copper salts are dissolved or leached from ground ores containing other minerals by sulfuric acid or ammoniacal solutions. Cobalt and nickel salts are leached from their ores by sulfuric acid-ammonia-oxygen mixtures. Gold is leached from its ore using an aqueous sodium cyanide solution. Sodium hydroxide is leached from a slurry of calcium carbonate and sodium hydroxide prepared by reacting  $\text{Na}_2\text{CO}_3$  with  $\text{Ca(OH)}_2$ .

### 12.8B Preparation of Solids for Leaching

**1. Inorganic and organic materials.** The method of preparation of the solid depends to a large extent upon the proportion of the soluble constituent present, its distribution throughout the original solid, the nature of the solid—i.e., whether it is composed of plant cells or whether the soluble material is completely surrounded by a matrix of insoluble matter—and the original particle size.

If the soluble material is surrounded by a matrix of insoluble matter, the solvent must diffuse inside to contact and dissolve the soluble material and then diffuse out. This occurs in many hydrometallurgical processes where metal salts are leached from mineral ores. In these cases crushing and grinding of the ores is used to increase the rate of leaching since the soluble portions are made more accessible to the solvent. If the soluble substance is in solid solution in the solid or is widely distributed throughout the whole solid, the solvent leaching action could form small channels. The passage of additional solvent is then made easier, and grinding to very small sizes may not be needed. Grinding of the particles is not necessary if the soluble material is dissolved in solution adhering to the solid. Then simple washing can be used as in washing of chemical precipitates.

**2. Animal and vegetable materials.** Biological materials are cellular in structure and the soluble constituents are generally found inside the cells. The rate of leaching may be comparatively slow because the cell walls provide another resistance to diffusion. However, to grind the biological material sufficiently small to expose the contents of individual cells is impractical. Sugar beets are cut into thin wedge-shaped slices for leaching so that the distance required for the water solvent to diffuse to reach individual cells is reduced. The cells of the sugar beet are kept essentially intact so that sugar will diffuse through the semipermeable cell walls, while the undesirable albuminous and colloidal components cannot pass through the walls.

For the leaching of pharmaceutical products from leaves, stems, and roots, drying of the material before extraction helps rupture the cell walls. Thus, the solvent can directly dissolve the solute. The cell walls of soybeans and many vegetable seeds are largely ruptured when the original materials are reduced in size to about 0.1 mm to 0.5 mm by rolling or flaking. Cells are smaller in size, but the walls are ruptured and the vegetable oil is easily accessible to the solvent.

# RHEOLOGY (Review)

*Rheology* is the study of the deformation and flow of matter under the influence of an applied stress ( $\tau$ ) at a certain shear rate ( $\dot{\gamma}$ ).

rate of transfer process = driving force ( $\Delta P$ )/resistance (Viscosity)

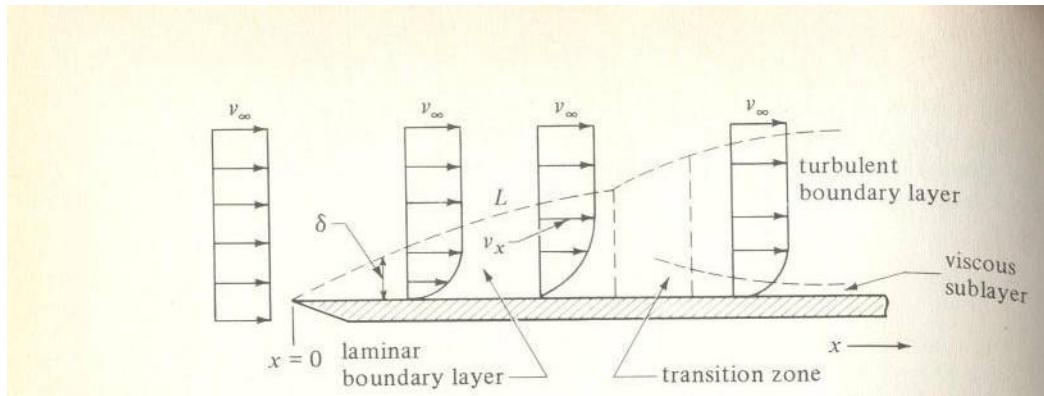
*The apparent of dynamic viscosity* ( $\mu$ ) is a measure of the resistance of a fluid to being deformed by either shear stress or extensional stress in units of Centipoise (cP). The kinematic viscosity ( $\nu$ ) measured in units of Stokes, is defined as the ratio of the apparent viscosity ( $\mu$ ) to the fluid density ( $\rho$ ) at a given temperature. Viscosity is also defined as the slope of the shear stress ( $\tau$ ) vs. shear rate ( $\dot{\gamma}$ ) curve. For a *Newtonian fluid* such as water, this slope is constant.

