Transport Phenomena in Biomedical Engineering (196 C)

DATES: January 28 to May 20, 2008
TIMES: 6:00-8:45 PM
ROOM: 333

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OUTLINE

- Review and Finish Mass Transfer in Biological Systems.

- Solve Assignment #5 Problems.

USEFUL REFERENCES

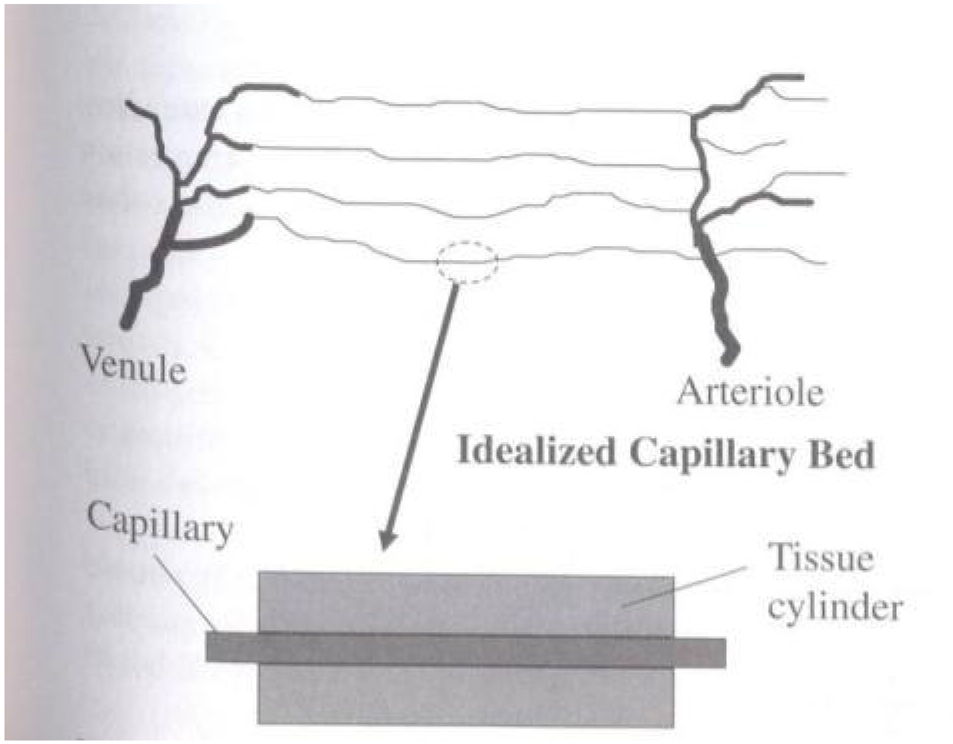
- *Basic Transport Phenomena in Biomedical Engineering* by R.L. Fournier
- *Transport Phenomena*, *Revised Second Edition* by Bird , Stewart and Lightfoot

http://bcs.wiley.com/he-bcs/Books?action=index&bcsId=3406&itemId=0470115394

- *Elementary Differential Equations and Boundary Value Problems* by Boyce and Diprima or any other Differential Equations Book.

TRANSPORT OF A SOLUTE BETWEEN A CAPILLARY AND THE SURROUNDING TISSUE

(REVIEW)



KROGH TISSUE CYLINDER MODEL DEVELOPED FOR OXYGEN UPTAKE

LINK

KROGH TISSUE CYLINDER

(REVIEW)

- Krogh (1919) used the cylindrical capillary tissue model to study the supply of oxygen to muscle. The assumption is that the tissue space surrounding the capillary is a continuous phase (not discrete cells).
- Solute diffusivity (D_T) is driven by consumption/production of the solute by the cells within the tissue space.
- The Michealis Menten equation (LINK) in enzyme kinetics will be used to describe the metabolic rate of the solute in the tissue space. This kinetic model is relevant to situations where the concentration of enzyme is much lower than the concentration of substrate.

$$R(\overline{C}) = \frac{V_{max}\overline{C}}{K_m + \overline{C}} \quad (36)$$

$R(\overline{C}) > 0 \quad$ *when produced*

$R(\overline{C}) < 0 \quad$ *when consumed*

$K_m \quad$ *Michaelis Menten constant*

The maximum reaction rate occurs when, $R(\overline{C}) = V_{max} = R_0$ *for* $K_m << \overline{C} \Rightarrow$ *zero order reaction (biological cases)*

