11.8.15

Homework 7

$$I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}$$

$x^2 + a^2 = 0 \quad x = \pm ia$ poles of order 2

Only $x = +ia$ will be considered which is in the upper $\frac{1}{2}$ plane.

$$a_{-1} = \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{(z-ia)^2 \frac{1}{(z-ia)^2(z+ia)^2}\right\}\right]_{z=ia}$$

$$a_{-1} = \left\{\frac{d}{dz}\frac{1}{(z+ia)^2}\right\}_{z=ia} = \left.\frac{-2}{(z+ia)^3}\right|_{z=ia}$$

$$a_{-1} = \frac{-2}{(2ia)^3} = \frac{-2}{-8ia^3} = \frac{1}{4ia^3}$$

$$I = 2\pi i a_{-1} = 2\pi i \frac{1}{4ia^3} = \frac{\pi}{2a^3}$$

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$$

11.8.15 — Another Method

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} \qquad z^2+a^2=0 \qquad z=\pm ia$$

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = 2\pi i \operatorname{Res}(z=ia) = 2\pi i \left.\frac{1}{2z}\right|_{z=ia}$$

$$= 2\pi i \frac{1}{2\,ia} = \frac{\pi}{a}$$

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \frac{\pi}{a}$$

$$\frac{\partial}{\partial a}\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \frac{\partial}{\partial a}\frac{\pi}{a}$$

$$-2a\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = -\frac{\pi}{a^2}$$

$$I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}$$

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$$

11.8.16

$$I = \int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx$$

$$1 + z^4 = 0 \qquad z^4 = -1 = e^{i(\pi + 2k\pi)}$$

$$z = e^{i\pi(2k+1)/4} \qquad k = 0, 1, 2, 3$$

$$z = \underbrace{e^{i\pi/4}, e^{i3\pi/4}}_{\text{upper } \frac{1}{2} \text{ plane}}, e^{i5\pi/4}, e^{i7\pi/4}$$

$$\text{Res}\left(e^{i\pi/4}\right) = \left.\frac{z^2}{4z^3}\right|_{e^{i\pi/4}} = \left.\frac{1}{4z}\right| = \frac{1}{4}e^{-i\pi/4} = \frac{1}{4}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$\text{Res}\left(e^{i3\pi/4}\right) = \left.\frac{z^2}{4z^3}\right|_{e^{i3\pi/4}} = \left.\frac{1}{4z}\right| = \frac{1}{4}e^{-i3\pi/4} = \frac{1}{4}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$I = 2\pi i \sum \text{Res} = 2\pi i \left\{ \frac{1}{4}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) + \frac{1}{4}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) \right.$$

$$= 2\pi i\,(2)\left(\frac{1}{4}\right)\left(-i\frac{\sqrt{2}}{2}\right) = \frac{\pi\sqrt{2}}{2}$$

(11.8.21)

$$I = \int_{-\infty}^{\infty} \frac{x^2\,dx}{x^4 - 2x^2\cos 2\theta + 1}$$

$$z^4 - 2z^2\cos 2\theta + 1 = 0$$

$$z^2 = +\cos 2\theta \pm \sqrt{\cos^2 2\theta - 1} = +\cos 2\theta \pm i\sin 2\theta = e^{\pm 2i\theta}$$

$$z^2 = +e^{2i\theta} \Rightarrow z_1 = e^{i\theta},\ z_2 = -e^{i\theta}$$

$$z^2 = e^{-2i\theta} \Rightarrow z_3 = e^{-i\theta},\ z_4 = -e^{-i\theta}$$

Two roots in the upper $\frac{1}{2}$ plane are $z_1 = e^{i\theta}$, $z_4 = -e^{-i\theta}$

$$\mathrm{Res}(z_1) = \left.\frac{z^2}{4z^3 - 4z\cos 2\theta}\right|_{z_1} = \left.\frac{z}{4\left[z^2 - \cos 2\theta\right]}\right|_{z_1}$$

z_1 is pole of order 1.

$$\mathrm{Res}(z_1) = \frac{e^{i\theta}}{4\left[e^{2i\theta} - \frac{e^{2i\theta} + e^{-2i\theta}}{2}\right]} = \frac{e^{i\theta}}{2\underbrace{\left[e^{2i\theta} - e^{-2i\theta}\right]}_{2i\sin 2\theta}}$$

$$\boxed{\mathrm{Res}(z_1) = \frac{e^{i\theta}}{4i\sin 2\theta}}$$

z_4 pole of order 1:

$$\mathrm{Res}(z_4) = \frac{-e^{-i\theta}}{4\left\{e^{-2i\theta} - \frac{e^{2i\theta} + e^{-i2\theta}}{2}\right\}} = \frac{-e^{-i\theta}}{2\underbrace{\left[e^{-2i\theta} - e^{2i\theta}\right]}_{-2i\sin 2\theta}}$$

z_4 is pole of order 1.

$$\boxed{\mathrm{Res}(z_4) = \frac{e^{-i\theta}}{4i\sin 2\theta}}$$

$$I = 2\pi i \sum \mathrm{Res} = 2\pi i\left\{\frac{e^{i\theta} + e^{-i\theta}}{4i\sin 2\theta}\right\} = \frac{\pi\cos\theta}{\sin 2\theta} = \frac{\pi\cos\theta}{2\sin\theta\cos\theta} = \underline{\frac{\pi}{2\sin\theta}}$$

11.8.10

$$I = \int_0^\infty \frac{x \sin x}{x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1}\,dx$$

$$= \frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2+1}\,dx$$

$x^2+1=0 \quad x = \pm i$ pole, order 1

$$\mathrm{Res}(z=i) = \left.\frac{z e^{iz}}{\frac{d}{dz}(z^2+1)}\right|_{z=i} = \left.\frac{z e^{iz}}{2z}\right|_{z=i} = \left.\frac{e^{iz}}{2}\right|_{z=i} = \frac{e^{-1}}{2}$$

$$I = \frac{1}{2}\,\mathrm{Im}\left\{2\pi i\left(\frac{e^{-1}}{2}\right)\right\} = \frac{\pi}{2e}$$