# BOUNDARY LAYER FLOW (Close to a Surface)

(Review)

(137-146, Examples 4.4 and 4.5)

- In fluid the boundary layer region near the solid, the fluid motion is greatly affected by the solid surface. In the thin boundary layer, viscosity is important. Since the region is thin, simplified solutions can be obtained for the boundary layer region.



$$N_{Re,x} = \frac{xv_{\infty}\rho}{\mu}$$

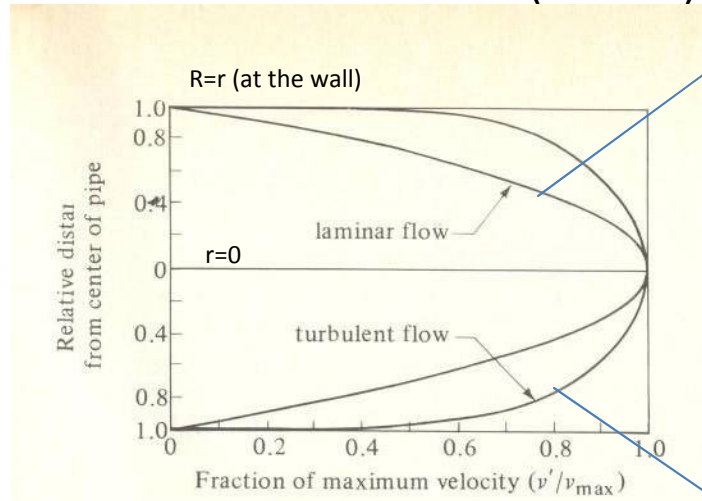
Laminar Flow :  $N_{Re,x} < 20000$

Transition Flow :  $20000 < N_{Re,x} < 3000000$

- The velocity at  $x=0$  is uniform across the fluid stream and has the value  $v$ . The velocity of the fluid at the interface is 0 and the velocity  $v_x$  increases in the  $x$  direction as one goes farther from the plate. When the flow is laminar, the thickness  $\delta$  of the boundary layer increases with  $(x)^{0.5}$  as we move in the  $x$  direction.  $L$  is the line when the velocity has reached 99% the bulk velocity  $v_{\infty}$ . The Reynolds number is defined as

# FLOW REGIMES IN BLOOD VESSELS

(Review) (Simplified)



