

Transport Phenomena in Biomedical Engineering (196 C)

DATES: January 28 to May 20, 2008

TIMES: 6:00-8:45 PM

ROOM: 333

INSTRUCTOR: Maryam Mobed-Miremadi, PhD

EMAIL: cysomir@sbcglobal.net

PHONE: (408) 718 7875

OUTLINE

- Mass Transfer in Biological Systems
- Solve Problems from Chapter 5 of the book .

USEFUL REFERENCES

- *Basic Transport Phenomena in Biomedical Engineering* by R.L. Fournier
- *Transport Phenomena, Revised Second Edition* by Bird , Stewart and Lightfoot

<http://bcs.wiley.com/he-bcs/Books?action=index&bcsId=3406&itemId=0470115394>

- *Elementary Differential Equations and Boundary Value Problems* by Boyce and Diprima or any other Differential Equations Book.

CAPILLARY FLOWRATES (REVIEW)

Property	Value
Inside diameter (D_i)	10 microns
Length (l)	0.1 cm
Wall thickness (t_w)	0.5 microns
Average blood velocity (V)	0.05 cm/sec
Pore fraction	0.001
Wall pore diameter (d_p)	6-7 nm
Inlet pressure	30 mm Hg
Outlet pressure	10 mm Hg
Mean pressure (P_c)	17.3 mm Hg
Colloid osmotic pressure (π_c)	28 mm Hg
Interstitial fluid pressure (P_{IF})	-3 mm Hg
Interstitial fluid colloid osmotic pressure (π_{IF})	8 mm Hg

$$L_p = \left(\frac{A_p}{S} \right) \frac{r^2}{8\mu t_m \tau} \quad (1)$$

L_p is the hydraulic conductance

A_p total pore area

S circumferential area (πDL)

$\left(\frac{A_p}{S} \right)$ is defined as porosity

t_m wall thickness

μ fluid viscosity

$$Q = L_p S [(P_c - P_{IF}) - (\pi_c - \pi_{IF})] \quad (2)$$

Q is the plasmapheresis rate

$$Q_t = V\pi \frac{D_c^2}{4} \quad (3)$$

Q_t is the total blood flowrate

BOUNDARY CONDITIONS AND MATHEMATICAL METHODS

- Concentration is a function of position (x,y,z) and time (t). Regular Integration [f(x)] involves a single lower and upper limit . For a multivariate function [f(x,t)] an initial condition and Boundary Conditions are required to “frame” the simulation in time an space. This is an example:
IC: $t=0, C(x,t)= C_0$
BC1: $x=0, dC/dx=0$
BC2: $x=L, C =0$ for $t>0$
- Depending on the complexity of the Diffusion Problem several methods are used alone or in conjunction with each other to model the flow by using Partial Differential Equation(s):
 - a) Laplace Transforms (i.e. Piece-wise continuous functions)
 - b) **Matrix Solutions** (i.e. Eigenvectors)
 - c) Numerical Integration (i.e Newton Raphson Method, Euler Method)
 - d) **Boundary Value Problems** (i.e., Fourier Series)
- **It is more important to understand the physical phenomenon in order to set up the equations correctly.** Many of these equations are solved by a software tool (Mathematica, Matlab, Maple, etc...)

SOLUTE DIFFUSION (REVIEW)

The solute can diffuse down its own concentration gradient from the higher to the lower concentration.

$$\text{Rate of transfer process} = \frac{\text{driving force}}{\text{resistance}}$$

Fick's First Law:

$$J_{AZ} = -D_{AB} \frac{dc_A}{dz} \quad (4)$$

J_{AZ} molar flux of component A in direction z in (kgmolA / s.m²)

D_{AB} molecular diffusivity of the molecule A in B in (m² / s)

c_A concentration of A in (kgmolA / m³)

For turbulent Mass Transfer

$$J_{AZ} = -(D_{AB} + \varepsilon_m) \frac{dc_A}{dz} \quad (5)$$

ε_m turbulent or eddy mass transfer diffusivity in (m² / s)

FICK'S SECOND LAW (REVIEW)

Unsteady State Solute Balance across a shell of thickness Δy

$$S\Delta y \frac{\partial c}{\partial t} = -DS \left(\left. \frac{\partial c}{\partial y} \right|_y - \left. \frac{\partial c}{\partial y} \right|_{y+\Delta y} \right)$$

as $\Delta y \rightarrow 0$ and dividing by Δy

$$\frac{\partial c}{\partial t} = \frac{-DS \left(\left. \frac{\partial c}{\partial y} \right|_y - \left. \frac{\partial c}{\partial y} \right|_{y+\Delta y} \right)}{\Delta y}$$

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial y^2} \quad (6)$$

IC: $t=0 \quad C=0$

BC1: $y=0 \quad C=C_0$

BC2: $y=\infty \quad C=0$

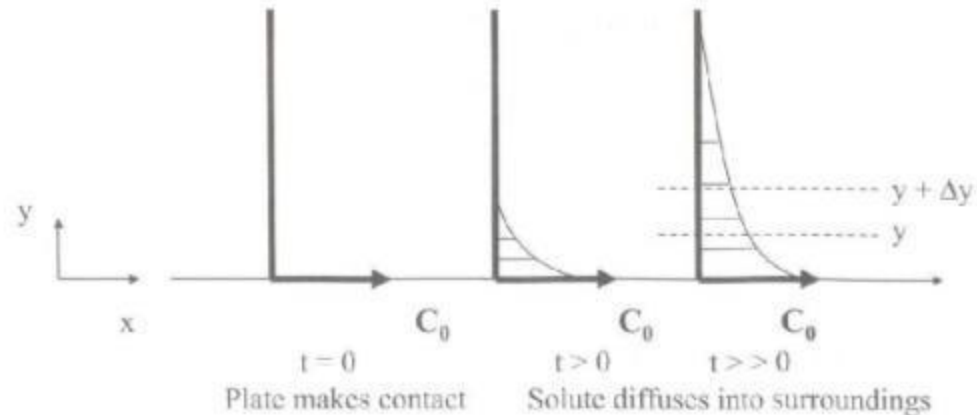
$$\frac{C(y,t)}{C_0} = 1 - \operatorname{erf} \left(\frac{y}{\sqrt{4Dt}} \right) \quad (7)$$

At $(C/C_0) = 0.01 \Rightarrow \delta_c(y,t) = 4\sqrt{Dt} \quad (8)$

$$j_s = -D \left. \frac{\partial c}{\partial y} \right|_{y=0} = k_m (C_{High} - C_{Low}) \quad (9)$$

