Course Syllabi and Supporting Instructional Material
Kyle Hambrook

Contents

- Website and syllabus for MTH 161 - Calculus IA - Fall 2017 at the University of Rochester. Also available here: https://web.math.rochester.edu/courses/current/161/

- A sample of lecture notes for MTH 161. The full set of notes is available on the course website (link above).

- A worksheet from a MTH 161 workshop. Workshops meet once a week for 75 minutes. Each workshop is led by a workshop TA. Students are given a worksheet of problems to solve. The problems aim to get students to think more deeply about the material they are learning in class. They work in small groups, analyzing, discussing and explaining as they work on the problems. Students turn in one set of solutions per group. But instead of being graded on correct answers, they are graded individually on their total engagement and effort in the workshop.

- Midterm 1 from MTH 161

- Website and syllabus for MTH 142 - Calculus II - Fall 2016 at the University of Rochester. Also available here: https://web.math.rochester.edu/people/faculty/khambroo/teaching/mth142F2016/

- A sample of lecture notes for MTH 142. The full set of notes is available on the course website (link above).

- A quiz from MTH 142. All the exercises are known to the students in advance. On the day of the quiz, two exercises are picked randomly.

- Website and syllabus for MTH 210 - Introduction to Financial Mathematics - Fall 2017 at the University of Rochester. Link: https://web.math.rochester.edu/courses/current/210/

- A sample of the lecture notes for MTH 210. The full set of notes is available on the course website (link above). My intention is to eventually publish these notes as a short textbook.

- A sample assignment (with solutions) for MTH 210.

- A supplement for MTH 210 - Introduction to Financial Mathematics - Fall 2016. It presents interest rates, forward rates, and forward rate agreements in a different, more unified way than in the 2016 lecture notes. This material has now been incorporated into the 2017 lecture notes.

- A sample of lecture notes for MATH 152 - Linear Systems at the University of British Columbia. The full set of notes is at: https://web.math.rochester.edu/people/faculty/khambroo/teaching/math152/LectureNotes/index.html
Fall 2017: MTH 161

MTH 161 - CALCULUS IA

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Instructors

Kyle Hambrook [http://web.math.rochester.edu/people/faculty/khambroo/] (khambroo@ur.rochester.edu)

Class Time/Location:  
   MW 10:25 – 11:40, Harkness 115 (CRN 31045)
   MW 2:00 – 3:15, Morey 321 (CRN 31147)

Office Hours: Hylan 905. Mon, Wed 12:00 – 1:00, 5:00 – 6:00, or by appointment.

Vitaly Lorman [http://web.math.rochester.edu/people/faculty/vlorman/] (vlorman@ur.rochester.edu)

Class Time/Location: TuTh 9:40 – 10:55, Lattimore 201

Office Hours: Hylan 919. Tu 11:00 – 12:00, Wed 11:00 – 12:00, Th 12:30 – 1:30, or by appointment.

Saul Lubkin [http://web.math.rochester.edu/people/faculty/lubkin/] (lubkin@math.rochester.edu)

Class Time/Location: MW 9:00 – 10:15, Bausch & Lomb 270

Office Hours: Hylan 705. Tu 11:00 – 2:00, or by appointment.

Yakun Xi [http://web.math.rochester.edu/people/faculty/yxi4/] (yxi4@math.rochester.edu)

Class Time/Location: TuTh 3:25 – 4:40, Hutchinson 140

Office Hours: Hylan 1019. Wed 1:30 – 3:00, Thur 1:30 – 2:30, or by appointment.

Note: Classes and office hours of all instructors are open to all MTH 161 students, regardless of which section they are registered for.
Textbook

*Calculus: Early Transcendentals*, 8th edition by James Stewart

Older editions of the textbook are acceptable, but section and exercise numbers in the schedule below are for the 8th edition.

Lecture Notes

Professor Hambrook’s Lecture Notes

Course Description

Official CDCS Description

Elementary real functions: algebraic, trionometric, exponentials and their inverses and composites; their graphs, derivatives and integrals; limits, l’Hopital’s rules, Mean value theorem, maxima and minima, curve plotting. The fundamental theorem of calculus, with geometric and physical applications. This course uses the Tuesday/Thursday 08:00-09:30am Common Exam time. This course cannot be taken for credit after completing any of MTH 141, 142, 143, or 162. Students who want to repeat a course for a grade need to secure the approval of the Dean by completing an online Repeat Course Request Form.

Overview

MTH 161 is designed to provide a detailed introduction to the fundamental ideas of calculus. It does not assume any prior calculus knowledge, but the student is expected to be proficient working with functions and their graphs as well as manipulating expressions involving variables and solving equations using algebra. There will be a brief review of algebra, trigonometry, and precalculus, but students are expected to have sufficient understanding of Chapter 1 and Appendices A, B, and D in the textbook.

Throughout most of this course, we will work to solve the following problem:

> **Given a changing quantity, how do you calculate the exact rate of change of that quantity at a given point in time?**

Or, equivalently:

> **Given a curve in the Cartesian plane, what is the slope of the curve at any given point?**

In order to solve this, we will first cover the notion of limits, as this is necessary for understanding the definition of the instantaneous rate of change, or derivative. We’ll then move on to the definition of the derivative. Even though we will learn many rules to simplify the task of calculating derivatives, there will be many times that you will be expected to use the limit definition in order to calculate the derivative of a function. We will also cover standard applications of derivatives, including answering the above questions, and all of the derivative rules. During the last few weeks of the semester, we’ll move on to answering the next big question in calculus – the area question:
Given a curve in the Cartesian plane, what is the area under the curve?

In order to solve this question, we will again use limits in approximating the area under the curve. This will lead us to the definition of the integral, which we will then relate back to the derivative using the Fundamental Theorem of Calculus. We’ll cover only a few methods for calculating integrals; other methods will be covered in Calculus IIA (MTH 162).

Objectives

At the end of this semester, you should be able to do the following:

- Calculate limits of functions; explain the relationship between a function and its graph and its limit at a point.
- Define a derivative using limits and explain its geometric significance; evaluate derivatives of various functions.
- Apply the concepts of limits and derivatives to real-world problems and curve sketching.
- Analyze the connection between derivatives and integrals in the context of the Fundamental Theorem of Calculus.
- Evaluate basic integrals using antiderivatives and substitution; recognize the geometric significance of an integral.

Grading

The course grades will be calculated based on the following percentages:

- **WeBWorK:** 10%
- **Workshops:** 10%
- **Midterm 1:** 20%
- **Midterm 2:** 20%
- **Final Exam:** 40%

The final exam will be cumulative and will test material from the entire semester. It will contain two parts, one covering material from the first two midterms, and the other covering material from after the second midterm. Each student’s lower midterm score will be replaced by that student’s score on the first part of the final exam if the latter is higher than the original midterm score. There will be no make-up exams.

Webwork Homework

Weekly online Webwork assignments are due on Fridays at 11:59 PM.

Your two lowest Webwork marks will be dropped when calculating your Webwork grade. For this reason, no late submissions will be allowed.

Access Webwork: Login to Blackboard [https://learn.rochester.edu/] , then go to Course Materials to find the Webwork link.

Tips:
- Start assignments as early as possible.
- Ask questions to instructors and TAs in office hours.
- Use the “E-mail Webwork TA” button in Webwork to ask questions via email.
- Detailed questions will get detailed answers from WebWork TAs.
- Don’t expect a response from the WebWork TAs if you email after 6 PM on Friday.

**WebWorK TAs**

Lizhi Lei (llei5@u.rochester.edu)

Yuxin Wang (ywang211@u.rochester.edu)

Use the “E-mail Webwork TA” button in Webwork to ask them questions via email.

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**Workshops**

You **must** sign-up for a workshop session. Sign-up opens Wednesday, September 6, 2017 at 5:30 PM.

**Sign-Up Instructions** [http://www.sas.rochester.edu/mth/undergraduate/recitations.html]

**Note:** You **must** attend the session you are registered for. You **cannot** bounce around to other workshop sessions. You should sign up for the one that best fits your schedule and stick to that one.

Workshops will meet once a week for 75 minutes. Workshops start the week of September 11.

Each workshop will be led by a workshop TA. In each workshop, you will be given a worksheet of problems to solve. The problems aim to get you to think more deeply about the material you are learning in class. They will generally be more challenging than WebWork problems or examples done in class. You will work in a small group, analyzing, discussing and explaining as you work on the problems. Explaining math to others is one of the **best** ways to learn it!

You will turn in one set of solutions per group. But instead of being graded on correct answers, you will be graded individually on your total engagement and effort in the workshop. The grading each week will be based on the following 0-2 point scale:

- **0:** if the student was absent
- **1:** if the student was not engaged, pulled out their phone, computer, etc. during the workshop, or was late
- **2:** if the student was engaged for the entire workshop, actively working on the problems, and made progress on most of the problems

At the end of the semester, your two lowest workshop scores will be dropped to account for absences.