Low velocity flow in veins

$$v = v_{max} \left( 1 - \left( \frac{r}{R} \right)^2 \right)$$

$$f = \frac{16}{N_{Re}} = \frac{16}{Dv\rho/\mu}$$

At  $r=0$ ,  $v=v_{max}$ ,  $\tau=dv/dr=0$   
 At  $r=R$ ,  $v=0$

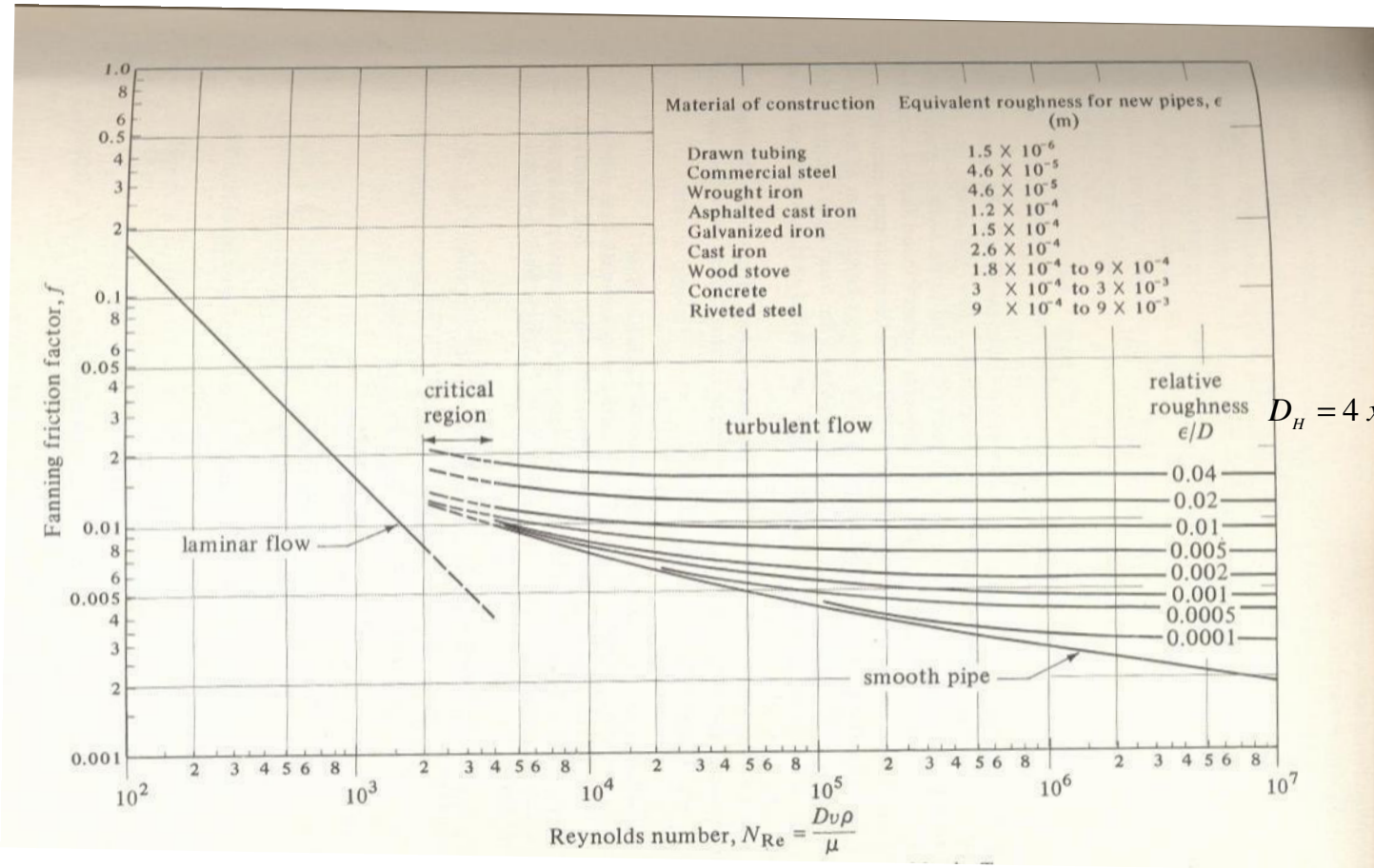
High velocity flow in arteries

$$v = v_{max} \left( \frac{R-r}{R} \right)^{1/7}$$

$$F_f = 4f \frac{\Delta L}{D} \frac{v^2}{2}$$

The Fanning friction factor ( $f$ ) is a ratio defined as the shear stress at the surface ( $\tau_s$ ) divided by the velocity head ( $\rho v^2/2$ ).