$C_{High} = C_0 \quad C_{Low} = 0$

$$k_m = \frac{D}{32\sqrt{\pi Dt}} = \frac{D}{8\sqrt{\pi}} \left(\frac{1}{\delta_c(t)} \right) \quad (10)$$



Boundary Layer Thickness

Mass Transfer Coefficient

Table 5.2 A selection of useful mass transfer coefficient correlations (from Cussler 1984)

Physical situation	Correlation
Laminar flow over a flat plate	$\frac{\bar{k}_m L}{D} = Sh = 0.62 \left(\frac{\rho V L}{\mu} \right)^{1/2} \left(\frac{\mu}{\rho D} \right)^{1/3} = 0.626 Re^{1/2} Sc^{1/3}$
Laminar flow in a circular tube, short contact time solution	$Sh = \frac{\bar{k}_m D_{tube}}{D} = \frac{3}{\Gamma(4/3)} \left(\frac{2}{9} \right)^{1/3} Re^{1/3} Sc^{1/3} \left(\frac{R}{L} \right)^{2/3}$
Laminar flow in a circular tube, undeveloped flow and concentration profiles	$Sh = 3.66 + \frac{0.104 \frac{ReSc}{L/D_{tube}}}{1 + 0.016 \left(\frac{ReSc}{L/D_{tube}} \right)^{0.4}}$
Laminar flow in a circular tube, fully developed flow and concentration profiles	$Sh = 3.66$
Turbulent flow within a horizontal slit [$d = \left(\frac{2}{\pi} \right)$ (slit width)]	$\frac{\bar{k}_m d}{D} = 0.026 \left(\frac{d v_{avg}}{v} \right)^{0.8} \left(\frac{v}{D} \right)^{1/3}$ with \bar{k}_m the average mass transfer coefficient
Turbulent flow through a circular tube (D_{tube} is the tube diameter)	$\frac{\bar{k}_m d}{D} = 0.026 \left(\frac{d v_{avg}}{v} \right)^{0.8} \left(\frac{v}{D} \right)^{1/3}$ with \bar{k}_m the average mass transfer coefficient
Laminar flow in a circular tube (D_{tube} is the tube diameter)	$\frac{\bar{k}_m D_{tube}}{D} = 1.86 \left(ReSc \frac{D_{tube}}{L} \right)^{1/3}$ with \bar{k}_m the average mass transfer coefficient
Spinning disc (d is the disc diameter and ω is the disc rotation rate in radians/sec)	$\frac{\bar{k}_m d}{D} = 0.62 \left(\frac{d^2 \omega}{v} \right)^{1/2} \left(\frac{v}{D} \right)^{1/3}$ with \bar{k}_m the average mass transfer coefficient
Packed beds (d is the particle diameter and v_0 is the superficial velocity defined as the volumetric flowrate divided by the unpacked tube cross section)	$\frac{\bar{k}_m}{v_0} = 1.17 \left(\frac{d v_0}{v} \right)^{-0.42} \left(\frac{v}{D} \right)^{-0.42}$
Falling film (z is position along the length of the film and v_{wc} is the average film velocity)	$\frac{\bar{k}_m z}{D} = 0.69 \left(\frac{z v_{wc}}{D} \right)^{1/2}$ where \bar{k}_m is the local mass transfer coefficient

Sherwood Number

Schmidt Number

Reynolds Number

$$D_H = 4 \times \frac{\text{channel cross sectional area}}{\text{wetted perimeter}}$$

The hydraulic or equivalent diameter

is used in the calculations of Dimensionless Numbers.

DIFFUSIVITY DETERMINATION

(REVIEW)

Renkin and Curry equation (37°C, dilute aqueous solution)

$$D = 1.013 \times 10^{-4} (MW)^{-0.46} \text{ cm}^2/\text{sec} \quad (11)$$

Stokes – Einstein equation (37°C, dilute solution)

$$D = \frac{RT}{6\pi\mu a N_A} \quad (12)$$

a Solute / Stokes' radius

N_A Avogadro Number

$$a = \left(\frac{3MW}{4\pi\rho N_A} \right)^{1/3} \quad (13)$$

The “Stokes radius”, “Stokes-Einstein radius”, or hydrodynamic radius is not the effective radius of a hydrated molecule in solution. It is the radius of a hard sphere that diffuses at the same rate as the molecule. The behavior of this sphere includes hydration and shape effects. (**Chromatography**: [chromatography link](#)).

Since most molecules are not perfectly spherical, the Stokes radius is smaller than the effective radius (or the rotational radius). A more extended molecule will have a larger Stokes radius compared to a more compact molecule of the same molecular weight. In liquids where there are considerable interactions between solute and solvent molecule, Stokes radius is proportional to frictional coefficient f and inverse proportional to viscosity. The frictional coefficient is determined by the size and shape of the molecule.

SOLUTE TRANSPORT BY CAPILLARY FILTRATION (REVIEW)

$$S_a = \frac{C_f}{C_{bs}} = (1 - \lambda)^2 \left[2 - (1 - \lambda)^2 \right] \left[1 - \frac{2}{3} \lambda^2 - 0.163 \lambda^3 \right] \quad (14)$$

C_{bs} solute concentration at surface of the capillary on the blood side

C_f filtered solute concentration

λ a/r ; (a) solute radius and (r) capillary pore radius

$$q = Q/S$$

q filtration flux

$$q(C - C_f) = -D \frac{dC}{dy}$$

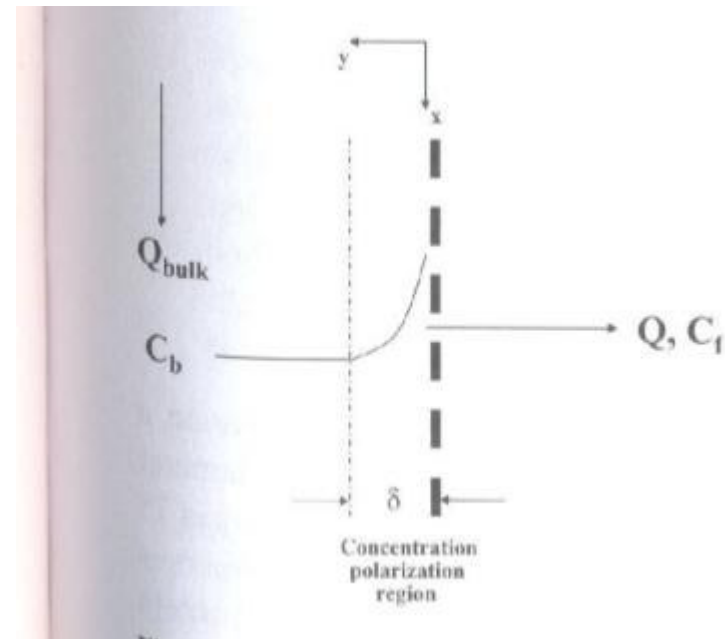
$$q \int_0^{\delta} dy = -D \int_{C_{bs}}^{C_b} \frac{dC}{C - C_f}$$

$$q = -\frac{D}{\delta_c} \ln \left(\frac{C_b - C_f}{C_{bs} - C_f} \right) = -k_m \ln \left(\frac{C_b - C_f}{C_{bs} - C_f} \right) \quad (15)$$