**Workshop TAs**

Hristijan Bosilkovski (hbosilko@u.rochester.edu)

**Workshop Time/Location:** Wed 3:25 PM Hylan 105

**Office Hours:** Fri 1:00 PM - 2:00 PM Gleason Library
Yiming Chen (ychen193@u.rochester.edu)
  Workshop Time/Location: Fri 10:25 AM Hutch 138
  Office Hours: Wed 7:30 PM – 8:30 PM Carlson Library 2nd Floor

Tanamaya Dahiya (tdahiya@u.rochester.edu)
  Workshop Time/Location: Thur 11:05 AM Hutch 138
  Office Hours: Tues 12:00 PM – 13:00 PM Gleason Library

Ekam Singh Gill (egill2@ur.rochester.edu)
  Workshop Time/Location: Mon 7:40 PM Hutch 138
  Office Hours: Mon 4:50 PM – 5:50 PM Carlson Library 2nd Floor

Gail Jardine (gjardine@ur.rochester.edu)
  Workshop Time/Location: Thur 3:25 PM Harkness 210
  Workshop Time/Location: Thur 4:50 PM Hylan 1106B
  Workshop Time/Location: Fri 2:00 PM Meliora 219
  Office Hours: Tues 4:00 PM – 5:00 PM Carlson Library 1st Floor

Arif Kodza (akodza@u.rochester.edu)
  Workshop Time/Location: Wed 10:25 AM Hylan 1104
  Office Hours: Fri 10:00 AM – 11:00 AM Wilson Commons Starbucks

Ishaan Kumar (ikumar@u.rochester.edu)
  Workshop Time/Location: Wed 7:40 PM Hylan 202
  Office Hours: Mon 10:45 AM to 11:45 AM Gleason Library

Yunlu Li (yli158@u.rochester.edu)
  Workshop Time/Location: Thur 2:00 PM Genesee 308
  Office Hours: Fri 10:00 AM – 11:00 AM Gleason Library

Jacob Lichtinger (j.lichtinger@rochester.edu)
  Workshop Time/Location: Tue 3:25 PM Hylan 1104
  Workshop Time/Location: Thur 3:25 PM Hylan 1104
  Workshop Time/Location: Fri 11:50 AM Hylan 202
  Office Hours: Fri 1:30 PM – 2:30 PM Hylan 908

Yuhan Liu (yliu142@u.rochester.edu)
  Workshop Time/Location: Fri 11:50 AM Hutch 138
  Office Hours: Mon 3:00 PM – 4:00 PM Gleason Library

Alana McGovern (amcgove2@u.rochester.edu)
  Workshop Time/Location: Tues 9:40 AM Hylan 1101
  Office Hours: Thur 2:00 PM – 3:00 PM Gleason Library

Viet-Duy Nguyen (vnguy14@u.rochester.edu)
  Workshop Time/Location: Fri 3:25 PM Hylan 1101
  Office Hours: Fri 9:00 AM – 10:00 AM Carlson Library 2nd Floor

Carlos Rojas (crojas3@u.rochester.edu)
  Workshop Time/Location: Tue 12:30 PM Mel 218
  Workshop Time/Location: Tue 4:50 PM Hylan 105
  Workshop Time/Location: Thur 4:50 PM Hylan 105
  Office Hours: Tues 11:15 AM – 12:15 PM Carlson Library 2nd Floor
Exams

General Information

- There will be two midterms and one final exam.
- You must write the exams with the section/instructor for which you are registered.
- Notes, books, calculators, phones or other electronic devices are not allowed; having any in your view during an exam is an academic honesty violation.

Dates, Times, Locations, and Other Details

Midterm 1: Thursday, Oct. 5, 8:00–9:15 AM
Location(s): Hambrook and Lorman: Hutchison Hall 141 (Hubbell Auditorium)
Location(s): Lubkin: Hutchison Hall 140 (Lander Auditorium)
Location(s): Xi: Dewey 1101
Midterm 1 Formula Sheet
Midterm 1 Solutions
Midterm 1 Average Score: 78.44
Interpreting Your Midterm 1 Score

Midterm 2: Tuesday, Nov. 7, 8:00–9:15 AM
Location(s): Hambrook and Lorman: Hutchison Hall 141 (Hubbell Auditorium)
Location(s): Lubkin: Hutchison Hall 140 (Lander Auditorium)
Location(s): Xi: Dewey 1101

Final Exam: Sunday, December 17, 4:00-7:00 PM
Location(s): TBA

Review for Midterm 1
Fall 2016 Midterm 1 (with solutions)
Spring 2016 Midterm 1 / Solutions
Fall 2015 Midterm 1 / Solutions
Fall 2010 Midterm 1 (with solutions)
Fall 2009 Midterm 1
Spring 2007 Midterm 1 (with solutions)
Fall 2006 Midterm 1
Fall 2004 Midterm 1
Fall 2001 Midterm 1 (with solutions)

Review for Midterm 2
Fall 2016 Midterm 2 (with solutions)
Spring 2016 Midterm 2 / Solutions
Fall 2015 Midterm 2 / Solutions
Fall 2010 Midterm 2 (with solutions)
Fall 2009 Midterm 2
Spring 2007 Midterm 2 (with solutions)
Fall 2006 Midterm 2
Fall 2001 Midterm 2
Fall 2000 Midterm 2

Review for Final Exam
Fall 2016 Final Exam
Fall 2015 Final Exam
Fall 2010 Final Exam (with solutions)
Fall 2009 Final Exam
Spring 2007 Final Exam
Policies, Expectations, and Additional Resources

Academic Integrity

Students are expected to abide by the University of Rochester Academic Honesty Policy, available at http://www.rochester.edu/college/honesty. All academic work should be done with the high level of honesty and integrity that this University demands. Academic misconduct of any kind may result in a grade penalty or the assignment of a failing grade.

Attendance

You are expected to be in class every day and come prepared to learn and work, and you are expected to arrive at exams on time. There will be no make-up exams.

Reading

On the course schedule, note that we have included readings – sections that you should read before coming to class and workshops. Reading the book and preparing for class and workshops ahead of time will help you follow the lecture better, complete your homework, and succeed in the workshops and on exams.

Extra Help

There are a variety of resources available to you to help you succeed in MTH 161. Office hours of all instructors and TAs are open to all MTH 161 students, regardless of which section and workshop they are registered for. There is a Math Department study hall each Monday through Friday, CETL runs a MTH 161-specific study group, and private tutoring is available.

Details and More Information

Disability Resources

The University of Rochester respects and welcomes students of all backgrounds and abilities. In the event you encounter any barrier(s) to full participation in this course due to the impact of a disability, please contact the Office of Disability Resources. The access coordinators in the Office of Disability Resources can meet with you to discuss the barriers you are experiencing and explain the eligibility process for establishing academic accommodations.

Office of Disability Resources (disability@rochester.edu; (585)275-9049; 1-154 Dewey Hall)

To be granted alternate testing accommodations, you (the student) must fill out forms with the Office of Disability Resources at least seven days before each and every exam. These forms are not sent
“automatically.” Professors are not responsible for requesting alternative testing accommodations at the Office of Disability Resources, and they are not obligated to make any accommodations on their own.

Schedule

Here is the tentative course schedule; there may be small changes as the semester progresses.

For each week, we have listed the textbook sections covered in class, the textbook sections associated with the workshop, and the WebWork assignment due on the Friday of that week. The worksheet for each week’s workshop will be posted at the end of the week.

Section and exercise numbers in the schedule below are for the 8th edition of the textbook.

**Week of Aug. 28**

*First day of classes: Wednesday, August 30*

**Topics**

- *Appendix A*: Numbers, Inequalities, and Absolute Values
  - *Supplementary Problems*: 11, 13, 21, 33, 44, 51
- *Appendix B*: Coordinate Geometry and Lines
  - *Supplementary Problems*: 3, 9, 23, 29, 34, 35, 45, 51, 57

**Workshops**

  - No workshops this week

**WeBWorK due Friday, Sept. 8**

  - Webwork Set 0: Getting started with WeBWorK

**Week of Sept. 4**

*Labor Day: Monday, September 4. No class.*

**Topics**

- *Appendix D*: Trigonometry
  - *Supplementary Problems*: 3, 9, 13, 23, 25, 65, 67
- *Section 1.3*: New Functions from Old Functions [Start]
  - *Supplementary Problems*: 3, 29, 32, 39, 41, 43, 50

**Workshops**

  - No workshops this week

**WeBWorK due Friday, Sept. 8**

  - Webwork Set 1: Appendices A and B

**Week of Sept. 11**

**Topics**

- *Section 1.3*: New Functions from Old Functions [Finish]
  - *Supplementary Problems*: 3, 29, 32, 39, 41, 43, 50
Section 1.4: Exponential Functions
Supplementary Problems: 7, 12, 15, 17, 29(abc)

Section 1.5: Inverse Functions and Logarithms
Supplementary Problems: 21, 23, 25, 35, 38, 49

Workshops
Appendices A, B, and D. WS1.pdf

Homework due Friday, Sept. 15
Webwork Set 2: Appendix D, Section 1.3

Week of Sept. 18

Topics
Section 2.1: The Tangent and Velocity Problems
Supplementary Problems: 3, 5
Section 2.2: The Limit of a Function
Supplementary Problems: 1, 3, 5, 9, 11, 15, 25, 31
Section 2.3: Calculating Limits Using the Limit Laws
Supplementary Problems: 1, 10, 11–23 (odd), 35, 37, 57, 63

Workshops
Sections 1.3, 1.4, 1.5. WS2.pdf

Homework due Friday, Sept 22
Webwork Set 3: Sections 1.4, 1.5, 2.1

Week of Sept. 25

Topics
Section 2.5: Continuity
Supplementary Problems: 3, 17, 20, 39, 43, 45, 47, 50
Section 2.6: Limits at Infinity; Horizontal Asymptotes
Supplementary Problems: 3, 5, 13–31 (odd), 63
Section 2.7: Derivatives and Rates of Change
Supplementary Problems: 5, 9, 11, 15, 17, 27, 29, 47

Workshops
Sections 2.1, 2.2, 2.3. WS3.pdf

Homework due Friday, Sept 29
Webwork Set 4: Sections 2.2, 2.3, 2.5

Week of Oct. 2

*Midterm 1: Date/Time/Location: See Exams above.