The combination of Newton's viscosity law, the Momentum Balance, Mechanical Energy, Bernoulli and Continuity equations and dimensionless analysis, allow Biomedical engineers to solve for  $(\Delta P)$ , Friction losses  $(F_f)$  and Pipe diameter  $(d)$ , Pipe length  $(L)$ .



$$N_{Re} = Lv\rho / \mu$$

$$N_{Eu} = \frac{P}{\rho v^2}$$

$$N_{Fr} = \frac{v^2}{gL}$$

$$D_H = 4 \times r_H = 4 \times \frac{(\text{cross sectional area})}{(\text{wetted perimeter})}$$

# BOUNDARY CONDITIONS AND MATHEMATICAL METHODS (REVIEW)

- Momentum is a function of position  $(x,y,z)$  and time  $(t)$ . Regular Integration  $[f(x)]$  involves a single lower and upper limit . For a multivariate function  $[f(x,t)]$  an initial condition and Boundary Conditions are required to “frame” the simulation in time an space. This is an example:  
IC:  $t=0, v(y)=0$  for all  $y$   
BC1:  $y=0, v=V$  for  $t>0$   
BC2:  $y=\infty, v=0$  for  $t>0$
- Depending on the complexity of the Momentum Balance equation(s) several methods are used alone or in conjunction with each other to model the flow by using Partial Differential Equation(s):
  - a) Laplace Transforms (i.e. Piece-wise continuous functions)
  - b) Matrix Solutions (i.e. Eigenvectors)
  - c) Numerical Integration (i.e Newton Raphson Method, Euler Method)
  - d) Boundary Value Problems (i.e., Fourier Series)
- **It is more important to understand the physical phenomenon in order to set up the equations correctly.** Many of these equations are solved by a software tool (Mathematica, Matlab, Maple, etc...)

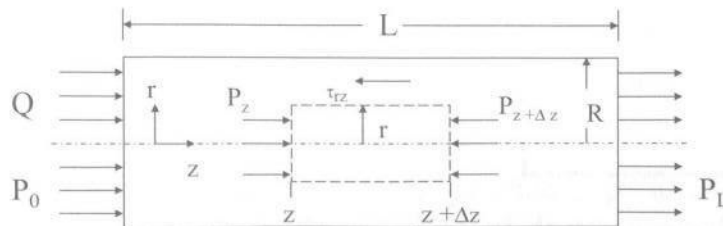
# KEY FLOW EQUATIONS (REVIEW)

**Table 4.3 Summary of key flow equations**

Relationship	Non-Newtonian	Newtonian
Shear Stress $\tau_{rz} =$	$\frac{(P_0 - P_L)r}{2L} = \tau_w \frac{r}{R}$	$\frac{(P_0 - P_L)r}{2L} = \tau_w \frac{r}{R}$
Wall Shear Stress $\tau_w =$	$\frac{(P_0 - P_L)R}{2L}$	$\frac{(P_0 - P_L)R}{2L}$
Shear Rate $\dot{\gamma} =$	$-\frac{dv_z}{dr} = \dot{\gamma}(\tau_{rz})$	$-\frac{dv_z}{dr} = \frac{\tau_{rz}}{\mu}$
Volumetric Flowrate $Q =$	$\pi \frac{R^3}{\tau_w^3} \int_0^{\tau_w} \dot{\gamma}(\tau_{rz}) \tau_{rz}^2 d\tau_{rz}$	$\frac{\pi R^4 \Delta P}{8\mu L}$
Wall Shear Stress $\tau_w =$	—	$\frac{4\mu Q}{\pi R^3}$
Wall Shear Rate $ \dot{\gamma}_w  =$	—	$\frac{4Q}{\pi R^3}$
Reduced Average Velocity $\bar{U} =$	$\frac{4Q}{\pi D^3}$	$\frac{4Q}{\pi D^3}$