The sieving coefficient (S_o) in the case of concentration polarization :

$$S_o = \frac{C_f}{C_b} = \frac{S_a}{(1 - S_a)(\exp - q/k_m) + S_a} \quad (16)$$

Because of the pores in the capillary wall, the filtration of the plasma by the capillary wall will tend to separate the solutes on the basis of their size (Sieving action).



DIFFUSIVITY DETERMINATION (REVIEW)

$$\frac{D_m}{D} = K\left(\frac{a}{r}\right) \times \omega_r\left(\frac{a}{r}\right) \quad (17)$$

D_m Pore Diffusivity

D Bulk Diffusivity

$K\left(\frac{a}{r}\right)$ Contribution of Steric exclusion

$\omega_r\left(\frac{a}{r}\right)$ Contribution of hydrodynamic drag

Substitute into Fick's First Law:

$$D_e = D \frac{A_p}{S} \left(\frac{K\omega_r}{\tau} \right) = \frac{\varepsilon D}{\tau} K\omega_r \quad (18)$$

D_e Effective Diffusivity

ε Porosity = $\frac{A_p}{S}$

LONGTERM DIFFUSION OF A SOLUTE A POLYMERIC MATERIAL

Unsteady State Solute Balance across a shell of thickness Δx

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}$$

$$IC: \quad t=0 \quad C(x,t) = C_0$$

$$BC1: \quad x=0 \quad \frac{dC}{dx} = 0$$

$$BC2: \quad x=L \quad C=0$$

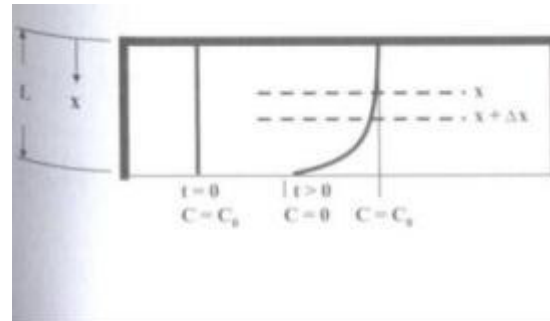


Figure 5.10 Diffusion of a solute within a microporous polymeric disk.

USING THE SEPARATION OF VARIABLES TECHNIQUE AND FOURIER SERIES

$$C(x,t) = \frac{4C_0}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} e^{-\frac{(2n+1)^2 \pi^2 D_e t}{4L^2}} \cos \frac{(2n+1)\pi x}{2L} \quad (19)$$

$$j_s = -D_e \frac{dC}{dx}$$

$$j_s \Big|_{x=L} = \frac{2D_e C_0}{L} \sum_{n=0}^{\infty} e^{-\frac{(2n+1)^2 \pi^2 D_e t}{4L^2}} \quad (20)$$

$$D(t) = S \int_0^L C(x,t) dx = \frac{8}{\pi^2} D_0 \sum_{n=0}^{\infty} \frac{(1)}{(2n+1)^2} e^{-\frac{(2n+1)^2 \pi^2 D_e t}{4L^2}} \quad (21)$$

$$f_R = \frac{D_0 - D(t)}{D_0} = 1 - \sum_{n=0}^{\infty} \frac{(1)}{(2n+1)^2} e^{-\frac{(2n+1)^2 \pi^2 D_e t}{4L^2}} \quad (22)$$

f_R Cumulative amount of drug released

$$D_0 = D(x,0) = C_0 SL$$

D_0 is the initial present in the total polymeric material volume including the void volume.



S is the Surface Area Normal to the Diffusion Direction

SHORT TERM DIFFUSION OF A SOLUTE A POLYMERIC MATERIAL

Unsteady State Solute Balance across a shell of thickness Δy

$$S\Delta y \frac{\partial c}{\partial t} = -DS \left(\left. \frac{\partial c}{\partial y} \right|_y - \left. \frac{\partial c}{\partial y} \right|_{y+\Delta y} \right)$$

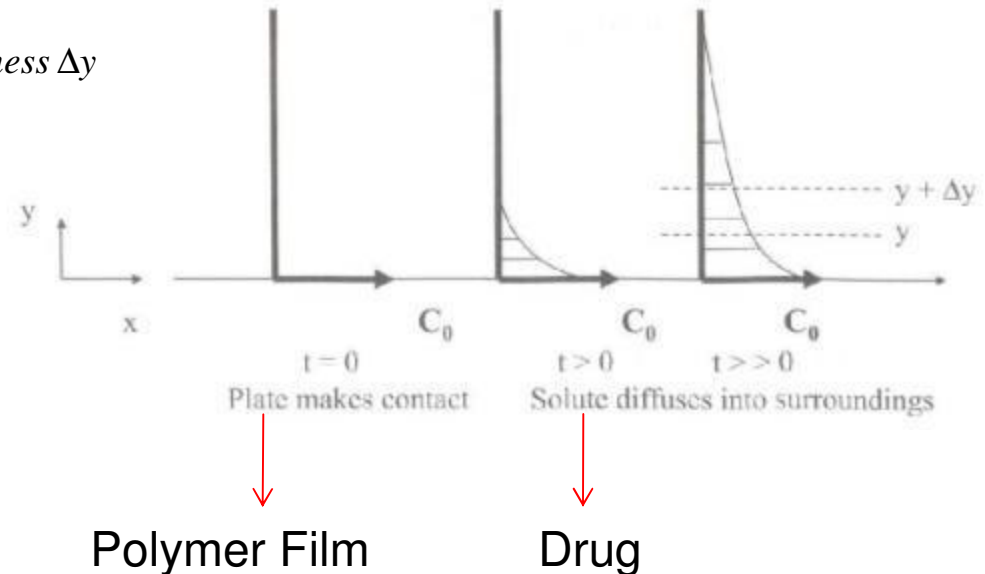
$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial y^2} \quad (6)$$

IC: $t = 0 \quad C = 0$

BC1: $y = 0 \quad C = C_0$

BC2: $y = \infty \quad C = 0$

$$\frac{C(y,t)}{C_0} = 1 - \operatorname{erf} \left(\frac{y}{\sqrt{4Dt}} \right) \quad (7)$$



Assume that the drug diffuses into boundary region .