Covers material up to (and including) Section 2.5.

Topics
Section 2.8: The Derivative as a Function
Supplementary Problems: 2, 5, 9, 13, 17, 25, 27, 29, 35, 37, 43, 47

Section 3.1: Derivatives of Polynomials and Exponential Functions
Supplementary Problems: 5, 6, 7, 15–23 (odd), 31, 49, 53, 65

Section 3.2: The Product and Quotient Rules
Supplementary Problems: 3–25 (odd), 44, 49, 51, 54

Section 3.3: Derivatives of the Trigonometric Functions
Supplementary Problems: 3, 5, 9, 14, 17, 38, 39, 41, 43, 44

Workshops
Sections 2.5–2.8. WS4.pdf

Homework due Friday, Oct. 6
Webwork Set 5: Sections 2.6–2.8

Week of Oct. 9

Fall Break: October 9–10

Topics
Section 3.4: The Chain Rule
Supplementary Problems: 5, 7, 9, 13, 15, 23, 32, 41, 43, 49, 53, 61, 65, 72, 80

Workshops
No workshops this week

Homework due Friday, Oct. 13
Due next week, but start now: Webwork Set 6: Sections 3.1–3.5

Week of Oct. 16

Topics
Section 3.5: Implicit Differentiation
Supplementary Problems: 3, 8, 11, 17, 21, 27, 39, 45, 50, 51, 59, 71

Section 3.6: Derivatives of Logarithmic Functions
Supplementary Problems: 3, 4, 7, 8, 11, 23, 37, 39, 40, 49

Section 3.7: Rates of Change in the Natural and Social Sciences
Supplementary Problems: 1, 8, 12, 13, 20, 26, 30

Workshops
Sections 3.1–3.4. WS5.pdf

Homework due Friday, Oct. 20
Webwork Set 6: Sections 3.1–3.5

Week of Oct. 23

Topics
Section 3.8: Exponential Growth and Decay
Supplementary Problems: 3, 7, 9, 12, 15, 19
Section 3.9: Related Rates  
Supplementary Problems: 3, 10, 13, 15, 17, 19, 24, 33, 41

Workshops  
Sections 3.5–3.7

Homework due Friday, Oct 27  
Webwork Set 7: Sections 3.4–3.6

Week of Oct. 30

Topics  
Section 3.10: Linear Approximation and Differentials  
Supplementary Problems: 2, 5, 11, 22, 23, 25, 27, 35, 39, 41(e)  
Section 4.1: Maximum and Minimum Values  
Supplementary Problems: 3, 7–19 (odd), 31, 34, 39, 50, 55, 59, 63, 70  
Section 4.2: The Mean Value Theorem  
Supplementary Problems: 4, 5, 11, 17, 23, 25

Workshops  
Sections 3.8, 3.9

Homework due Friday, Nov. 3  
Webwork Set 8: Sections 3.7–3.9

Week of Nov. 6

*Midterm 2: Date/Time/Location: See Exams above.  
Covers material up to (and including) Section 3.9

Topics  
Section 4.3: How Derivatives Affect the Shape of a Graph  
Supplementary Problems: 5, 8, 11, 15, 23, 25, 31, 41, 45, 67, 86  
Section 4.4: Indeterminate Forms and L’Hospital’s Rule  
Supplementary Problems: 5–11 (odd), 17–23 (odd), 37, 42, 49, 53, 55, 56, 76  
Section 4.5: Summary of Curve Sketching (Ignore Slant Asymptotes) [Start]  
Supplementary Problems: 3, 13, 18, 33, 44

Workshops  
Sections 3.10, 4.1

Homework due Friday, Nov. 10  
Webwork Set 9: Sections 3.10, 4.1, 4.2

Week of Nov. 13

Topics  
Section 4.5: Summary of Curve Sketching (Ignore Slant Asymptotes) [Finish]  
Supplementary Problems: 3, 13, 18, 33, 44
Section 4.7: Optimization Problems
Supplementary Problems: 2, 5, 12, 18, 23, 27, 39, 42, 53

Workshops
Sections 4.2, 4.3, 4.4

Homework due Friday, Nov 17
Webwork Set 10: Sections 4.3, 4.4

Week of Nov. 20

Thanksgiving Break: November 22 (noon)–26

Topics
Section 4.9: Antiderivatives
Supplementary Problems: 3, 13, 15, 21, 35, 37, 45, 49, 53, 59, 61, 66, 73

Workshops
No workshops this week

Homework due Friday, Nov. 24
Due next week, but start now: Webwork Set 11: Sections 4.5, 4.7

Week of Nov. 27

Topics
Section 5.1: Areas and Distances
Supplementary Problems: 4, 15, 17, 20, 23
Section 5.2: The Definite Integral
Supplementary Problems: 1, 5, 7, 19, 29, 33, 35, 36, 42, 43, 47, 49, 54, 55
Section 5.3: The Fundamental Theorem of Calculus [Start]
Supplementary Problems: 4, 5, 7–17 (odd), 23, 31, 36, 37, 43, 57, 63, 67, 78

Workshops
Sections 4.7, 4.9

Homework due Friday, Dec. 1
Webwork Set 11: Sections 4.5, 4.7
Webwork Set 12: Sections 4.9

Week of Dec. 4

Topics
Section 5.3: The Fundamental Theorem of Calculus [Finish]
Supplementary Problems: 4, 5, 7–17 (odd), 23, 31, 36, 37, 43, 57, 63, 67, 78
Section 5.4: Indefinite Integrals and the Net Change Theorem
Supplementary Problems: 7, 10, 12, 16, 27, 31, 37, 43, 49, 52, 59, 69
Section 5.5: The Substitution Rule
Supplementary Problems: 7, 10, 12, 19, 27, 31, 35, 43, 59, 65, 72, 81
Week of Dec. 11

Last day of classes: Wednesday, December 13

*Final Exam: Date / Time / Location: See Exams above.

Topics
  Catch-up/Review

Workshops
  No workshops this week

Homework due Friday, Dec 15
  Webwork Set 14: Sections 5.3–5.5 (For practice, highly recommended, but not for credit.)
Limit of a Function

If \( f(x) \) approaches a (single, finite) number \( L \) as \( x \) approaches the number \( a \), then we write

\[
\lim_{x \to a} f(x) = L
\]

and we say that \( \lim_{x \to a} f(x) \) exists.

The number \( L \) is called the limit of \( f(x) \) as \( x \) approaches \( a \).

**Quiz**

\[ y = f(x) \]

Find

\[
\begin{align*}
f(4) \\
\lim_{x \to 4} f(x) \\
f(5) \\
\lim_{x \to 5} f(x) \\
f(6) \\
\lim_{x \to 6} f(x)
\end{align*}
\]

**Solution**

\[
\begin{align*}
f(4) & \text{ undefined} \\
\lim_{x \to 4} f(x) & = 2 \\
f(5) & = 3 \\
\lim_{x \to 5} f(x) & = 3 \\
f(6) & = 3 \\
\lim_{x \to 6} f(x) & = 4
\end{align*}
\]
One-Sided Limits

**Limit from Right**

\[ \lim_{{x \to a^+}} f(x) = L \]

\[ a < x \]

\( f(x) \) approaches \( L \) as \( x \) approaches \( a \) from the right

**Limit from Left**

\[ \lim_{{x \to a^-}} f(x) = L \]

\( f(x) \) approaches \( L \) as \( x \) approaches \( a \) from the left

Example

\[ y = f(x) \]

\[ \lim_{{x \to -2^-}} f(x) = 4 \]

\[ f(-2) = 2 \]

\[ \lim_{{x \to -2^+}} f(x) = 2 \]

\[ \lim_{{x \to -2}} f(x) \text{ DNE (does not exist)} \]
Relationship Between One-Sided and Two-Sided Limits

\[ \lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L \]

Example ("Death by Oscillation")

Investigate \( \lim_{x \to 0} \cos\left(\frac{1}{x}\right) \)

<table>
<thead>
<tr>
<th>(x)</th>
<th>(\cos\left(\frac{1}{x}\right))</th>
</tr>
</thead>
<tbody>
<tr>
<td>+0.001</td>
<td>0.5628...</td>
</tr>
<tr>
<td>-0.001</td>
<td>0.95216...</td>
</tr>
<tr>
<td>+0.0001</td>
<td>-0.99936...</td>
</tr>
<tr>
<td>-0.0001</td>
<td>0.93675...</td>
</tr>
<tr>
<td>+0.00001</td>
<td>-0.90727...</td>
</tr>
<tr>
<td>-0.00001</td>
<td>0.36338...</td>
</tr>
</tbody>
</table>

\(\cos(-x) = \cos x\)

If we let \(x\) go to 0 through values of the form \(x = \pm \frac{1}{n\pi}\) (\(n\) integer), then

\[ \cos\left(\frac{1}{x}\right) = \cos\left(n\pi\right) = \begin{cases} 1 & \text{if } n \text{ even} \\ -1 & \text{if } n \text{ odd} \end{cases} \]

\(\cos\left(\frac{1}{x}\right)\) oscillates between \(-1\) and \(1\) infinitely often as \(x \to 0\).