Rabinovitch ←

→ Poiseuille-Hagen



**Figure 4.5** Forces acting on a cylindrical fluid element within a capillary viscometer.

# CASSON EQUATION (REVIEW)

THE PHYSICAL AND FLOW PROPERTIES OF BLOOD AND OTHER FLUIDS 125

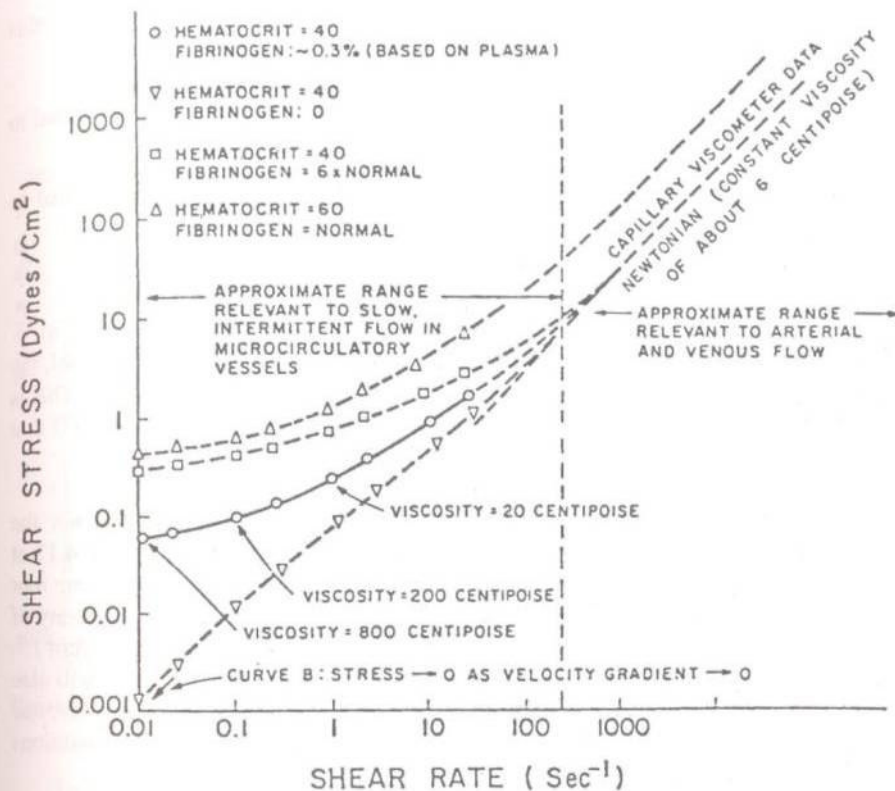


Figure 4.6 Shear stress versus shear rate for blood (from Replogle *et al.* 1967, with permission).

$$\tau^{1/2} = \tau_y^{1/2} + s\dot{\gamma}^{1/2}$$

$\tau_y$  is the yield stress

Substituting for  $\dot{\gamma}$  in the Rabinovitch equation and solving for the reduced average velocity

$$\bar{U} = \frac{1}{2s^2} \left[ \frac{\tau_w}{4} - \frac{4}{7} \sqrt{\tau_y} \sqrt{\tau_w} - \frac{1}{84} \frac{\tau_y^4}{\tau_w^3} + \frac{\tau_y}{3} \right]$$

$$\tau_y = 0.0289 \text{ dynes/cm}^2$$

$$s = 0.229 \text{ dynes.s/cm}^2$$

Equation 4.23 page 129 for the velocity profile applies for

$$r_{critical} \leq r \leq R \text{ where } r_{critical} \tau_w = R \tau_y$$

At lower Shear Rates, RBC aggregate and the assumption of blood being a homogeneous fluid no longer applies.