DIFFUSION IN BLOOD AND TISSUE

Solute can diffuse through both the continuous fluid space (Blood) as well as through the cells themselves (Tissue). The transport mechanism in blood or the tissue space is by diffusion and D_T represents the effective diffusivity.

Maxwell Model(1873)

$$\frac{D_T}{D_0} = \frac{2D_0 + D_{cell} - 2\phi(D_0 - D_{cell})}{2D_0 + D_{cell} + \phi(D_0 - D_{cell})} \quad (23)$$

D_0 Diffusivity of solute through interstitial space or blood (D_{plasma})

D_{cell} Diffusivity of solute in the cells.

ϕ Volume fraction of the cells ($\phi = \pi\alpha_f^2 L$)

This equation does not depend on particle size

For $0.04 < \phi < 0.95$, Riley et al.(1994 – 1996)

$$\frac{D_T}{D_0} = 1 - \left(1 - \frac{D_{cell}}{D_0}\right) (1.727\phi - 0.8177\phi^2 + 0.09075\phi^3) \quad (24)$$

The interstitial fluid has a gel like structure (macromolecule obstacles for solute).

Brinkman equation (1947)

$$\frac{D_0}{D} = \omega_r = \frac{1}{1 + \kappa\alpha + 1/3(\kappa\alpha)^2} \quad (25)$$

D Solute Diffusivity in pure water

κ Experimentally obtained as a function of gel microstructure

$$K = \exp \left[-\phi \left(1 + \frac{a}{a_f} \right)^2 \right] \quad (26)$$

a_f represents the radius of macromolecules

Curry and Michel (1980)

$$\frac{D_T}{D} = \exp \left[- \left(1 + a/a_f \right) \nu^{1/2} C_F^{1/2} \right] \quad (27)$$

ν specific volume of the fibers

C_F fiber concentration in the interstitial fluid

$C_F \nu$ related fiber volume fraction

Renkin equation (1954)

$$\frac{D_m}{D} = K \left(\frac{a}{r} \right) \times \omega_r \left(\frac{a}{r} \right) \quad (17)$$

D_m Pore Diffusivity

D Bulk Diffusivity

$K \left(\frac{a}{r} \right)$ Contribution of Steric exclusion

$\omega_r \left(\frac{a}{r} \right)$ Contribution of hydrodynamic drag

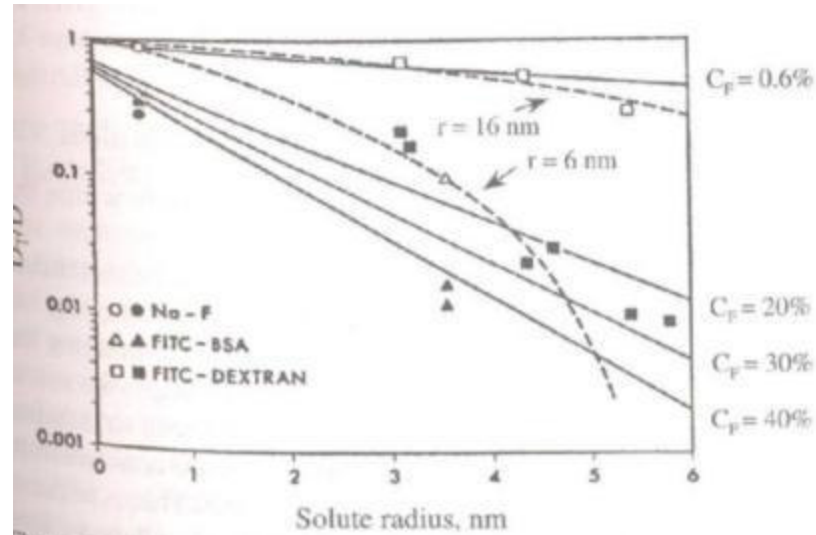
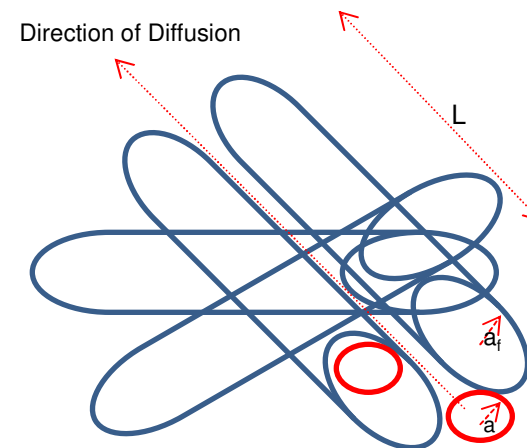


Figure 5.12 Ratios of the effective diffusivity in tissue to the aqueous diffusivity as a function of the solute radius. Open and closed symbols represent data for tumor and normal tissue, respectively. Solid lines represent the fiber matrix model and dashed lines represent the Renkin pore model equation (from Nugent and Jain 1984a, with permission).



SOLUTE PERMEABILITY

Permeability (P_m) is used to describe solute transport (N_s) across a membrane. Using the total surface area S .

$$N_s = P_m S (C_{high} - C_{low}) \quad (28)$$

$$P_m = \frac{D}{t_m} \left(\frac{A_p}{S} \right) \left(\frac{K \omega_r}{\tau} \right) \quad (29)$$

For cylindrical pores :

$K \omega_r$ would be given by equation (17) and τ (tortuosity) is determined experimentally

For a gel like membrane :

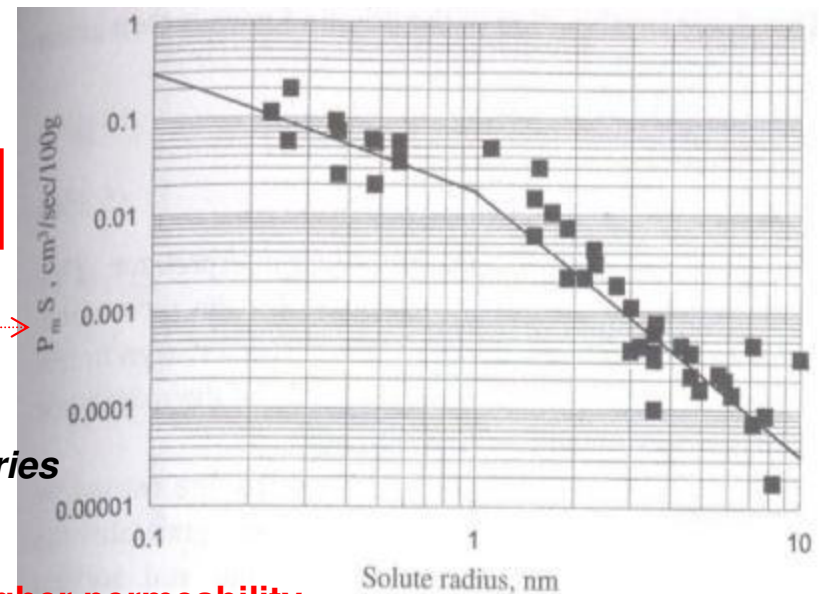
$\left(\frac{A_p}{S} \right) \left(\frac{K \omega_r}{\tau} \right)$ would be given by equations (25) and (26)

