15.3. 1. We assume $\frac{1}{4\pi\epsilon_0} = 1$

$$V(F) = \frac{q}{|r-a|} - \frac{2q}{|r|} + \frac{q}{|r+a|}$$

$$|r| = r \quad |a| = \alpha, \quad |a| = \alpha$$

$$\frac{1}{|r|^2} = \sum_{n=0}^{\infty} \frac{r^n}{n+1} P_n(\cos \theta)$$

A) We assume $|r| > \alpha$

$$r_1 < \alpha$$
$$r_2 > r$$

$$\frac{1}{|r-a|} = \sum_{n=0}^{\infty} \frac{a^n}{r^{n+1}} P_n(\cos \theta), \quad \frac{1}{|r|} = \sum_{n=0}^{\infty} \frac{a^n}{r^{n+1}} P_n(\cos \theta)$$

$$V(F) = q \sum_{n=0}^{\infty} \frac{a^n}{r^{n+1}} P_n(\cos \theta) - \frac{2q}{r} + q \sum_{n=0}^{\infty} \frac{a^n}{r^{n+1}} P_n(-\cos \theta)$$

$$= q \sum_{n=0}^{\infty} \frac{a^n}{r^{n+1}} P_n(\cos \theta) - \frac{2q}{r} + q \sum_{n=0}^{\infty} \frac{a^n}{r^{n+1}} (-1)^nP_n(\cos \theta)$$

$$= \frac{q}{r} \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^n \left[1 + (-1)^n\right] P_n(\cos \theta) - \frac{2q}{r}$$

$$\frac{2q}{r} + \frac{q}{r} \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \left[2\right] P_n(\cos \theta) = \frac{2q}{r} \sum_{n: \text{ even } \text{ not zero}} \left(\frac{a}{r}\right)^n P_n(\cos \theta)$$

$$V(r) = \frac{2q}{r} \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \left[2\right] P_n(\cos \theta)$$
B) If we assume \( r < a \), then

\[
\begin{align*}
  r_< &= r, \\
  r_> &= a
\end{align*}
\]

\[
\frac{1}{|r^2 - a^2|} = \sum_{n=0}^{\infty} \frac{r^n}{a^{n+1}} P_n(C_n(\theta))
\]

\[
\frac{1}{|r^2 + a^2|} \cdot \frac{1}{|r^2 - (-a)^2|} = \sum_{n=0}^{\infty} \frac{r^n}{a^{n+1}} P_n(C_n(\pi - \theta))
\]

\[
V(r) = q \sum_{n=0}^{\infty} \frac{r^n}{a^{n+1}} P_n(C_n(\theta)) - \frac{2q}{r} + q \sum_{n=0}^{\infty} \frac{r^n}{a^{n+1}} (-1)^n P_n(C_n(\theta))
\]

For \( n \) odd term 1 + 3 = 0

For \( n \) even \( \nu \cdot 1 + 3 = 2 \cdot 1 = 2 \cdot 3 

\[
V(r) = 2q \sum_{n=0}^{\infty} \frac{r^n}{a^{n+1}} P_n(C_n(\theta)) - \frac{2q}{r}
\]

\[
V(r) = \frac{2q}{a} \sum_{n=0}^{\infty} \frac{r^n}{a^n} P_n(C_n(\theta)) - \frac{2q}{r}
\]

\[
V(r) = \frac{2q}{a} \sum_{l=0}^{\infty} \frac{(r/a)^{2l}}{2^l} P_n(C_n(\theta)) - \frac{2q}{r} \quad r > a
\]
15.3.2

We show this only for $r > 2a$

\[ V(r) = \frac{-q}{\sqrt{r^2 + 2a^2}} + \frac{2q}{\sqrt{r^2 + a^2}} + \frac{2q}{\sqrt{r^2 - a^2}} + \frac{q}{\sqrt{r^2 - 2a^2}} \]

\[ \frac{1}{\sqrt{r_1 - r_2}} = \frac{1}{r} \sum_{n=0}^{\infty} \left( \frac{r}{r_1} \right)^n P_n (c_{10}) \quad r > = r \quad r > 2a \]

\[ V(r) = (-q) \sum_{n=0}^{\infty} \frac{\left(2a\right)^n}{n!} P_n (c_{10}) + 2q \sum_{n} \frac{a^n}{r^{n+1}} (-1)^n P_n (c_{10}) \]

\[ + q \sum_{n} \frac{\left(2a\right)^n}{n!} P_n (c_{10}) - 2q \sum_{n} \frac{a^n}{r^{n+1}} P_n (c_{10}) \]

