**Physics 230**

**Homework Set 3** Solution

**Due Date: Tuesday September 8, 2015**

Textbook problems listed below:

11.2.3, 11.2.9 (a, b, and c) 11.2.11, 11.3.5, 11.4.2, 11.4.7

Reading Assignment 3 (Ch. 11, Complex Variable Theory)

List of pages: 469 - 492

# HW III

11.2.3

$u(x,y) = x^3 - 3xy^2$

a) $\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}$

$v(x,y) = \int (3x^2 - 3y^2)\, dy + \phi(x)$

$v(x,y) = 3x^2 y - y^3 + \phi(x)$

$\frac{\partial v}{\partial x} = 6xy + \phi'(x) = -\frac{\partial u}{\partial y} = +6xy$

$\phi'(x) = 0 \qquad \phi(x) = \text{Const}$

$v(x,y) = 3x^2 y - y^3 + \text{Const}$

$f(z) = x^3 - 3xy^2 + i(3x^2 y - y^3) + \text{Const}$

$f(z) = z^3 + \text{Const}$

b) $v(x,y) = e^{-y} \sin x$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -e^{-y} \sin x$

$u(x,y) = \int -e^{-y} \sin x \, dx + \phi(y)$

$u(x,y) = +e^{-y} \cos x + \phi(y)$

$\frac{\partial u}{\partial y} - e^{-y} \cos x + \phi'(y) = -\frac{\partial v}{\partial x} = -e^{-y} \cos x$

$\phi'(y) = 0 \quad \phi = \text{Const} \quad u(x,y) = e^{-y} \cos x + C$

$\phi(z) = e^{-y} \cos x + i e^{-y} \sin x +$
$= e^{-y}(\cos x + i \sin x) +$
$= e^{-y+ix} + C_1 =$
$\phi(z) = e^{iz} + C_1$

11.2.9

a) $f(z) = \dfrac{\sin z}{z}$ function is analytic everywhere including $z = 0$

$$\lim_{z\to 0} f'(z) = \lim_{z\to 0} \frac{z\cos z - \sin z}{z^2} = \lim_{z\to 0} \frac{z\left(1 - z^2/2! + \frac{z^4}{4!} + \cdots\right) - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)}{z^2}$$

$$= \lim_{z\to 0} \frac{\left(-\frac{1}{2!} + \frac{1}{3!}\right)z^3}{z^2} = 0$$

b) $f(z) = \dfrac{1}{z^2+1}$ analytic anywhere except $z^2+1=0$ or $z = \pm i$

$$f'(z) = \frac{-2z}{(z^2+1)^2}$$

c) $f(z) = \dfrac{1}{z(z+1)}$ analytic everywhere except $z=0$ & $z=-1$.

$$f'(z) = \frac{-2z-1}{z^2(z+1)^2}$$

11.2.11

a) $$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = (\vec{\nabla} u)_x - i(\vec{\nabla} u)_y$$

$$\frac{df}{dz} = V_x - iV_y$$

b) $$\vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = \frac{\partial}{\partial x}(\vec{\nabla} u)_x + \frac{\partial}{\partial y}(\vec{\nabla} u)_y$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

c) $$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & 0 \end{vmatrix} = \hat{e}_3\left(\frac{\partial}{\partial x}V_y - \frac{\partial}{\partial y}V_x\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 0$$

11.3.5

$\oint_C (x^2 - i y^2)\, dz$

unit circle $z = re^{i\theta}$ where $r=1$

$z=e^{i\theta}$, $dz = ie^{i\theta}\, d\theta$

$x = \frac{1}{2}(e^{i\theta}+e^{-i\theta})$, $y = \frac{1}{2i}(e^{i\theta}-e^{-i\theta})$

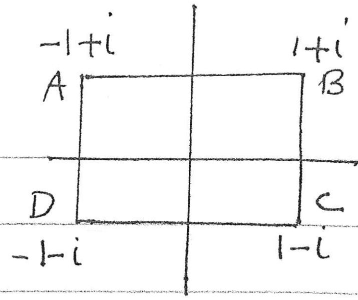
$x^2 = \frac{1}{4}(e^{2i\theta}+e^{-2i\theta}+2)$, $y^2 = -\frac{1}{4}(e^{2i\theta}+e^{-2i\theta}-2)$

$$\oint_C dz(x^2-iy^2) = \int_{2\pi}^{0} \frac{1}{4}\{(e^{2i\theta}+e^{-2i\theta}+2) + i(e^{2i\theta}+e^{-2i\theta}-2)\}\, ie^{i\theta}\, d\theta$$

$$= -\frac{i}{4}\int_0^{2\pi}\{(1+i)[e^{2i\theta}+e^{-2i\theta}]+2(1-i)\}\, e^{i\theta}\, d\theta$$

$$= -\frac{i}{4}\int_0^{2\pi}\{(1+i)(e^{3i\theta}+e^{-i\theta})+2(1-i)e^{i\theta}\}\, d\theta = 0$$