If $K = 1$, the solute has already diffused through the membrane.

$$P_m S = 0.0184 a^{-1.223}, \quad a < 1 \text{ nm} \quad (30a)$$

$$P_m S = 0.0287 a^{-2.92}, \quad a > 1 \text{ nm} \quad (30b)$$

**Renkin and Curry
(1979)
Continuous Capillaries**



Discontinuous and Fenestrated capillaries have much higher permeability

THE REFLECTION COEFFICIENT

The application of irreversible dynamics to membrane processes was developed by Staverman (1948) and Kedem and Katchalsky (1958). This effort was originated to describe the interaction of convection (bulk flow, driving force ΔP) and diffusion (driving force ΔC).

$$Q = SL_p \Delta P + SL_{ps} RT \Delta C \quad (31a)$$

$$J_v = SL_{sp} \Delta P + SL_s RT \Delta C \quad (31b)$$

L_{ps} Cross Coefficient for Secondary driving force (Osmotic flow)

L_{sp} Cross Coefficient for Secondary driving force (Ultrafiltration by sieving)

L_p Hydraulic Conductance

L_s Related to membrane permeability

$$\frac{N_s}{QC}$$

ONSAGER'S THEOREM

$$L_{ps} = L_{sp}$$

$$Q = SL_p (\Delta P - \sigma RT \Delta C) \quad (31a)$$

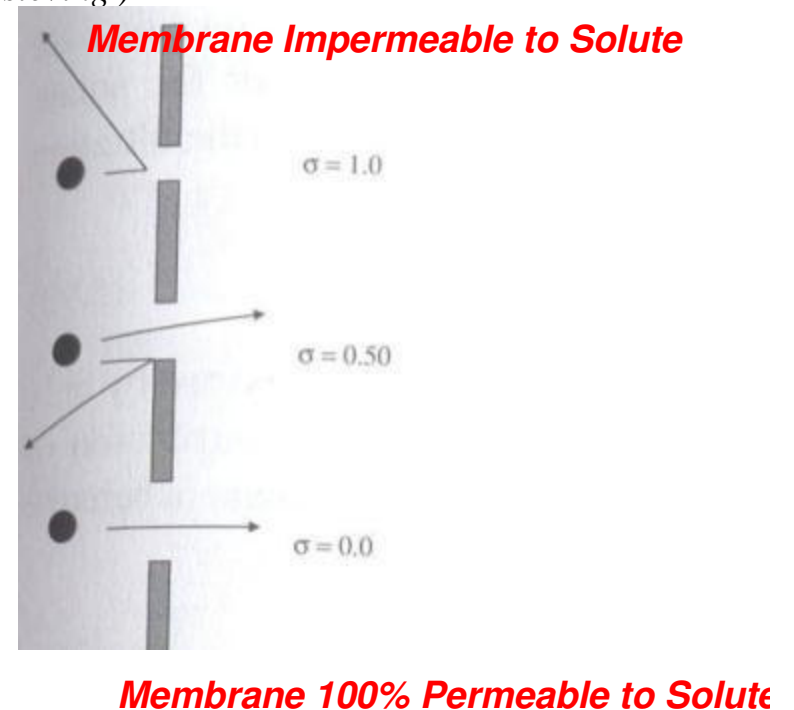
$$J_v = SL_p \left[-\sigma \Delta P + \frac{L_s}{L_p} RT \Delta C \right] \quad (31b)$$

σ ($-L_{sp}/L_p$) or ($-L_{ps}/L_p$) is the Staverman Reflection Coefficient

$$0 \leq \sigma \leq 1$$

$$N_s = C(Q + J_v)$$

$$N_s = C(1 - \sigma)Q + P_m S \Delta C \quad (32)$$



L_p , P_m and σ ESTIMATIONS

$$Q = SL_p([\Delta P - \sigma RT\Delta C]) \quad (31a)$$

$$\sigma = 0 \Leftrightarrow L_p = \frac{Q}{S\Delta P}$$

$$N_s = C(1 - \sigma)Q + P_m S\Delta C \quad (32)$$

$$Q = 0 \Leftrightarrow P_m = \frac{N_s}{S\Delta C}$$

σ :

1) Let $\Delta C = 0$ experimentally and measure Q

$$N_s = C(1 - \sigma)Q + P_m S\Delta C \Leftrightarrow (1 - \sigma) = \frac{N_s}{QC}$$

2) Let $\Delta P = 0$ experimentally and measure Q under controlled ΔC

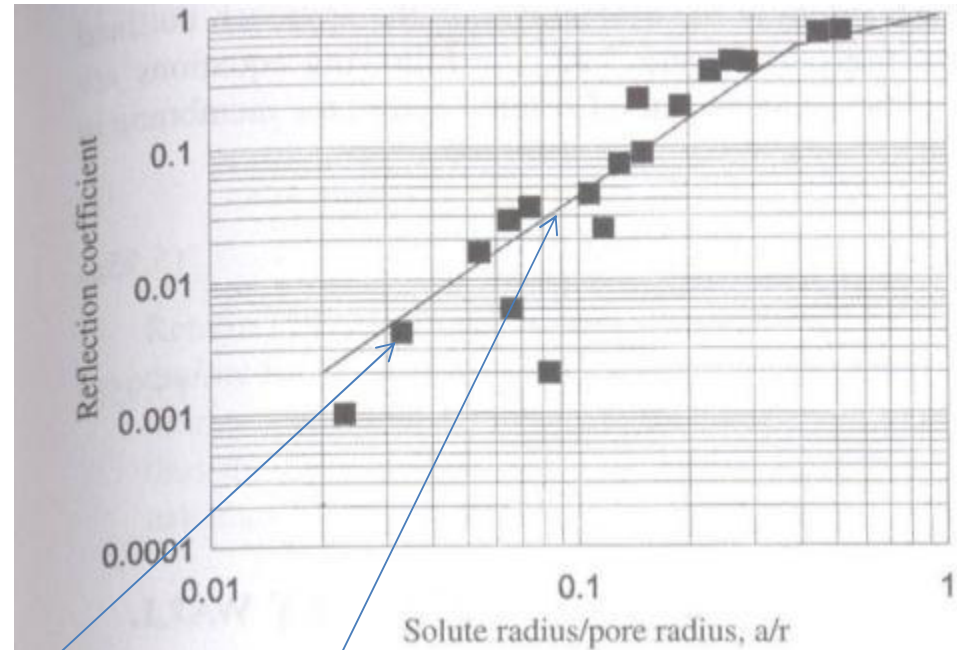
$$Q = SL_p([\Delta P - \sigma RT\Delta C]) \Leftrightarrow \sigma = \frac{-Q}{SL_p RT\Delta C}$$