If $n$ is even \(1 + 3 = 0\) \(\equiv 0\) \(\mod 2\) \(\equiv 0\) \(\mod 2\), $V(r) = 0$

If $n$ is odd \(1 + 3 = 2(1)\) \(\equiv 2\) \(\mod 2\) \(\equiv 2\) \(\mod 2\), $V(r) \neq 0$

\[ V(r) = 2q \sum_{l=0}^{\infty} \frac{(2a)^{2l+1}}{r^{2l+2}} P_{2l+1} (c_{10}) - 2q \sum_{l=0}^{\infty} \frac{a^{2l+1}}{r^{2l+2}} P_{2l+1} (c_{10}) \]

\[ V(r) = 2q \sum_{l=0}^{\infty} \frac{(2a)^{2l+1}}{r^{2l+2}} P_{2l+1} (c_{10}) \left[ \frac{2l+1}{2} \right] \]

\[ V(r) = (4q/r) \sum_{l=1}^{\infty} \frac{(a/r)^{2l+1}}{2l+1} P_{2l+1} (c_{10}) \left[ \frac{2l}{2l+1} \right] \]

For $r > 2a$

The lowest order term

\[ V(r) = \frac{12q}{r} \left( \frac{a}{r} \right)^3 P \rightarrow 0 \]

There is no term with powers less than \(1/r^4\).
15.3.3

\[ V(r) = \frac{q}{|r-a|} \]

\[ \frac{1}{|r_1-r_2|} = \sum_{l=0}^{\infty} \frac{r_l}{r_l+1} P(\cos \theta) \]

\[ r < a \quad r = r_1 \quad \xi = \frac{1}{r} \quad \Gamma = \alpha \]

\[ V(r) = \frac{q}{|r-a|} = q \sum_{l=0}^{\infty} \frac{r_l}{a_{l+1}} \frac{1}{l} P(\cos \theta) = \frac{q}{a} \sum_{l=0}^{\infty} \left( \frac{r}{\alpha} \right)^l P(\cos \theta) \]

15.3.4

\[ V(r) = -\frac{q}{|r+a|} + \frac{q}{|r-a|} = \frac{-q}{|r-a|} + \frac{q}{|r+a|} \]

For \( r < a \), \( r = r_1 \) \( (r > a) \)

\[ V(r) = -q \sum_{l=0}^{\infty} \frac{a^l}{r^{l+1}} (-1)^l P(\cos \theta) + q \sum_{l=0}^{\infty} \frac{a^l}{r^{l+1}} P(\cos \theta) \]

If \( l \) is even, then \( V(r) = 0 \)

If \( l \) is odd \( (l = 2n+1) \) \( V(r) \neq 0 \)

\[ V(r) = 2q \sum_{n=0}^{\infty} \frac{2^{n+1}}{r^{2n+2} 2n+1} P(\cos \theta) = \frac{2q}{r} \sum_{n=0}^{\infty} \left( \frac{a}{r} \right)^{2n+1} P(\cos \theta) \]

If \( r \gg a \), we only keep the lowest order term.

\[ V(r) = \frac{2q}{r} \left( \frac{a}{r} \right) P(\cos \theta) = \frac{2qa \cos \theta}{r^2} = \frac{P \cos \theta}{r^2} \]

where \( P = 2q \mu \); dipole moment
\[ \vec{E} = - \nabla \phi = - \nabla \left( \frac{p \cos \phi}{r^2} \right) \]

\[ E_r = - \frac{\partial}{\partial r} \left( \frac{p \cos \phi}{r^2} \right) = \frac{2p \cos \phi}{r^3} \]

\[ E_\theta = - \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{p \cos \phi}{r^2} \right) = \frac{p \sin \phi}{r^3} \]

\[ E_\phi = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left( \frac{p \cos \phi}{r^2} \right) = 0 \]