When $K_m >> \overline{C}, \quad R(\overline{C}) = \frac{V_{max}\overline{C}}{K_m} \quad \Rightarrow$ *first order reaction*

A MODEL OF THE KROGH TISSUE CYLINDER (REVIEW)

Assume flux in the axial direction (z) is constant

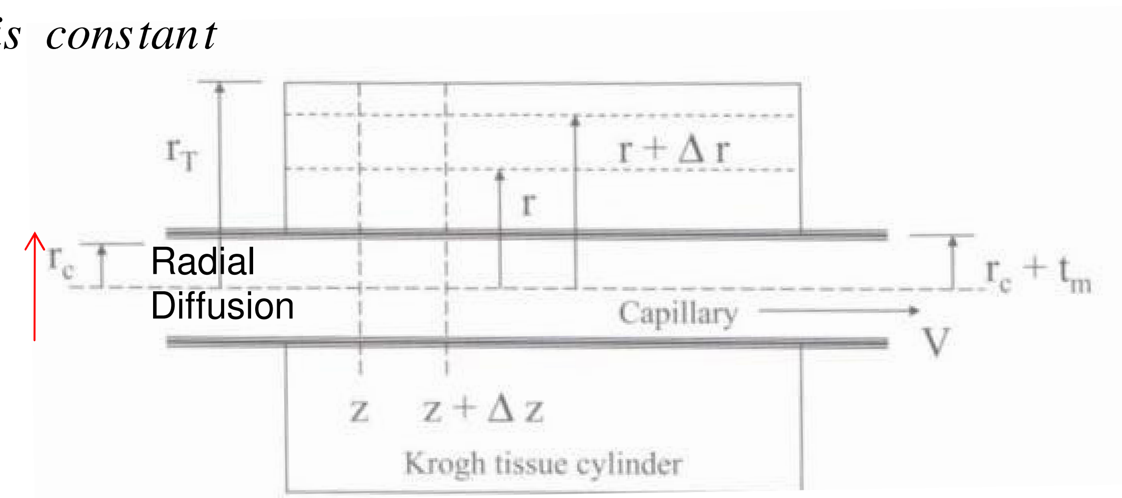
$$N_s = C(1-\sigma)Q + P_m S\Delta C \quad (32)$$

$$If \quad C(1-\sigma)Q << P_m S\Delta C$$

$$Then \quad N_s = P_m S\Delta C$$

$$S = 2\pi r_c \Delta z$$

$$N_s = 2\pi r_c \Delta z K_0 \left(C - \overline{C}\ \big|_{r_c+t_m}\right)$$



Axial Convection (neglected)

Assume blood as a continuous phase

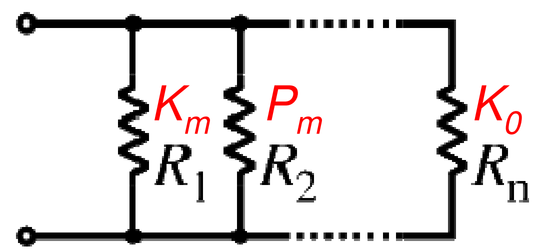
$$V\pi r_c^2 C\ \big|_z - V\pi r_c^2 C\ \big|_{z+\Delta z} = 2\pi r_c \Delta z K_0 \left(C - \overline{C}\ \big|_{r_c+t_m}\right)$$

Divide Δz *and taking the limit at* $\Delta z \to 0$

for SS Shell Balance in z direction

$$-V\frac{dC}{dz} = \frac{2}{r_c} K_0 \left(C - \overline{C}\ \big|_{r_c+t_m}\right) \quad (37)$$

Solute exchange for blood /capillary exchange occurs at r_c
Solute exchange for capillary tissue exchange occurs at ($r_c + t_m$)