3) For $0.01 \leq (a/r) \leq 1$

Durbin's experimental data (1960)

Anderson and Quinn (1974) showed that $S_a \equiv \frac{N_s}{QC} = (1 - \sigma)$ when $\Delta C = 0$

$$\sigma = 1 - \left(1 - \frac{a}{r}\right)^2 \left[2 - \left(1 - \frac{a}{r}\right)^2\right] \times \left[1 - \frac{2}{3}\left(\frac{a}{r}\right)^2 - 0.163\left(\frac{a}{r}\right)^3\right] \quad (33)$$



MULTICOMPONENT MEMBRANE TRANSPORT

The derivation of the filtration flowrate and the solute transport rate of solute i across the membrane in a multicomponent system are based on those of a single solute/solvent system.

$$Q = SL_p \Delta P + SL_{PS} RT \Delta C \quad (31a)$$

$$J_V = SL_{SP} \Delta P + SL_S RT \Delta C \quad (31b)$$

$$Q = SL_p \Delta P + SRT \sum_i L_{PS_i} \Delta C_i \quad (34a)$$

$$J_V = S \Delta P \sum_i L_{SP_i} + SRT \sum_i L_{S_i} \Delta C_i \quad (34b)$$

$$\text{Let } L_{SP_i} = L_{PS_i}$$

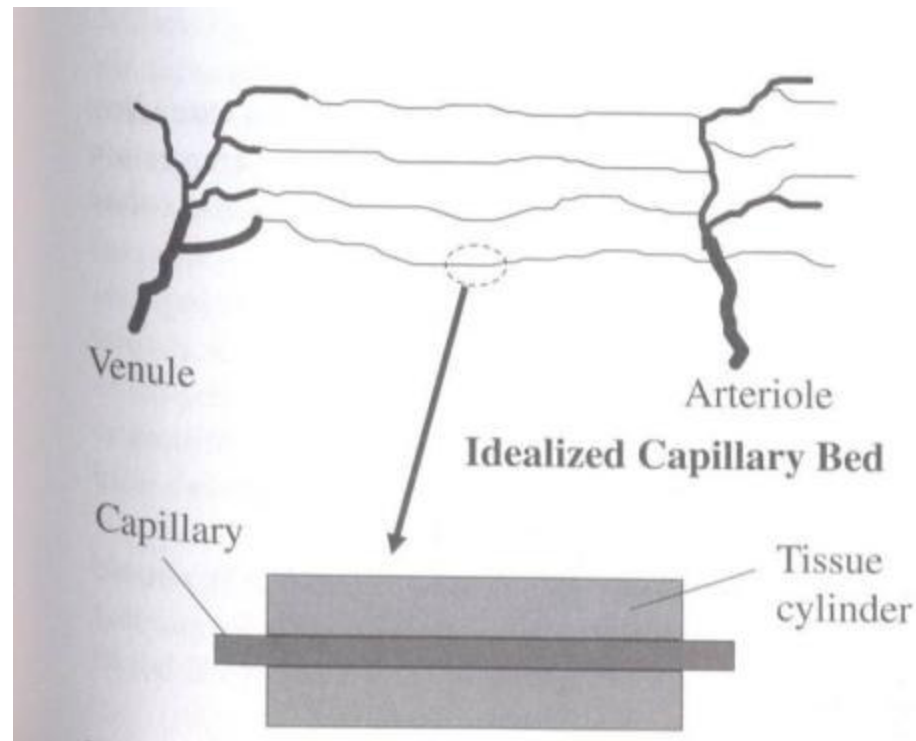
$$Q = SL_p \left[\Delta P - RT \sum_i \sigma_i \Delta C_i \right] \quad (35a)$$

$$N_i = C_i (1 - \sigma_i) Q + P_{m_i} S \Delta C_i \quad (35b)$$

TRANSPORT OF SOLUTE ACROSS THE CAPILLARY WALL

- Example 5.8 : Transport of glucose (small solute, water soluble=hydrophilic)
- Example 5.9: Transport of oxygen (gas, lipid soluble=lipophilic).
- Example 5.10: Transport of proteins (hydrophilic macromolecules)
- Example 5.11: Effect of ΔP on the transport of proteins in example 5.10

TRANSPORT OF A SOLUTE BETWEEN A CAPILLARY AND THE SURROUNDING TISSUE



KROGH TISSUE CYLINDER MODEL DEVELOPED FOR OXYGEN UPTAKE

[LINK](#)

KROGH TISSUE CYLINDER

- Krogh (1919) used the cylindrical capillary tissue model to study the supply of oxygen to muscle. The assumption is that the tissue space surrounding the capillary is a continuous phase (not discrete cells).
- Solute diffusivity (D_T) is driven by consumption/production of the solute by the cells within the tissue space.
- The Michealis Menten equation ([LINK](#)) in enzyme kinetics will be used to describe the metabolic rate of the solute in the tissue space. This kinetic model is relevant to situations where the concentration of enzyme is much lower than the concentration of substrate.

$$R(\bar{C}) = \frac{V_{max} \bar{C}}{K_m + \bar{C}} \quad (36)$$

$R(\bar{C}) > 0$ when produced

$R(\bar{C}) < 0$ when consumed

K_m Michaelis Menten constant

The maximum reaction rate occurs when, $R(\bar{C}) = V_{max} = R_0$ for $K_m \ll \bar{C} \Rightarrow$ zero order reaction (biological cases)

When $K_m \gg \bar{C}$, $R(\bar{C}) = \frac{V_{max} \bar{C}}{K_m} \Rightarrow$ first order reaction