In fact, \(\cos\left(\frac{1}{x}\right)\) takes on every value between \(-1\) and \(1\) infinitely often as \(x \to 0\).
Since the values of $\cos\left(\frac{1}{x}\right)$ do not approach a single number $L$ as $x \to 0$,

$$\lim_{{x \to 0}} \cos\left(\frac{1}{x}\right) \text{ DNE}$$
Problem Set Instructions: Work through the following problems with your group. Pick a scribe who will write out your solutions on a separate sheet of paper and turn it in at the end of the session. You might not finish all of the problems, but be sure to work on all of them together and gain a good idea of how to proceed.

Discussion Questions: Discuss the following question with your group.

- Find an example to show that the derivative of \( f(x)g(x) \) is not equal to \( f'(x)g'(x) \).
- Discuss how the product, quotient, and chain rules let you take the derivative of a complicated function by breaking that function into simpler parts.
- How could you see that \( \frac{d}{dx} \sin^2 x = -\frac{d}{dx} \cos^2 x \) without using the chain rule?

1. Use the quotient rule to prove that \( \frac{d}{dx} \sec x = \sec x \tan x \).

2. Find \( \frac{d}{dx} e^{\text{ex}e^{x}} \).

3. With your group, come up with formulas for the following derivatives:
   (a) (A sort of double chain rule) Suppose \( f, g, \) and \( h \) are differentiable functions. If \( F(x) = f(g(h(x))) \), find a formula for \( F'(x) \).
   (b) (Inverse functions) Suppose \( f \) is a differentiable, one-to-one function, and let \( f^{-1} \) be its inverse. Show that
   \[
   \frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.
   \]
   (Hint: Start by writing \( f(f^{-1}(x)) = x \), then take derivatives of both sides.)

4. An object with weight \( W \) is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle \( \theta \) with the plane, then the magnitude of the force is
   \[
   F = \frac{\mu W}{\mu \sin \theta + \cos \theta},
   \]
   where \( \mu \) is a constant called the coefficient of friction.
   (a) Find the rate of change of \( F \) with respect to \( \theta \).
   (b) When is this rate of change equal to 0?

5. Recall that a function \( f(x) \) is even if \( f(-x) = f(x) \) and odd if \( f(-x) = -f(x) \).
   (a) If \( f(x) \) is an even function, is \( f'(x) \) even, odd, or neither?
   (b) If \( f(x) \) is an odd function, is \( f'(x) \) even, odd, or neither?

6. Let \( r \) be a constant. For what value(s) of \( r \) does the function \( y = e^{rx} \) satisfy the differential equation \( y'' - 4y' + y = 0 \)?
7. Consider the function \( f(x) = 2 \sin x + \sin^2 x \). Below are two solutions to find all points on the graph such that the tangent line is horizontal. Is either of them correct? With your group, (1) pick the correct solution (if any); (2) point out the error(s) in the incorrect solution(s); and (3) write a correct solution if neither solution is correct.

(a) First, find \( f'(x) \):
\[
f'(x) = 2 \cos x + 2 \sin x.
\]
Now, set \( f'(x) = 0 \) and solve for \( x \):
\[
f'(x) = 0
\]
\[
2 \cos x + 2 \sin x = 0
\]
\[
2 \sin x = -2 \cos x
\]
\[
\tan x = -1
\]
\[
x = \frac{3\pi}{4} + \pi k, \text{ where } k \text{ is any integer}
\]

(b) First, find \( f'(x) \):
\[
f'(x) = 2 \cos x + 2 \sin x \cos x.
\]
Now, set \( f'(x) = 0 \) and solve for \( x \):
\[
f'(x) = 0
\]
\[
2 \cos x + 2 \sin x \cos x = 0
\]
\[
2 \sin x \cos x = -2 \cos x
\]
\[
2 \sin x = -2
\]
\[
\sin x = -1
\]
\[
x = \frac{3\pi}{2} + 2\pi k, \text{ where } k \text{ is any integer}
\]

**Challenge:** For a function \( f(x) \), the \( n \textbf{th iterate of} \ f \) is the function \( f^{\text{on}}(x) \) defined as follows:
\[
f^{\text{on}}(x) = (f \circ f \circ \cdots \circ f)(x). \text{n times}
\]
For example, \( f^{\circ 4}(x) = (f \circ f \circ f \circ f)(x) = f(f(f(f(x)))) \).
(a) Use the chain rule to find the derivatives of \( f^{\circ 2}(x) \) and \( f^{\circ 3}(x) \).
(b) Can you come up with a general formula for the derivative of \( f^{\text{on}}(x) \) for any positive integer \( n \)?
Please read the following instructions very carefully:

• You have **75 minutes** to complete this exam.

• Only pens/pencils are allowed. The presence of notes, calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden. You may refer to the **formula sheet** on the last page of this exam.

• Show your work and justify your answers. If you need extra space, use the back of the previous page and **clearly indicate** that you have done so. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given. **Clearly circle or label your final answers.**

• Sign the following academic honesty statement: _I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own._

Signature: 

<table>
<thead>
<tr>
<th>QUESTION</th>
<th>VALUE</th>
<th>SCORE</th>
</tr>
</thead>
<tbody>
<tr>
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<td></td>
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<tr>
<td>2</td>
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<td></td>
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<tr>
<td>3</td>
<td>16</td>
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<tr>
<td><strong>TOTAL</strong></td>
<td><strong>100</strong></td>
<td></td>
</tr>
</tbody>
</table>
1. (20 points)

(a) Find the solution set of the following inequality:

\[ x^2 - 3x - 18 \geq 0. \]

(b) Find the solution set of the following inequality:

\[ |x - 2| > |x|. \]
(c) Solve for $x$:
\[3^x = 5^{x-1}\]

(d) Solve for $x$:
\[\ln \left( x^2 + \frac{1}{2} \right) = 0.\]

(e) Find the value of:
\[\tan \left( \cos^{-1} \left( \frac{-4}{5} \right) \right).\]
2. (12 points) Consider the line $L$ given by $2x + 5y + 17 = 0$.

(a) Find the slope of this line.

(b) Find an equation for the line that is perpendicular to $L$ and passes through the point $(0, 1)$.

(c) Find an equation for the line that is parallel to $L$ and passes through the point $(0, 1)$. 
3. (16 points) Let \( f(x) = \frac{2x}{x+1}, \ g(x) = x^2 - 1, \) and \( h(x) = \sqrt{x}. \)

(a) Find \((f - g + h)(4)\).

(b) Find \(h \circ g(x) = h(g(x))\) and its domain.
(c) Find $g \circ f(x) = g(f(x))$ and its domain.

(d) Find $f^{-1}(x)$ and its domain.
4. (20 points) Compute the following limits.
   • If the limit does not exist, write “DNE.”
   • When appropriate, write $\infty$ or $-\infty$ instead of “DNE.”
   • You may only use methods discussed up to this point in the course.

(a) $\lim_{x \to 4} \frac{x}{3x - 8}$

(b) $\lim_{x \to 2} \frac{|x - 2|}{x^2 - 4}$
(c) \( \lim_{x \to 0^+} \frac{2 - \sqrt{x + 4}}{x} \)

(d) \( \lim_{x \to 5} \frac{\frac{1}{x} - \frac{1}{5}}{x - 5} \)
(e) \( \lim_{x \to 0^-} \left( \frac{1}{x} + \frac{1}{|x|} \right) \)
5. (16 points) Let

\[ f(x) = \begin{cases} 
  x, & \text{if } x < 0 \\
  e^x - 1, & \text{if } 0 \leq x \leq 2 \\
  2 - x, & \text{if } x > 2. 
\end{cases} \]

(a) Find the numbers at which \( f \) is discontinuous.

(b) At each number where \( f \) is discontinuous, determine if \( f \) is continuous is from the right, from the left, or neither.
(c) Sketch the graph of $f$ below.
6. (16 points)

(a) State the Intermediate Value Theorem.

(b) Use the Intermediate Value Theorem to show that the equation

\[ \sin x = 1 - x \]

has a solution \( x \) between 0 and \( \frac{\pi}{2} \).
Formula Sheet

Midterm 1

\[
\sin(x + y) = \sin x \cos y + \cos x \sin y
\]
\[
\sin(x - y) = \sin x \cos y - \cos x \sin y
\]
\[
\cos(x + y) = \cos x \cos y - \sin x \sin y
\]
\[
\cos(x - y) = \cos x \cos y + \sin x \sin y
\]

\[
\sin 2x = 2 \sin x \cos x
\]
\[
\cos 2x = \cos^2 x - \sin^2 x
\]
\[
= 2 \cos^2 x - 1
\]
\[
= 1 - 2 \sin^2 x
\]

\[
\cos^2 x = \frac{1 + \cos 2x}{2}
\]
\[
\sin^2 x = \frac{1 - \cos 2x}{2}
\]
Math 142 - Calculus II

Instructors

Kyle Hambrook
Lectures: MW 15:25-16:40 MOREY 321
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Help Hours: MW 17:00-18:00

Yu Zeng
Lectures: MW 9:00-10:15 GAVET 312
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TAs

Scott Kirila (Head TA)
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Help Hour: Friday, 2pm-3pm, Hylan 717

Carlos Rojas
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Help Hour: Friday, 10:30am-11:30am, Carlson Library, 2nd floor

Akif Hosain
E-mail: ahosain at u dot rochester dot edu
Help Hour: Thursday, 3:30pm-4:30pm, Hylan 206

AJ Vargas
E-mail: avargas7 at u dot rochester dot edu
Help Hour: Monday, 4:50pm-5:50pm, Hylan 206

Saifeddin Abdalrahman (Webwork TA)
E-mail: Use the “E-mail Webwork TA” button in Webwork

Prerequisites

MTH 141

Description

Calculus of algebraic, logarithmic, exponential, and trigonometric functions and their inverses.
The definite integral, the fundamental theorem of calculus, geometric and physical applications including areas, volumes, work, and arc length. Techniques of integration including substitution rule, integration by parts, trigonometric substitution, partial fractions. Improper integrals.