$$K_0 = \frac{1}{\frac{1}{K_m} + \frac{1}{P_m}}$$

Fick's First Law in Radial Direction

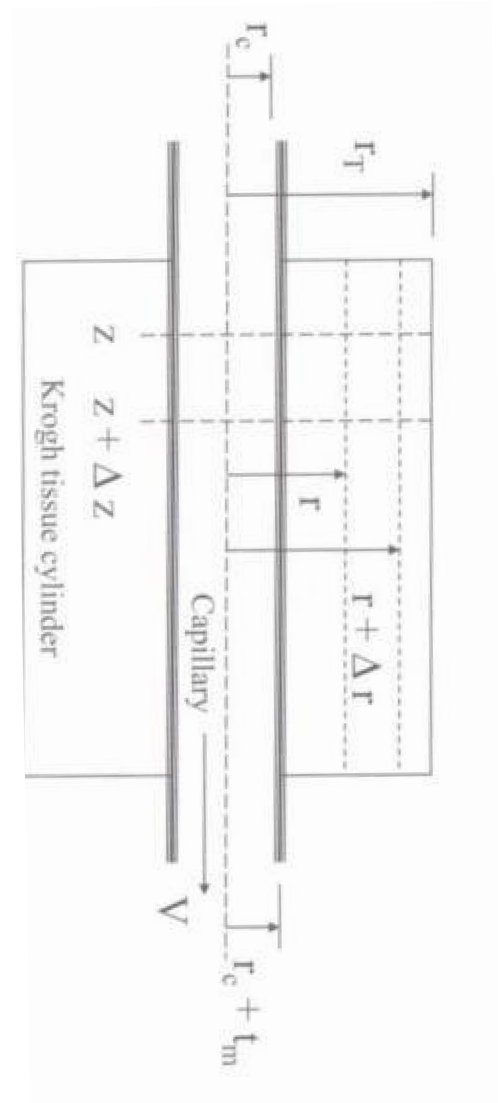
$$J_{Ar} = -(D_T)\frac{dC}{dr} \qquad (5)$$

A SS shell balance for a given z at r to (r + Δr) for the solute concentration in tissue space:

$$-D_T\frac{d\overline{C}}{dr}2\pi r\Delta z\Big|_r + D_T\frac{d\overline{C}}{dr}2\pi r\Delta z\Big|_{r+\Delta r} = R(\overline{C})2\pi r\Delta z\Delta r$$

After dividing by $2\pi r\Delta r$ and taking the limit as $\Delta r \to 0$:

$$D_T\frac{d}{dr}\left(r\frac{d\overline{C}}{dr}\right) - R(\overline{C}) = 0 \qquad (38)$$



$$-V\frac{dC}{dz}=\frac{2}{r_c}K_0\left(C-\bar{C}\,\big|_{r_c+t_m}\right) \qquad (37)$$

$$D_T\frac{d}{dr}\left(r\frac{d\bar{C}}{dr}\right)-R(\bar{C})=0 \qquad (38)$$

$$BC1: \quad z=0, \quad C=C_0$$

$$BC2: \quad r=r_c+t_m, \quad \bar{C}=\bar{C}\,\big|r_c+t_m$$

$$BC3: \quad r=r_T, \quad \frac{d\bar{C}}{dr}=0$$

$$\bar{C}(r,z)=\bar{C}(z)\Big|_{r_c+t_m}+\frac{R_0}{4D_T}\left[\left(r^2-\left(r_c+t_m\right)^2\right)\right]-\frac{R_0 r_T^2}{2D_T}\ln\left(\frac{r}{r_c+t_m}\right) \qquad (5.104)$$

$$V\pi r_c^2 C_0 - V\pi r_c^2 C\big|_z = \pi\left[r_T^2 - \left(r_c + t_m\right)^2\right] z\, R_0 \tag{5.105}$$

This equation may be rearranged to give the following equation that provides for the axial variation of the solute concentration in the capillary.

$$C(z) = C_0 - \frac{R_0}{V r_c^2}\left[r_T^2 - \left(r_c + t_m\right)^2\right] z \tag{5.106}$$

Equation 5.106 can now be used to find $\frac{dC}{dz}$ in Equation 5.100 with the result that we can solve for $\overline{C}(z)\big|_{r_c + t_m}$ which is given by the equation below.

$$\overline{C}(z)\Big|_{r_c + t_m} = C(z) - \frac{R_0}{2 r_c K_0}\left(r_T^2 - \left(r_c + t_m\right)^2\right) \tag{5.107}$$

Note that $C(z)$ in Equation 5.107 makes $\overline{C}(z)\big|_{r_c + t_m}$ depend on z, and by Equation 5.104, the tissue space solute concentration then depends on both r and z as discussed earlier. Equations 5.104, 5.106, and 5.107 can be combined to give the following equation for the solute concentration in the tissue space.