A MODEL OF THE KROGH TISSUE CYLINDER

Assume flux in the axial direction (z) is constant

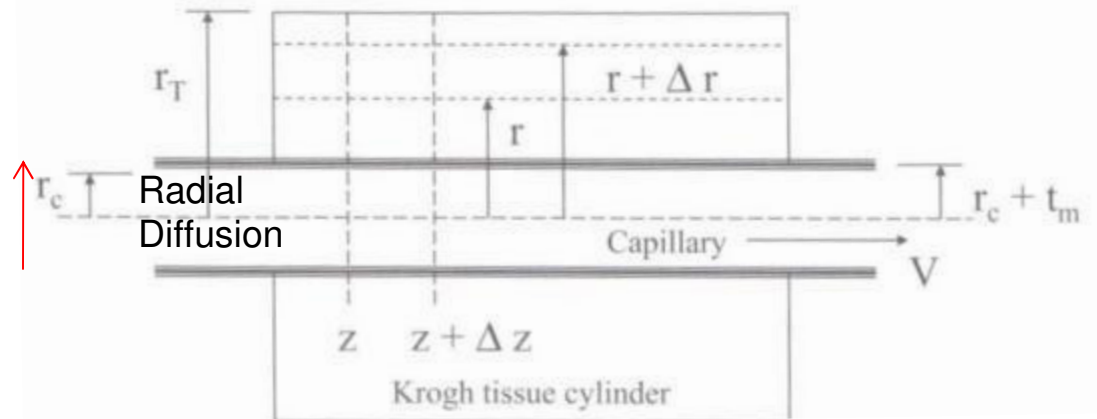
$$N_s = C(1 - \sigma)Q + P_m S \Delta C \quad (32)$$

If $C(1 - \sigma)Q \ll P_m S \Delta C$

Then $N_s = P_m S \Delta C$

$$S = 2\pi r_c \Delta z$$

$$N_s = 2\pi r_c \Delta z K_0 \left(C - \bar{C} \Big|_{r_c + t_m} \right)$$



Axial Convection (neglected)

Solute exchange for blood /capillary exchange occurs at r_c
 Solute exchange for capillary tissue exchange occurs at $(r_c + t_m)$

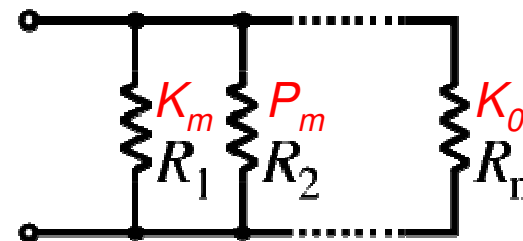
Assume blood as a continuous phase

$$V\pi r_c^2 C \Big|_z - V\pi r_c^2 C \Big|_{z+\Delta z} = 2\pi r_c \Delta z K_0 \left(C - \bar{C} \Big|_{r_c + t_m} \right)$$

Divide Δz and taking the limit at $\Delta z \rightarrow 0$

for SS Shell Balance in z direction

$$-V \frac{dC}{dz} = \frac{2}{r_c} K_0 \left(C - \bar{C} \Big|_{r_c + t_m} \right) \quad (37)$$



$$K_0 = \frac{1}{\frac{1}{K_m} + \frac{1}{P_m}}$$

Fick's First Law in Radial Direction

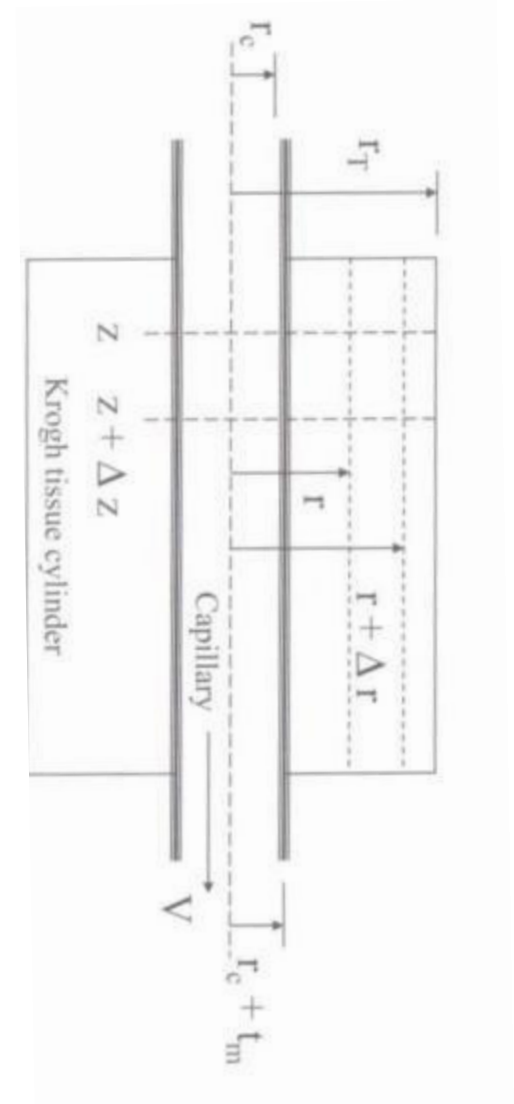
$$J_{Ar} = -(D_T) \frac{dC}{dr} \quad (5)$$

A SS shell balance for a given z at r to $(r + \Delta r)$ for the solute concentration in tissue space :

$$-D_T \frac{d\bar{C}}{dr} 2\pi r \Delta z \Big|_r + D_T \frac{d\bar{C}}{dr} 2\pi r \Delta z \Big|_{r+\Delta r} = R(\bar{C}) 2\pi r \Delta z \Delta r$$

After dividing by $2\pi r \Delta r$ and taking the limit as $\Delta r \rightarrow 0$:

$$D_T \frac{d}{dr} \left(r \frac{d\bar{C}}{dr} \right) - R(\bar{C}) = 0 \quad (38)$$



$$-V \frac{dC}{dz} = \frac{2}{r_c} K_0 (C - \bar{C} |_{r_c+t_m}) \quad (37)$$

$$D_T \frac{d}{dr} \left(r \frac{d\bar{C}}{dr} \right) - R(\bar{C}) = 0 \quad (38)$$

$$BC1: \quad z=0, \quad C = C_0$$

$$BC2: \quad r=r_c+t_m, \quad \bar{C} = \bar{C} |_{r_c+t_m}$$

$$BC3: \quad r=r_T, \quad \frac{d\bar{C}}{dr} = 0$$

$$\bar{C}(r, z) = \bar{C}(z) |_{r_c+t_m} + \frac{R_0}{4D_T} \left[\left(r^2 - (r_c+t_m)^2 \right) \right] - \frac{R_0 r_T^2}{2D_T} \ln \left(\frac{r}{r_c+t_m} \right) \quad (5.104)$$

$$V\pi r_c^2 C_0 - V\pi r_c^2 C \Big|_z = \pi \left[r_c^2 - (r_c + t_m)^2 \right] z R_0 \quad (5.105)$$