**Textbook**

*Calculus: Early Transcendentals* (8th edition) by James Stewart.

Older editions of the textbook are acceptable, but section and exercise numbers below are for the 8th edition.

Exercises for relevant sections from 8th edition

**Recitations**

You must sign-up for a recitation session. Sign-up opens Friday, September 2, at 15:00. Go here for instructions: [http://www.sas.rochester.edu/mth/undergraduate/recitations.html](http://www.sas.rochester.edu/mth/undergraduate/recitations.html)

Recitations are 75 minutes long and are held weekly. Recitations start the week of September 5.

There will usually be a quiz in the last 15 minutes of each recitation session. More information about the quizzes is below.

During the first part of the recitation session, the TA will answer your questions about the course material, Webwork questions, and possible quiz questions.

**Grading**

- Webwork: 10%
- Quizzes: 10%
- First Midterm Exam: 20%
- Second Midterm Exam: 20%
- Final Exam: 40%
- If it results in a higher course grade for you, your lowest midterm mark will be worth 0% and your final exam mark will be worth 60%.

**Webwork**

Webwork assignments are due on Fridays at 11:59 PM. No late submissions will be allowed. Your two lowest Webwork marks will be dropped when calculating your Webwork grade.

To ask questions about Webwork problems, use the “E-mail Webwork TA” button in Webwork. Do not expect a response if you ask a question less than 6 hours before the deadline.

**Access Webwork** [https://math.webwork.rochester.edu/webwork2/fall16mth142/]

- Username = UR NetID (i.e. the first part of your Rochester e-mail address)
- Password = student ID number (you can change it after you log in)

Cannot log in? Email the following information to your instructor:
• Full name
• Rochester email address
• Student ID number
• Course number (MTH142)
• Instructor’s name (Hambrook or Zeng)
• Time class meets (MW 15:25-16:40 or MW 9:00-10:15)

Quizzes

• There will usually be a quiz in the last 15 minutes of each recitation session.
• The quiz questions will be selected randomly from the lists of exercises in the schedule below.
• You must show your work.
• Your two lowest quiz marks will be dropped when calculating your quizzes grade.
• No accommodation for missed quizzes will be given.
• There are some weeks with a recitation but no quiz. For such weeks, you will be marked for recitation attendance.

First Midterm Exam

• Tuesday, October 11, 8:00-9:30 (AM), Dewey 1101
• No notes, textbooks, phones, calculators, or other electronic devices.
• Bring your student ID.
• No make-up midterm will be given.
• The questions will be very similar to the webwork and quiz questions, so study those to do well.
• The layout will be similar to the following practice midterm.
• Textbook Sections: 4.5, 4.7, 4.9, 5.1, 5.2, 5.3, 5.4
• Topics: Curve Sketching, Optimization, Antiderivatives, Indefinite Integrals, Areas and Distances, Definite Integrals, Fundamental Theorem of Calculus, Net Change Theorem
• Practice Midterm 1
• Practice Midterm 1 - Solutions
• Old Exams. Not all with solutions. Not all content matches current course.
• Midterm 1 - solutions
• Midterm 1 scores are posted on Blackboard. It is a score out of 75.
• Here is rough approximation of the letter grade corresponding to your midterm score. “B” means “B- or B or B+”, etc.
  o A: 54-75
  o B: 41-53
  o C: 25-40
  o D: 18-24
  o F: 0-17

Second Midterm Exam

• Thursday, November 17, 8:00-9:30 (AM), Dewey 1101
• No notes, textbooks, phones, calculators, or other electronic devices.
• Bring your student ID.
• No make-up midterm will be given.
• The questions will be very similar to the webwork and quiz questions, so study those to do well.
• The layout will be similar to the practice midterm below.
• A formula sheet is included with the practice midterm.
• The same formula sheet will be included with the midterm.
• Textbook Sections: 5.5, 6.5, 6.1, 6.2, 6.3, 6.4, 7.1, 7.2
• Topics: Substitution Rule, Average Value of a Function, Areas Between Curves, Volumes, Volumes by Cylindrical Shells, Work, Integration by Parts, Trigonometric Integrals
• Practice Midterm 2
• Practice Midterm 2 - Solutions
• Old Exams. Not all with solutions. Not all content matches current course.
• Midterm 2 - solutions
• Midterm 2 scores are posted on Blackboard. It is a score out of 75.
• Here is rough approximation of the letter grade corresponding to your midterm score. “B” means “B- or B or B+”, etc.
  o A: 50-75
  o B: 35-49
  o C: 23-34
  o D: 17-22
  o F: 0-16

Final Exam

• Saturday, December 17, 8:30 (AM), Location TBA
• No notes, textbooks, phones, calculators, or other electronic devices.
• Bring your student ID.
• No make-up final with be given.

Academic Honesty

All assignments and activities associated with this course must be performed in accordance with the University of Rochester’s Academic Honesty Policy
[http://www.rochester.edu/college/honesty].

You may work together on homework, but copying on homework or exams is NOT allowed, and it will be considered academic dishonesty.

Additional Help

Math Study Hall, Tutoring, and Other Resources
[http://www.math.rochester.edu/undergraduate/tutoring/]

If you are having any difficulties, seek help immediately - do not wait until it is too late to recover from falling behind or failing to understand a concept. Ask an instructor or TA either in class, during office hours, or during an appointment. Email your instructor or TA, or use the WebWork “email WeBWorK TA” button. Work with your classmates (this is always a good idea). It is essential not to fall behind because each lecture is based on previous work.
Disability Resources

If you have an academic need related to a disability, contact the Center for Excellence in Teaching and Learning (CETL) [http://www.rochester.edu/college/cetl/].

The University of Rochester is committed to providing equal educational opportunities for individuals with disabilities. Costs of required auxiliary services are to be borne by the university, not by the student. For more complete information about disabilities, please read the disability handbook which is available on the CETL page.

To be granted alternate testing accommodations, you (the student) must fill out forms with CETL at least seven days before each and every exam. These forms are not sent “automatically.” Professors are not responsible for requesting alternative testing accommodations at CETL, and they are not obligated to make any accommodations on their own.

Lecture Notes (by Professor Hambrook)

- Curve Sketching
- Optimization
- Antiderivatives and Indefinite Integrals
- Definite Integral - Part 1
- Definite Integral - Part 2
- Substitution Rule
- Average Value of Function
- Areas Between Curves
- Volumes
- Volumes by Cylindrical Shells
- Work
- Integration by Parts
- Trigonometric Integrals
- Trigonometric Substitution
- Partial Fractions
- Arc Length (Not Available)
- Improper Integrals - Part 1

Schedule

The schedule may change according to our progress through the topics.

Week of Aug 29
Topics: Curve Sketching
Reading: 4.5 (Ignore Slant Asymptotes. Helpful to review 4.1 and 4.3, which were covered in MTH 141)
No class on Monday. No recitation this week.

Week of Sep 5
Topics: Curve Sketching
Reading: 4.5 (Ignore Slant Asymptotes. Helpful to review 4.1 and 4.3, which were covered in MTH 141)
Quiz Exercises for Sep 12: 4.5: 13, 19, 20, 27, 35, 43.  
The quiz next week will be one of these exercises.  
No class on Monday. No quiz this week. Recitation graded on attendance.

Week of Sep 12  
Topics: Optimization Problems  
Reading: 4.7  
The quiz next week will be two of these exercises.

Week of Sep 19  
Topics: Antiderivatives, Indefinite Integrals  
Reading: 4.9, 5.4 to Example 2  
Quiz Exercises for Sep 26: 4.9: 16, 36, 47, 61, 63. 5.4: 3, 9, 11, 17.  
The quiz next week will be two of these exercises.

Week of Sep 26  
Topics: Areas and Distances, Definite Integrals  
Reading: 5.1, 5.2  
Quiz Exercises for Oct 3: 5.1: 3, 13. 5.2: 1, 6, 21, 40, 47, 68.  
The quiz next week will be two of these exercises.

Week of Oct 3  
Topics: Fundamental Theorem of Calculus, Net Change Theorem  
Reading: 5.3, 5.4  
Practice Exercises: 5.3: 3, 11, 21, 25, 37, 43, 47, 55, 59, 68. 5.4: 32, 45, 51, 61.  
These exercises should be done as practice for the midterm.