$$\begin{aligned}\overline{C}(r,z) = {} & C_0 - \frac{R_0}{V r_c^2}\left(r_T^2 - \left(r_c + t_m\right)^2\right) z \\ & - \frac{R_0}{2 r_c K_0}\left[r_T^2 - \left(r_c + t_m\right)^2\right] + \frac{R_0}{4 D_T}\left[\left(r^2 - \left(r_c + t_m\right)^2\right)\right] - \frac{R_0 r_T^2}{2 D_T} \ln\left(\frac{r}{r_c + t_m}\right)\end{aligned} \tag{5.108}$$

Under some conditions either the delivery of the solute to the capillary may be limited by the capillary flowrate, or the transport rate of the solute across the capillary wall is limited, or the consumption of the solute by the tissue is very rapid. Any one of these conditions may lead to regions of the tissue that have no solute. We can then define a *critical radius* in the tissue, $r_{\text{critical}}(z)$, defined as the distance beyond which no solute is present in the tissue. For this situation we need to modify boundary condition 3 in Equation 5.103 to the following:

$$\text{BC3}': \ r = r_{\text{critical}}(z), \ \frac{d\bar{C}}{dr} = 0 \text{ and } \bar{C} = 0 \tag{5.109}$$

Under these conditions, the solute concentrations in the capillary, i.e. $C(z)$, at the interface between the capillary and the tissue space, i.e. $\bar{C}(z)|_{r_c+t_m}$, and in the tissue space itself, i.e. $\bar{C}(r,z)$, would still be given respectively by Equations 5.106, 5.107, and 5.108, however, the Krogh tissue cylinder radius, r_T, is replaced with $r_{\text{critical}}(z)$ once the solute concentration in the tissue at a particular location has reached zero. The critical radius may be obtained by recognizing that at $r_{\text{critical}}(z)$, $\bar{C}(r, z) = 0$. Thus we may use Equation 5.108, with $r_T = r_{\text{critical}}(z)$ and $\bar{C}(r, z) = 0$, to obtain the following expression for the critical radius.

$$\left(\frac{r_{\text{critical}}(z)}{r_c+t_m}\right)^2 \ln\left(\frac{r_{\text{critical}}(z)}{r_c+t_m}\right)^2 - \left(\frac{r_{\text{critical}}(z)}{r_c+t_m}\right)^2 + 1 = \left(\frac{4D_T C_0}{R_0(r_c+t_m)^2}\right)$$
$$- \frac{4D_T}{Vr_c^2}\left[\left(\frac{r_{\text{critical}}(z)}{r_c+t_m}\right)^2 - 1\right]z - \frac{2D_T}{r_c K_0}\left[\left(\frac{r_{\text{critical}}(z)}{r_c+t_m}\right)^2 - 1\right] \tag{5.110}$$

THE PECLET NUMBER (REVIEW)

- The magnitude of the Peclet (Pe) dimensionless number represents the importance of axial convection in comparison to axial diffusion. **The criterion for ignoring axial diffusion is given by *Pe=VL/D>>1.***

- A similar line of reasoning could be applied to the tissue space to support the neglect of **axial diffusion in comparison to radial diffusion:**

$$\left(\frac{D_T C_0}{r_T - r_C}\right) 2\pi r_C L = \quad Trasnport \ by \quad Radial \quad Diffusion \quad (39)$$

$$D_T \frac{C_0}{L} \pi\left(r^2{}_T - r^2{}_C\right) = \quad Trasnport \ by \quad Axial \quad Diffusion \quad (40)$$

RENKIN-CRONE EQUATION (REVIEW)

$$-V\frac{dC}{dz}=\frac{2}{r_c}K_0\left(C-\bar{C}\,\big|_{r_c+t_m}\right) \qquad (37)$$

$$When \quad \bar{C}\,\big|_{r_c+t_m} << C \qquad K_0 \approx P_m$$

$$\frac{dC}{dz}=-\frac{2}{Vr_c}P_mC=-\frac{2\pi r_c}{Q}P_mC \qquad (41)$$

Integrating eq (41) and rearranging to obtain the solute concentration at any axial position z,

The RENKIN−CRONE EQUATION gives the solute extraction E

$$E=\frac{[C_0-C(z)]}{C_0}=1-exp\left[-\frac{2\pi r_cP_mz}{Q}\right] \qquad (42)$$