11.3.5 Cont'd




$$I = \oint (x^2 - iy^2)\, dz$$

AB: $dz = dx$, $y = 1$, $dy = 0$

$$\boxed{\int_{-1}^{+1} x^2 dx - i(1)^2 \int_{-1}^{1} dx = \frac{2}{3} - 2i}$$

BC: $dz = i\,dy$, $x = 1$ $dx = 0$

$$\boxed{\int_{1}^{-1} (1)^2 (i\,dy) - i \int_{1}^{-1} y^2 (i\,dy) = -2i - \frac{2}{3}}$$

CD: $dz = dx$, $y = -1$, $dy = 0$

$$\boxed{\int_{1}^{-1} x^2 dx - i(-1)^2 \int_{1}^{-1} dx = -\frac{2}{3} + 2i}$$

DA: $dz = i\,dy$. $x = -1$ $dx = 0$

$$\boxed{(-1)^2 \int_{-1}^{+1} i\,dy - i \int_{-1}^{1} y^2 (i\,dy) = 2i + \frac{2}{3}}$$

$$I = \left(\frac{2}{3} - 2i\right) + \left(\frac{-2}{3} - 2i\right) + \left(-\frac{2}{3} + 2i\right) + \left(2i + \frac{2}{3}\right)$$

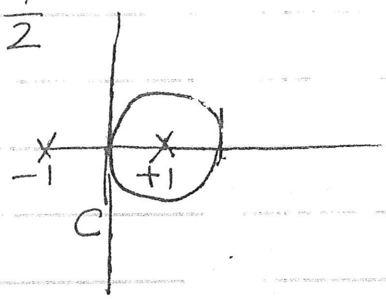
$$I = 0$$

The integrals over paths AB & CD add up to zero since the function is an even function and paths are in opposite directions. Same is true for paths BC & DA. Same argument is true for integration on the circular path with origin as center.

11.4.2

$$I = \oint \frac{dz}{z^2-1} = \oint dz\left[\frac{1}{z-1} - \frac{1}{z+1}\right]\frac{1}{2}$$

$$I = \oint_C \frac{\frac{1}{2}\,dz}{z-1} - \frac{1}{2}\oint \frac{dz}{z+1}$$



$$\oint_C \frac{f(z)\,dz}{z-z_0} = f(z_0)\cdot 2\pi i \qquad z_0 \text{ inside } C$$

$$\oint \frac{f(z)}{z-z_0}\,dz = 0 \qquad z_0 \text{ outside}$$

$I = I_1 + I_2$

In $I_1$ $f(z) = \frac{1}{2}$ $z_0 = 1$ inside $C$

therefore $I_1 = 2\pi i\, f(z_0) = 2\pi i \times \frac{1}{2} = \pi i$

In $I_2$ $f(z) = -\frac{1}{2}$ $z_0 = -1$ outside $C$

Therefore $I_1 = 0$

$I = I_1 + I_2 = \pi i$

---

<u>Second Method</u>

$$I = \oint_C \frac{dz}{z^2-1} = \oint_C \frac{\left(\frac{1}{z+1}\right)dz}{z-1} = \oint \frac{f(z)\,dz}{z-z_0} = 2\pi i f(z_0)$$

$f(z) = \frac{1}{z+1}$ $\qquad z_0 = 1$ $\quad f(z_0) = f(1) = \frac{1}{1+1} = \frac{1}{2}$

$I = 2\pi i \times \frac{1}{2} = \pi i$

11.4.7

$$I = \oint \frac{\sin^2 z - z^2}{(z-a)^3}\, dz$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}}$$

$$\frac{d^2}{dz^2}\left[\sin^2 z - z^2\right] = \frac{d}{dz}\left[\underbrace{2\cos z \sin z}_{\sin 2z} - 2z\right]$$

$$= 2\cos 2z - 2$$

$$I = \int \frac{\sin^2 z - z^2}{(z-a)^3}\, dz = \frac{2\pi i}{2!} \frac{d^2}{dz^2}\left(\sin^2 z - z^2\right)\Big|_{z=a} = 2\pi i(\cos 2z - 1)\Big|_{z=a}$$

$$= \frac{2\pi i}{2}\left[2\cos 2a - 2\right] = 2\pi i\,(\cos 2a - 1)$$