Week of Oct 10  
Topics: Substitution Rule, Average Value of a Function  
Reading: 5.5, 6.5  
Quiz Exercises for Oct 17: 5.5: 16, 45, 47, 56, 61, 73. 6.5: 5, 9, 14.  
The quiz next week (except Tuesday) will be three of these exercises.  
No quiz this week. No recitation this week.  
First Midterm Exam. Tuesday, October 11, 8:00-9:30 (AM), DEWEY 1101.

Week of Oct 17  
Topics: Areas Between Curves  
Reading: 6.1  
The quiz next week will be two of these exercises.  
No class on Monday. No quiz on Tuesday. No recitation on Tuesday.

Week of Oct 24  
Topics: Volumes, Volumes by Cylindrical Shells  
Reading: 6.2, 6.3  
The quiz next week will be two of these exercises.

Week of Oct 31  
Topics: Work
Reading: 6.4  
The quiz next week will be two of these exercises.

**Week of Nov 7**  
Topics: Integration by Parts, Trigonometric Integrals  
Reading: 7.1, 7.2  
These exercises should be done as practice for the midterm.

**Week of Nov 14**  
Topics: Trigonometric Substitution, Partial Fractions  
Reading: 7.3, 7.4  
Quiz Exercises for Nov 28: 7.3: 5, 7, 8, 16, 21, 23.  
The quiz for the week of Nov 28 will be two of these exercises.  
No quiz this week. No recitation this week.  
Second Midterm Exam. Thursday, Nov 17, 8:00-9:30 (AM), DEWEY 1101.

**Week of Nov 21**  
No class on Monday. No class on Wednesday. No quiz this week. No recitation this week. No office hours.

**Week of Nov 28**  
Topics: Partial Fractions (continued if necessary), Arc Length  
Reading: 7.4, 8.1  
Quiz Exercises for Dec 5: 7.4: 5, 19, 23, 41, 8.1: 11, 13, 15.  
The quiz next week will be two of these exercises, one from each section.

**Week of Dec 5**  
Topics: Improper Integrals  
Reading: 7.8  
Practice Exercises: 7.8: 2, 11, 13, 35, 36, 37, 49, 53.  
These exercises should be done as practice for the final exam.

**Week of Dec 12**  
Topics: Review  
No class on Wednesday. No recitation this week.  
Final Exam. Saturday, December 17, 16:00, Location TBA.

Page URL: http://web.math.rochester.edu/courses/current/142/  
Modified: Thu Dec 1 00:14:48 2016
Areas Between Curves

Suppose $f$ and $g$ are continuous functions.

$A = \text{area of the region bounded by the curves } y = f(x), \ y = g(x), \ x = a, \ x = b.$

Find $A$.

To start assume $f(x) \geq g(x)$ on $[a, b]$.

\[
\begin{array}{c}
\text{y = f(x)} \\
\text{y = g(x)} \\
\end{array}
\]

Divide $[a, b]$ into $n$ subintervals

\[
[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]
\]

with equal width $\Delta x = \frac{b-a}{n}$

Choose sample points $x^*_i$ in $[x_{i-1}, x_i]$. 

[Diagram of the area between two curves with labeled points and intervals]
The area of the $i^{th}$ rectangle is
\[(f(x_i^*) - g(x_i^*)) \Delta x\]

The total area $A$ of the region is approximately the sum of the areas of the rectangles

\[A \approx \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*)) \Delta x\]

The right-hand side is a Riemann sum.

\[A = \lim_{n \to \infty} \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*)) \Delta x\]

\[A = \int_{a}^{b} (f(x) - g(x)) \, dx = \int_{a}^{b} (\text{upper} - \text{lower}) \, dx\]
Example

Find the area bounded by

\[ y = \sqrt{x}, \quad y = \frac{1}{x^2}, \quad x = 1, \quad x = 4 \]

\[ \text{Area} = \int_{1}^{4} \left( \sqrt{x} - \frac{1}{x^2} \right) \, dx = \left[ \frac{2}{3} x^{3/2} - \left( -\frac{1}{x} \right) \right]_{1}^{4} \]

\[ = \left( \frac{2}{3} (4)^{3/2} + \frac{1}{4} \right) - \left( \frac{2}{3} (1)^{3/2} + \frac{1}{1} \right) \]

\[ = \frac{14}{3} \]
The following table shows the velocity for Car A and Car B, which start side by side.

<table>
<thead>
<tr>
<th>t (s)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
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<td>54</td>
<td>67</td>
<td>76</td>
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<tr>
<td>( v_B )</td>
<td>0</td>
<td>21</td>
<td>34</td>
<td>44</td>
<td>51</td>
<td>56</td>
<td>60</td>
<td>63</td>
<td>65</td>
</tr>
<tr>
<td>( v_A - v_B )</td>
<td>0</td>
<td>13</td>
<td>20</td>
<td>23</td>
<td>25</td>
<td>28</td>
<td>29</td>
<td>29</td>
<td>30</td>
</tr>
</tbody>
</table>

Use a Riemann sum with midpoints to estimate the distance between the cars after 16 seconds.

The area between the velocity curves \( t = 0 \) and \( t = 16 \) is the distance between the cars at \( t = 16 \).
Use $n = 4$ subintervals. So $\Delta t = \frac{b - a}{n} = \frac{16 - 0}{4} = 4$

The midpoints of the intervals are

\[ t_1 = 2, \quad t_2 = 6, \quad t_3 = 10, \quad t_4 = 14 \]

\[
distance \approx \sum_{i=1}^{n} (v_A(t_i) - v_B(t_i)) \Delta t
\]

\[
= \Delta t \cdot \sum_{i=1}^{n} (v_A(t_i) - v_B(t_i))
\]

\[
= 4 \left( 13 + 23 + 28 + 29 \right)
\]

\[
= 372 \text{ ft}
\]

\[
distance = \int_{0}^{16} (v_A(t) - v_B(t)) dt
\]
Suppose $f$ and $g$ are continuous.
Final $A = \text{area bounded by the curves } y = f(x), \ y = g(x), \ x = a, \ x = b$.

Now we don't assume $f(x) \geq g(x)$ for all $x$ in $[a, b]$.

\[ A = A_1 + A_2 + A_3 \]
\[ = \int_a^c (g(x) - f(x)) \, dx + \int_c^d (f(x) - g(x)) \, dx + \int_d^b (g(x) - f(x)) \, dx \]
\[ = \int_a^b (\text{upper} - \text{lower}) \, dx \]
\[ = \int_a^b |f(x) - g(x)| \, dx = \lim_{n \to \infty} \sum_{i=1}^n |f(x_i^*) - g(x_i^*)| \Delta x \]

\[ b/c \ |f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{when } f(x) \geq g(x) \\ g(x) - f(x) & \text{when } g(x) \geq f(x) \end{cases} \]
Example
Find the area of the region bounded by the curves
\( y = \sin x, \quad y = \cos x, \quad x = -\frac{\pi}{2}, \quad x = \frac{\pi}{2} \)

\[ \text{Area} = A_1 + A_2 \]

Points of intersection:
\( \sin x = \cos x \)
\[ \frac{\sin x}{\cos x} = 1 \]
\[ \tan x = 1 \]
\[ x = \frac{\pi}{4} \left( \frac{\pi}{4} + n\pi, \quad n \in \mathbb{Z} \right) \]
Area = \( A_1 + A_2 \)

\[
= \int_{-\pi/2}^{\pi/4} (\text{upper} - \text{lower}) \, dx + \int_{\pi/4}^{\pi/2} (\text{upper} - \text{lower}) \, dx
\]

\[
= \int_{-\pi/2}^{\pi/4} (\cos x - \sin x) \, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) \, dx
\]

\[
= (\sin x + \cos x) \bigg|_{-\pi/2}^{\pi/4} + (-\cos x - \sin x) \bigg|_{\pi/4}^{\pi/2}
\]

\[
= (\sin (\pi/4) + \cos (\pi/4)) - (\sin (-\pi/2) + \cos (-\pi/2))
\]

\[
+ (-\cos (\pi/2) - \sin (\pi/4)) - (-\cos (\pi/4) - \sin (\pi/4))
\]

\[
= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \left(-1 + (0)\right)
\]

\[
+ \left(-2 + 1\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)
\]

\[
= \frac{4}{\sqrt{2}}
\]
Example
Find the area of the region bounded by
\[ y = x^3 - 16x \quad \text{and} \quad y = -8x \]
\[ y = x(x^2 - 16) \]
\[ y = x(x + 4)(x - 4) \]

Intersection Points:
\[ x^3 - 16x = -8x \]
\[ x^3 - 8x = 0 \]
\[ x(x^2 - 8) = 0 \]
\[ x = 0 \quad \text{or} \quad x = \sqrt{8} \quad \text{or} \quad x = -\sqrt{8} \]

\[ \sqrt{8} = \sqrt{2 \cdot 4} = 2\sqrt{2} \]

Area:
\[ \int_{-\sqrt{8}}^{0} (x^3 - 16x - (-8x)) \, dx \]
\[ + \int_{0}^{\sqrt{8}} (-8x - (x^3 - 16x)) \, dx \]

\[ = \int_{-\sqrt{8}}^{0} (x^3 - 8x) \, dx \quad + \quad \int_{0}^{\sqrt{8}} (-x^3 + 8x) \, dx \]
\[
= \left( \frac{1}{4} x^4 - 4x^2 \right) \bigg|_0^{\sqrt{8}} + \left( -\frac{1}{4} x^4 + 4x^2 \right) \bigg|_0^{\sqrt{8}}
\]
\[
= \frac{1}{4} (0)^4 - 4(0)^2 - \left( \frac{1}{4} (\sqrt{8})^4 - 4(\sqrt{8})^2 \right)
\]
\[
+ \left( -\frac{1}{4} (\sqrt{8})^4 + 4(\sqrt{8})^2 \right) - \left( -\frac{1}{4} (0)^4 + 4(0)^2 \right)
\]
\[
= - \left( \frac{1}{4} \cdot 64 - 4 \cdot 8 \right) - \left( \frac{1}{4} \cdot 64 + 4 \cdot 8 \right)
\]
\[
= -2 \left( \frac{1}{4} \cdot 64 - 4 \cdot 8 \right)
\]
\[
= -2 \left( 16 - 32 \right)
\]
\[
= 32
\]
Region bounded by 
\[ y = f(x), \quad y = g(x), \quad x = a, \quad x = b \]

Area = \[ \int_a^b |f(x) - g(x)| \, dx = \int_a^b (\text{upper} - \text{lower}) \, dx \]

Case where 
\[ f(x) \geq g(x) \]
for \( a \leq x \leq b \).