# FAHRAEUS AND THE FAHRAEUS-LINDQUIST EFFECTS AT HIGH SHEAR RATES

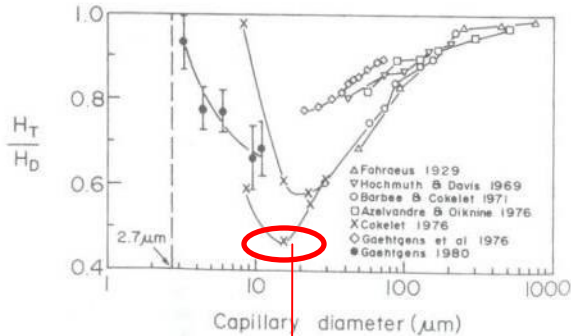
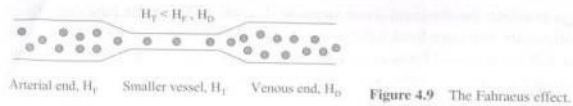


Figure 4.10 The Fahraeus effect (from Goehrigs 1980, with permission, data source identity may be found in this reference).

Plasma Skimming , @HF=HD, Reduced Hematocrit

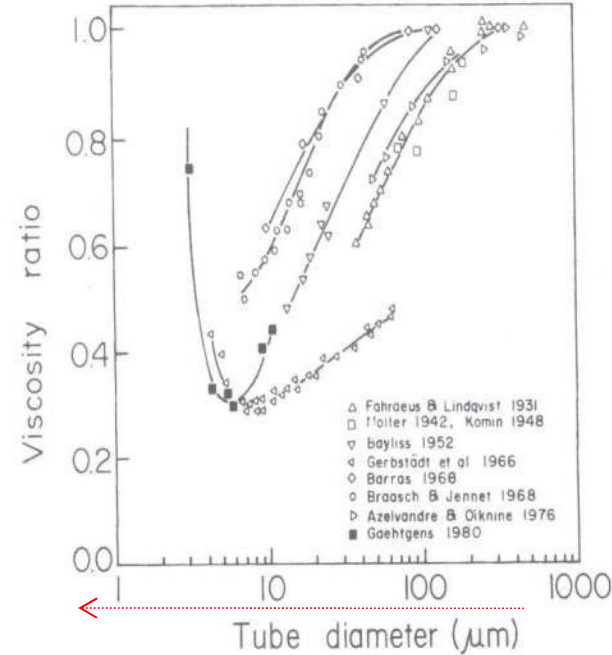
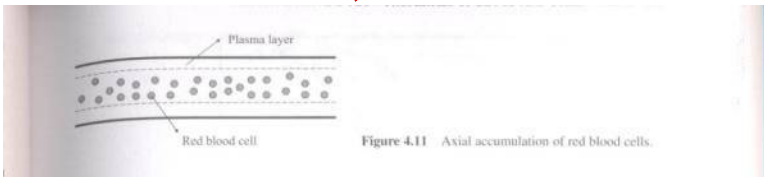


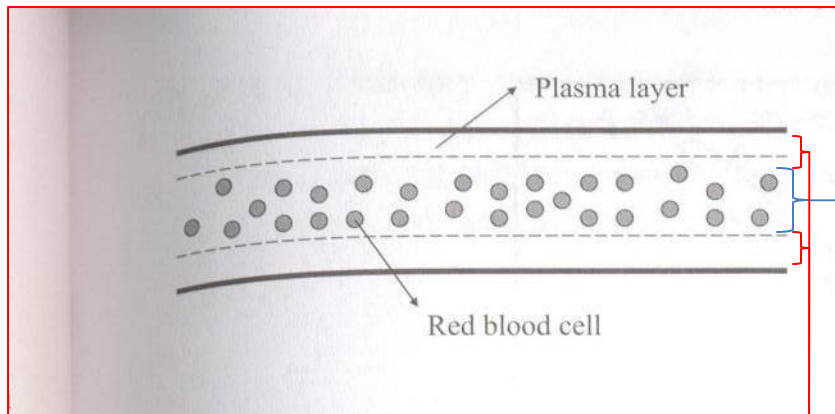
Figure 4.12 The Fahraeus-Lindquist effect (from Goehrigs 1980, with permission, data source identity may be found in this reference).