QC_0 *is the maximum amount of solute that can be transported,* $QC_0 \times E$ *is the actual amount.*

$$If \quad \frac{2\pi r_cP_mz}{Q}>>1, \quad flow\ limited \quad regime$$

$$If \quad \frac{2\pi r_cP_mz}{Q}<<1, \quad Diffusion\ limited \quad regime$$

For regions of tissue with multiple capillaries

$2\pi r_c z = S,\quad S$ *total surface area of capillaries within the tissue region of interest.*

DETERMINING THE VALUE OF SOLUTE PERMEABILITY (P_mS)

- The multiple tracer indicator diffusion technique is used to obtain permeability across the capillary wall in organs, large tissue regions and artificial membranes (hemodialysis and bioartificial organs). At time equal to zero, the solutes are injected into a main artery feeding the region of interest.
- The clearance is calculated by obtaining venous blood samples for the concentration of test (permeable, C) and reference (non-permeable, C_o) solutes over a few seconds (Figure 5.20).

Clearance (CL), is the volumetric flowrate of the fluid that has been totally cleared of the solute at a certain time.

$$CL(z) \equiv \frac{Q[C_0 - C(z)]}{C_0} = QE = Q\left(1 - e^{-\frac{PmS}{Q}}\right) \quad (43)$$

$$When \ \ P_mS >> Q \ \ (flow \ \ limited) \ \ and \ \ E \approx 1, \quad CL = Q$$

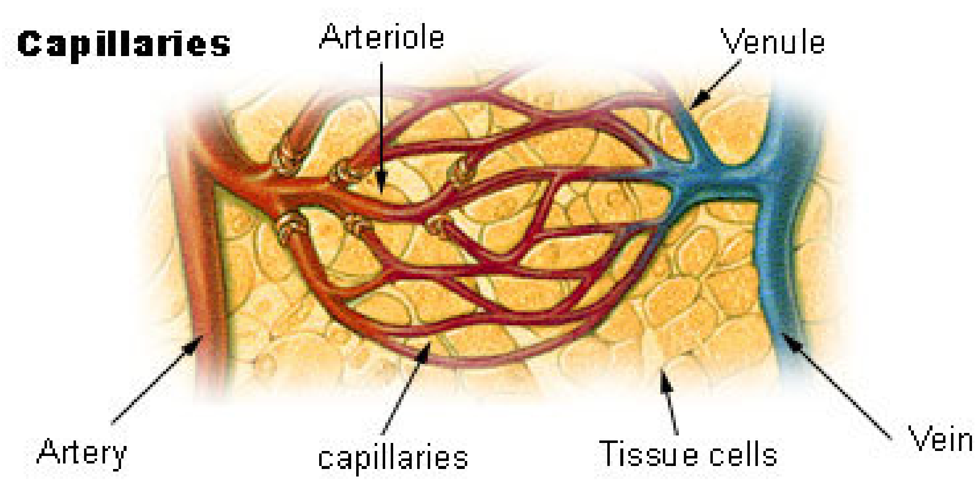
$$When \ \ P_mS << Q \ \ (diffusion \ \ limited), \ CL = P_mS$$

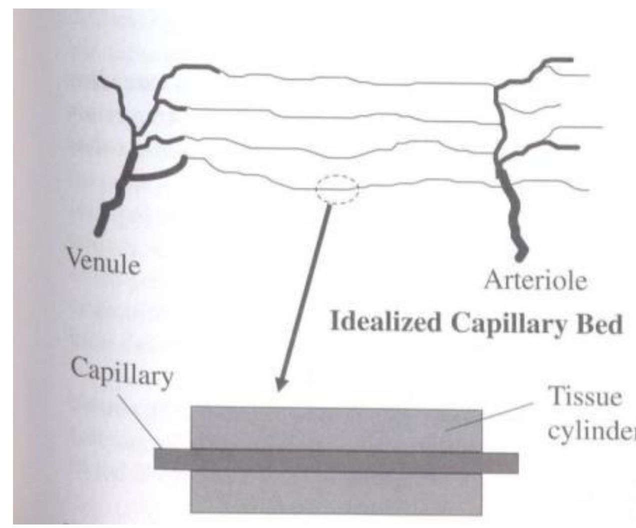
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{for all } x$$

Example 5.16 in textbook.