Region bounded by 
\[ x = f(y), \quad x = g(y), \quad y = c, \quad y = d \]

Area = \[ \int_c^d |f(y) - g(y)| \, dy = \int_c^d (\text{right} - \text{left}) \, dx \]

Case where 
\[ f(y) \geq g(y) \]
for \( c \leq y \leq d \).
Example
Find the area enclosed by the line \( y = x - 1 \) and the parabola \( y^2 = 2x + 6 \).

Points of intersection:
\( y = x - 1 \) and \( y^2 = 2x + 6 \)
\[(x - 1)^2 = 2x + 6\]
\[x^2 - 2x + 1 = 2x + 6\]
\[x^2 - 4x - 5 = 0\]
\[(x - 5)(x + 1) = 0\]
\[x = 5 \quad \text{or} \quad x = -1\]
\[y = 5 - 1 = 4 \quad \text{or} \quad y = -1 - 1 = -2\]
\((5, 4)\) \quad \((-1, -2)\)

\[
\begin{align*}
y &= x - 1 \\
x &= y + 1
\end{align*}
\]
\[
\begin{align*}
y^2 &= 2x + 6 \\
x &= \frac{1}{2}y^2 - 3
\end{align*}
\]

We can compute the area either by integrating in \( y \) or in \( x \).
we can compute the area in two ways. 

Computation 1. (Integrate in $y$):

$$\text{area} = \int_{-2}^{4} \left( \text{right} - \text{left} \right) dy$$

$$= \int_{-2}^{4} \left( (y + 1) - \left( \frac{1}{2} y^2 - 3 \right) \right) dy$$

$$= \int_{-2}^{4} \left( -\frac{1}{2} y^2 + y + 4 \right) dy$$

$$= \left. \left( -\frac{1}{6} y^3 + \frac{1}{2} y^2 + 4y \right) \right|_{-2}^{4}$$

$$= \left( -\frac{1}{6} (4)^3 + \frac{1}{2} (4)^2 + 4(4) \right)$$

$$- \left( -\frac{1}{6} (-2)^3 + \frac{1}{2} (-2)^2 + 4(-2) \right)$$

$$= 18$$
Computation 2 (integrate in $x$):

\[
\text{area} = \int_{-3}^{5} (\text{upper} - \text{lower}) \, dx
\]

\[
= \int_{-3}^{1} \left( \sqrt{2x+6^2} - (\sqrt{2x+6}) \right) \, dx
\]

\[
+ \int_{-1}^{5} \left( \sqrt{2x+6^2} - (x-1) \right) \, dx
\]

\[
= 2 \int_{-3}^{1} \sqrt{2x+6^2} \, dx + \int_{-1}^{5} \sqrt{2x+6^2} \, dx - \int_{-1}^{5} (x-1) \, dx
\]

\[
u = 2x+6 \quad \text{du} = 2 \, dx
\]

\[
x = -1 \Rightarrow u = 2(-1)+6 = 4
\]

\[
x = -3 \Rightarrow u = 2(-3)+6 = 0
\]

\[
u = 2x+6 \quad \text{du} = 2 \, dx
\]

\[
x = -1 \Rightarrow u = 2(-1)+6 = 4
\]

\[
x = 5 \Rightarrow u = 2(5)+6 = 16
\]

\[
= \frac{4}{3} \int_{0}^{4} u^{3/2} \, du + \frac{1}{2} \int_{4}^{16} u^{3/2} \, du - \int_{-1}^{5} (x-1) \, dx
\]

\[
= \frac{2}{3} u^{3/2} \bigg|_{0}^{4} + \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \bigg|_{4}^{16} - \frac{1}{2} (x^2 - x) \bigg|_{-1}^{5}
\]

\[
= \frac{2}{3} (4)^{3/2} - \frac{2}{3} (0)^{3/2} + \frac{1}{3} (16)^{3/2} - \frac{1}{3} (4)^{3/2}
\]

\[
- \left( \left( \frac{1}{2} (5)^2 - (5) \right) - \left( \frac{1}{2} (-1)^2 - (-1) \right) \right)
\]

\[
= 18
\]
Week of Sep 26.
Quiz is two of these exercises.

1–22 Find the most general antiderivative of the function. (Check your answer by differentiation.)
16. \( r(\theta) = \sec \theta \tan \theta - 2e^\theta \)
25–48 Find \( f \).
36. \( f'(x) = (x + 1)/\sqrt{x}, \quad f(1) = 5 \)
47. \( f''(x) = x^{-2}, \quad x > 0, \quad f(1) = 0, \quad f(2) = 0 \)
59–64 A particle is moving with the given data. Find the position of the particle.
61. \( a(t) = 2t + 1, \quad s(0) = 3, \quad v(0) = -2 \)
63. \( a(t) = 10 \sin t + 3 \cos t, \quad s(0) = 0, \quad s(2\pi) = 12 \)

Name:
ID Number:
Instructor:

1–4 Verify by differentiation that the formula is correct.
3. \( \int \tan^2 x \, dx = \tan x - x + C \)
5–18 Find the general indefinite integral.
9. \( \int (u + 4)(2u + 1) \, du \)
11. \( \int \frac{1 + \sqrt{x} + x}{x} \, dx \)
17. \( \int 2t(1 + 5t) \, dt \)
Fall 2017: 210

Math 210 - Introduction to Financial Mathematics

Instructors

Kyle Hambrook
  Lectures: MW 15:25-16:40 LATT 201
  E-mail: kyle.hambrook@rochester.edu
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  Office Hours: Mon Wed 12:00 – 1:00, 5:00 – 6:00, or by appointment.

Carl Mueller
  Lectures: MW 12:30-13:45 MOREY 321
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  Office Hours: Mon 2:00 - 3:30, Tues 10:30 - 12:00

Note: Office hours of all instructors are open to all MTH 210 students, regardless of which section they are registered for.

TAs

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Kathy Luo
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  Office Hours: Tuesday 10:00-11:00, 2nd floor of Carlson

Prerequisites

FIN 205 and (MTH 143 or 162) and (one of STT 211, 212, 213, ECO 230, or MTH 201).

Description
Mathematical concepts and techniques underlying finance theory; arbitrage pricing theory and option pricing. The emphasis will be on methods and proofs, rather than on plugging numbers into formulas.

**Textbook and Lecture Notes**

Textbook: *An Introduction to Quantitative Finance* by Stephen Blyth

**LECTURE NOTES**

You will find that the lectures and lectures notes very closely match the presentation of the textbook, except with more examples, more exercises, and more thorough explanations. The lecture notes also contain background material on probability theory that the textbook does not.

**Other Resources**

Here is a table to find values of Phi (standard normal cdf):

Table of Values for Phi (Standard Normal CDF)

Example for Table: Phi(1.96) is the value in row 1.9 and column 0.06.

Here is a widget to find probabilites for the normal distribution, including values of Phi (standard normal cdf).

![Normal Distribution Probability Calculations](http://www.wolframalpha.com)

Example for Widget: The input to Calculate Phi(1.96) is

- mean = 0
- standard deviation = 1
- lower bound = -infty
- upper bound = 1.96

Here are links for another app to find probabilites for the normal distribution, including values of Phi (standard normal cdf). It works for other distributions as well, and it has IOS and Andriod versions for your phone.

- **IOS** [https://itunes.apple.com/us/app/probability-distributions/id889106396]
- **Andriod** [https://play.google.com/store/apps/details?id=com.mbognar.probdist]
- **Author’s Website** [http://homepage.divms.uiowa.edu/~mbognar/]
Here is a spreadsheet to estimate the volatility of a stock from historical price data:

**Estimating Volatility Spreadsheet**

[https://docs.google.com/spreadsheets/d/1xv9mffWIFBqd6VPhvh0jXoJxFcNIOAR9aRVZ76oik/edit?usp=sharing]

**Grading**

Final Grade = 20% Homework + 35% Midterm Exam + 45% Final Exam

If it would result in a higher final grade for you, your final exam will be worth 80% and your midterm will be worth 0%. This will be done automatically.

After the midterm, we will convert your raw midterm score to a curved midterm score between 0 and 100.