Because of the Hematocrit reduction,  
The Poiseuille –Hagen Flow can be applied to define  
the apparent viscosity for tube diameters < 500 microns

$$\mu_{\text{apparent}} = \frac{\pi R^4 \Delta P}{8LQ}$$

# APPLICATION OF THE MARGINAL ZONE THEORY

The marginal zone theory may be used to characterize the Fahraeus-Lindquist effect in the diameter range of 4-6 microns to 500 microns to obtain an expression for the apparent viscosity in terms of plasma layer thickness ( $\delta$ ), tube diameter, and the hematocrit, using the equations for Newtonian flow.



$$\tau_{rz} = -\mu_c \frac{dv_z^c}{dr}$$

$$BC1: r=0, \frac{dv_z^c}{dr} = 0$$

$$BC2: r=R-\delta, \tau_{rz}|_c = \tau_{rz}|_p$$

For Hematocrit  $\leq 0.6$  in order to find  $H_c$  and  $\delta$

$$\frac{\mu_{\text{apparent}}}{\mu_F} = \frac{1 - \alpha_F H_F}{1 - \sigma \alpha_c H_c} \text{ where } \sigma = 1 - \frac{\delta}{R}$$

$\alpha(H, T)$  is a function

used for  $H \leq 0.6$

$$\tau_{rz} = -\mu_p \frac{dv_z^p}{dr}$$

$$BC1: r=0, v_z^p = v_z^c$$

$$BC2: r=R-\delta, v_z^p = 0$$

$$\mu_{\text{apparent}} = \frac{\mu_p}{1 - \left(1 - \frac{\delta}{R}\right)^4 \left(1 - \frac{\mu_p}{\mu_c}\right)}$$

$$\frac{\delta}{R} \rightarrow 0, \mu_{\text{apparent}} \rightarrow \mu_c \rightarrow \mu_p$$

# CAPILLARY RISE

$$r = R \cos \theta$$

$$P_4 - P_3 = \frac{2\gamma}{r} \quad \text{This is Laplace - Young equation}$$

$$P_1 = P_4 + \varphi_v gh \quad \text{and} \quad P_2 = P_3 + \varphi_L gh$$

$$P_1 = P_2 \quad \text{and assume that} \quad \varphi_v \ll \varphi_L$$

$$h = \frac{2\gamma \cos \theta}{R \varphi_L g}$$

Assume that flow in the capillary tube is laminar ( Poiseuille flow )

where the fluid enters the capillary at  $P_2$ ,  $h(t) > R$  :

$$Q(t) = \frac{\pi R^4 (P_2 - P_3)}{8\mu h(t)}$$

Since  $P_1 = P_4$  ( $\varphi_v \ll \varphi_L$ ) and  $P_1 = P_2$  then  $P_4 = P_2$

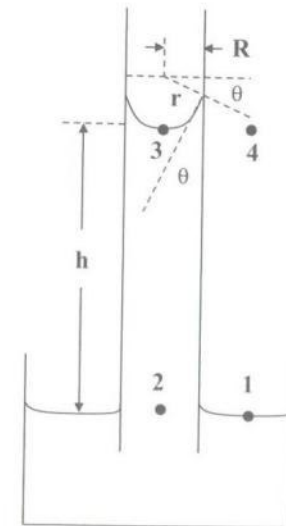
Re placing  $P_4$  by  $P_2$  in the Young's equation :

$$Q(t) = \frac{\pi R^3 \gamma \cos \theta}{4\mu h(t)}$$

$$Q(t) = \pi R^2 \frac{d(h(t))}{dt} = \frac{\pi R^3 \gamma \cos \theta}{4\mu h(t)}$$

Integrating with the I.C.  $h(t) = 0$  at  $t = 0$

$$h(t) = \sqrt{\frac{R \gamma \cos \theta t}{2\mu}}$$



# ASSIGNMENT

- Read Chapter 5 of the Textbook.
- Redo problem 2-23
- Problems 4.18, 4.20 and 4.30