SOLUTE TRANSPORT IN VASCULAR BEDS

Two main routes for circulation are the pulmonary (to and from the lungs) and the systemic (to and from the body). Pulmonary arteries carry blood from the heart to the lungs. In the lungs gas exchange occurs. Pulmonary veins carry blood from lungs to heart. The aorta is the main artery of systemic circuit. The vena cavae are the main veins of the systemic circuit. Coronary deliver oxygenated blood, food, etc. to the heart. The Krogh tissue cylinder model provides a description/capillary. This is not an appropriate model for the areas of high vascularization.





Two parameters are used to describe the degree of vascularization:

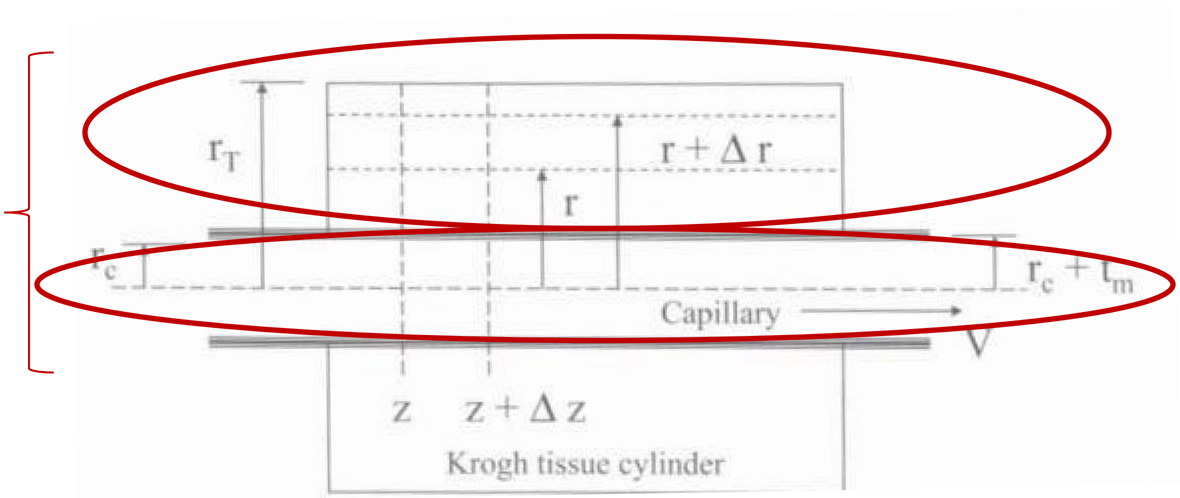
Surface area capillary density (s) is the capillary surface area / unit volume of tissue

$$s = \frac{2r_c}{r_T^2} \qquad (44)$$

(v) is the volume of capillaries / unit volume of tissue

$$v = \left(\frac{r_c}{r_T}\right)^2 \qquad (45)$$

Separate Well-Mixed Regions



A steady state balance around the capillary wall assuming that diffusion is the only driving force yields :

$$QC_0 \big|_{in} = QC_0 \big|_{out} - P_m S(C - \bar{C}) \qquad (46)$$

$$q_b = \frac{Q}{V_T} = \left(\frac{r_c}{r_T}\right)^2 \left(\frac{V}{L}\right) = \frac{v}{\tau} \qquad (47)$$

q_b *is defined as the tissue blood perfusion rate*

V_T *is the volume of the tissue region considered*

V *is the velocity within the capillary*

L *is the capillary length*

τ *is the blood residence time*

$$P_m S(C - \bar{C}) = R_0 V_T \qquad (\text{For a zero order reaction})$$

Solving eq.(46)

$$C = C_0 - \frac{V_T R_0}{Q} = C_0 - \frac{R_0}{q_b} \qquad (48)$$

C is the solute concentration in the blood in the tissue of interest

$$\bar{C} = C_0 - R_0 \left[\frac{1}{q_b} + \frac{V_T}{P_m S}\right] \qquad (49)$$

Example 5.15 is a good illustration of Solute Transport in Vascular Beds.

ASSIGNMENT

- Get your questions ready for the 4/14/08 exam by the 4/09/08 lecture. The office hours will be extended on Monday on 4/14/08 (1:00-5:00 PM).
- Study Chapters 4 and 5 (except for Vascular Beds section 5.10.3) for 4/14/08 Midterm. The format will be the same as the First Midterm (1/3 definitions, 2/3 Problem Solving).
- Do not forget the 4/9/08 lecture.