This equation may be rearranged to give the following equation that provides for the axial variation of the solute concentration in the capillary:

$$C(z) = C_0 - \frac{R_0}{Vr_c^2} \left[r_c^2 - (r_c + t_m)^2 \right] z \quad (5.106)$$

Equation 5.106 can now be used to find $\frac{dC}{dz}$ in Equation 5.100 with the result that we can solve for $\bar{C}(z) \Big|_{r_c+t_m}$ which is given by the equation below.

$$\bar{C}(z) \Big|_{r_c+t_m} = C(z) - \frac{R_0}{2r_c K_0} \left(r_c^2 - (r_c + t_m)^2 \right) \quad (5.107)$$

Note that $C(z)$ in Equation 5.107 makes $\bar{C}(z) \Big|_{r_c+t_m}$ depend on z , and by Equation 5.104, the tissue space solute concentration then depends on both r and z as discussed earlier. Equations 5.104, 5.106, and 5.107 can be combined to give the following equation for the solute concentration in the tissue space.

$$\begin{aligned} \bar{C}(r, z) = & C_0 - \frac{R_0}{Vr_c^2} \left(r_c^2 - (r_c + t_m)^2 \right) z \\ & - \frac{R_0}{2r_c K_0} \left[r_c^2 - (r_c + t_m)^2 \right] + \frac{R_0}{4D_r} \left[\left(r^2 - (r_c + t_m)^2 \right) \right] - \frac{R_0 r_c^2}{2D_r} \ln \left(\frac{r}{r_c + t_m} \right) \end{aligned} \quad (5.108)$$

Under some conditions either the delivery of the solute to the capillary may be limited by the capillary flowrate, or the transport rate of the solute across the capillary wall is limited, or the consumption of the solute by the tissue is very rapid. Any one of these conditions may lead to regions of the tissue that have no solute. We can then define a *critical radius* in the tissue, $r_{\text{critical}}(z)$, defined as the distance beyond which no solute is present in the tissue. For this situation we need to modify boundary condition 3 in Equation 5.103 to the following:

$$\text{BC3}^* : r = r_{\text{critical}}(z), \quad \frac{d\bar{C}}{dr} = 0 \text{ and } \bar{C} = 0 \quad (5.109)$$

Under these conditions, the solute concentrations in the capillary, i.e. $C(z)$, at the interface between the capillary and the tissue space, i.e. $\bar{C}(z)|_{r_c+t_m}$, and in the tissue space itself, i.e. $\bar{C}(r,z)$, would still be given respectively by Equations 5.106, 5.107, and 5.108, however, the Krogh tissue cylinder radius, r_T , is replaced with $r_{\text{critical}}(z)$ once the solute concentration in the tissue at a particular location has reached zero. The critical radius may be obtained by recognizing that at $r_{\text{critical}}(z)$, $\bar{C}(r, z) = 0$. Thus we may use Equation 5.108, with $r_T = r_{\text{critical}}(z)$ and $\bar{C}(r, z) = 0$, to obtain the following expression for the critical radius.

$$\left(\frac{r_{\text{critical}}(z)}{r_c+t_m}\right)^2 \ln\left(\frac{r_{\text{critical}}(z)}{r_c+t_m}\right) - \left(\frac{r_{\text{critical}}(z)}{r_c+t_m}\right)^2 + 1 = \left(\frac{4D_T C_0}{R_0(r_c+t_m)^2}\right) - \frac{4D_T}{Vr_c^2} \left[\left(\frac{r_{\text{critical}}(z)}{r_c+t_m}\right)^2 - 1\right]z - \frac{2D_T}{r_c K_0} \left[\left(\frac{r_{\text{critical}}(z)}{r_c+t_m}\right)^2 - 1\right] \quad (5.110)$$

APPLICATIONS OF KROGH'S MODEL

- Example 5.12
- Example 5.13

THE PECLET NUMBER

- The magnitude of the Peclet (Pe) dimensionless number represents the importance of axial convection in comparison to axial diffusion. **The criterion for ignoring axial diffusion is given by $Pe=VL/D \gg 1$.**
- A similar line of reasoning could be applied to the tissue space to support the neglect of **axial diffusion in comparison to radial diffusion:**

$$\left(\frac{D_T C_0}{r_T - r_C} \right) 2\pi r_C L = \text{Transport by Radial Diffusion} \quad (39)$$

$$D_T \frac{C_0}{L} \pi (r_T^2 - r_C^2) = \text{Transport by Axial Diffusion} \quad (40)$$

RENKIN-CRONE EQUATION

$$-V \frac{dC}{dz} = \frac{2}{r_c} K_0 (C - \bar{C} |_{r_c+t_m}) \quad (37)$$

When $\bar{C} |_{r_c+t_m} \ll C$ $K_0 \approx P_m$

$$\frac{dC}{dz} = -\frac{2}{V r_c} P_m C = -\frac{2\pi r_c}{Q} P_m C \quad (41)$$

Integrating eq (41) and rearranging to obtain the solute concentration at any axial position z ,

The RENKIN-CRONE EQUATION gives the solute extraction E

$$E = \frac{[C_0 - C(z)]}{C_0} = 1 - \exp\left[-\frac{2\pi r_c P_m z}{Q}\right] \quad (42)$$

QC_0 is the maximum amount of solute that can be transported, $QC_0 \times E$ is the actual amount.

$$\begin{aligned} \text{If } \frac{2\pi r_c P_m z}{Q} \gg 1, & \text{ flow limited regime} \\ \text{If } \frac{2\pi r_c P_m z}{Q} \ll 1, & \text{ Diffusion limited regime} \end{aligned}$$

For regions of tissue with multiple capillaries

$2\pi r_c z = S$, S total surface area of capillaries within the tissue region of interest.

ASSIGNMENT

- Follow through all the examples in Chapter 5.
- Solve problems 5 (1,4,5,9,13,15,21,25).
Solutions will be posted on 4/7/08 and reviewed on 4/9/08.
- Study Chapters 4 and 5 (except for Vascular Beds section 5.10.3) for 4/14/08 Midterm.
- The format will be the same as the First Midterm (1/3 definitions, 2/3 Problem Solving).