At the end of the course, we will convert your raw final exam score to a curved final exam score between 0 and 100. We will do the same with your raw average homework score.

Your final grade will be computed using curved scores in the formula above.

Your letter grade will be determined by the following scale:

- 93-100 A
- 90-93 A-
- 87-90 B+
- 83-87 B
- 80-83 B-
- 77-80 C+
- 73-77 C
- 70-73 C-
- 67-70 D+
- 63-67 D
- 60-63 D-
- 0-60 E

We will always round up at the borders.

**Homework**

- Homework will be due Wednesdays at the start of class.
- Homework assignments and due dates will be posted in the schedule below.
- No late submissions will be allowed.
- Your two lowest homework marks will be dropped when calculating your homework grade.

**Midterm Exam**

- Date, Time, Location: Thursday, Oct. 26, 8:00–9:15 AM, Dewey 1101
- Duration: 75 minutes
- No notes, textbooks, calculators, phones, or other electronic devices.
- No accommodation for missing the midterm will be given.
- Covers up to and including Section 4.2 of the text.
- Last year’s midterm
- Solution to last year’s midterm
- This year’s midterm
- Solution to this year’s midterm

**Final Exam**

- Date, Time, Location: Sunday, Dec. 17, 4:00–7:00 PM, Location TBA
- Duration: 3 hours
- No notes, textbooks, calculators, phones, or other electronic devices.
- No make-up final exam with be given.
- Bring your student ID.

**About The Exams**

You should know and be able to work with the main ideas of the course. For example, you may be asked to: State a definition or prove a result from class; Solve a problem similar to a class example or homework exercise; Solve a problem that is somewhat different from what you’ve seen in class and homework, but that still uses the concepts we have studied.

A sample exam will be posted about a week in advance of the midterm and final.

**Academic Honesty**

All assignments and activities associated with this course must be performed in accordance with the University of Rochester’s Academic Honesty Policy [http://www.rochester.edu/college/honesty].

You may work together on homework, but copying on homework or exams is **NOT** allowed, and it will be considered academic dishonesty.

**Additional Help**

**Math Study Hall, Tutoring, and Other Resources**

[http://www.math.rochester.edu/undergraduate/tutoring/]

Office hours of all instructors and TAs are open to all MTH 210 students, regardless of which section they are registered for. Work with your classmates (but don’t copy assignments). It is essential to not fall behind because each lecture is based on previous work. If you are having any difficulties, seek help immediately.

**Disability Resources**

The University of Rochester respects and welcomes students of all backgrounds and abilities. In the event you encounter any barrier(s) to full participation in this course due to the impact of a disability, please contact the Office of Disability Resources. The access coordinators in the Office of Disability Resources can meet with you to discuss the barriers you are experiencing and explain the eligibility process for establishing academic accommodations.
Office of Disability Resources [https://www.rochester.edu/college/disability/]
(disability@rochester.edu; (585)275-9049; 1-154 Dewey Hall)

To be granted alternate testing accommodations, you (the student) must fill out forms with the Office of Disability Resources at least seven days before each and every exam. These forms are not sent “automatically.” Professors are not responsible for requesting alternative testing accommodations at the Office of Disability Resources, and they are not obligated to make any accommodations on their own.

**Schedule**

This schedule is tentative. It may change according to our progress through the topics. Textbook and lecture notes reading is listed by section number.

**Week of Aug 28**
Topics: Basic probability
Textbook Reading: None
Lecture Notes Reading: 1.1, 1.2
HW1 posted Wednesday night. HW1 Solution posted next Wednesday night.
No class on Monday.

**Week of Sep 4**
Topics: Basic probability, Arbitrage
Textbook Reading: 6.1
Lecture Notes Reading: 1.3, 1.4, 2.1, 2.2, 2.3
HW1 due Wednesday. HW2 posted Wednesday night. HW2 Solution posted next Wednesday night.
No class on Monday.

**Week of Sep 11**
Topics: Monotonicity and replication, Interest rates and compounding, Zero coupon bonds and discounting, Time value of money
Textbook Reading: 6.2, 1.1, 1.2, 1.3
Lecture Notes Reading: 2.4, 3.1, 3.2
HW2 due Wednesday. HW3 posted Wednesday night. HW3 Solution posted next Wednesday night.
No class on Monday.

**Week of Sep 18**
Topics: Annuities, Stocks, Bonds, Foreign exchange, Derivative contracts, Forward contracts, Forward on asset paying no income
Textbook Reading: 1.3, 1.5, 2.1, 2.2, 2.3
Lecture Notes Reading: 3.3, 3.4, 3.5, 3.6, 4.1, 4.2, 4.3, 4.4, 4.5, 4.6
HW3 due Wednesday. HW4 posted Wednesday night. HW4 Solution posted next Wednesday night.

**Week of Sep 25**
Topics: Forward on asset paying known income, Value of forward contract, Forward on stock paying dividends and on currency
Textbook Reading: 2.4, 2.5, 2.6, 2.7
Lecture Notes Reading: 4.7, 4.8, 4.9, 4.10
HW4 due Wednesday. HW5 posted Wednesday night. HW5 Solution posted next Wednesday night.

**Week of Oct 2**
Topics: Forward zero coupon bond prices, Forward interest rates, Libor, Forward rate agreements and forward libor, Value of floating and fixed cashflows
Textbook Reading: 3.1, 3.2, 3.3, 3.4, 3.5
Lecture Notes Reading: 5.1, 5.2, 5.3, 5.4, 5.5, 5.6, 5.7
HW5 due Wednesday.
HW6 posted Wednesday night. HW6 Solution posted night of Wednesday, Oct 18.
HW7 posted early (content covered next week). HW7 Solution posted night of Wednesday night, Oct 18.

Week of Oct 9
Topics: Swap definition, Forward swap rate and swap value
Textbook Reading: 4.1, 4.2
Lecture Notes Reading: 6.1, 6.2, 6.3, 6.4
HW6 and HW7 due next Wednesday.
No class on Monday. No office hours this week.

Week of Oct 16
Topics: Spot-starting swaps, Swaps as difference between bonds, Physical versus cash settlement, Futures definition, Futures versus forward prices
Textbook Reading: 4.3, 4.4, 2.8, 5.1, 5.2
Lecture Notes Reading: 6.5, 6.6, 7.1, 7.2, 7.3, 7.4, 7.5
HW6 and HW7 due Wednesday. HW8 not posted; study for midterm instead.

Week of Oct 23
Topics: Option definitions, Put-call parity, Bounds on call prices
Textbook Reading: 7.1, 7.2, 7.3
Lecture Notes Reading: 8.1, 8.2, 8.3, 8.4, 8.5, 8.6
No HW due. HW8 posted Wednesday night. HW8 Solution posted next Wednesday night.
Midterm Exam: Thursday, October 26, 8:00-9:15 (AM), Dewey 1101

Week of Oct 30
Topics: Bounds on call prices, Call and put spreads, Call butterflies, Digital options
Textbook Reading: 7.3, 7.4, 7.5, 7.6
Lecture Notes Reading: 8.7, 8.8, 8.9, 8.10, 8.11, 8.12, 8.13
HW8 due Wednesday. HW9 posted Wednesday night. HW9 Solution posted next Wednesday night.

Week of Nov 6
Topics: Some advanced probability, Fundamental Theorem of Asset Pricing
Textbook Reading: 9.5
Lecture Notes Reading: 9.1, 9.2, 9.3, 9.4, 10.1, 10.2, 10.3, 10.4
HW9 due Wednesday. HW10 posted Wednesday night. HW10 Solution posted next Wednesday night.

Week of Nov 13
Topics: Binomial Tree
Textbook Reading: 8.1, 8.2, 8.3, 8.4
Lecture Notes Reading: 11.1, 11.2, 11.3, 11.4, 12.1, 12.2, 12.3, 12.4
HW10 due Wednesday. HW11 posted Wednesday night. HW11 Solution posted next next Wednesday night.
Week of Nov 20
Topics: Binomial Tree [Continued]
Textbook Reading: 8.1, 8.2, 8.3, 8.4
Lecture Notes Reading: 11.1, 11.2, 11.3, 11.4, 12.1, 12.2, 12.3, 12.4
No class on Wednesday. No homework or office hours.

Week of Nov 27
Topics: Normal distribution and central limit theorem, Lognormal limit, Risk-neutral limit, Black-Scholes Formula
Textbook Reading: 10.1, 10.2, 10.3
HW11 due Wednesday. HW12 posted Wednesday night. HW12 Solution posted next Wednesday night.

Week of Dec 4
Topics: Properties of Black-Scholes formula, Delta and vega, Volatility
Textbook Reading: 10.4, 10.5
Lecture Notes Reading: 14.4, 14.5, 14.6
HW12 due Wednesday. HW13 posted Wednesday night. HW13 Solution posted this Wednesday night.
HW13 is not to be handed in, but the material may appear on the final exam.

Week of Dec 11
Topics: Review
No class on Wednesday or Friday.
Final Exam: Sunday, Dec. 17, 4:00–7:00 PM, Location TBA

Page URL: http://web.math.rochester.edu/courses/current/210/
Modified: Sun Oct 29 16:35:44 2017
Chapter 6
Interest Rate Swaps

Interest rate swaps are the most widely traded and most liquid of all over-the-counter derivative contracts.

6.1 Swap Definition

A swap is an agreement between two counterparties to exchange a sequence of cash flows.

Parameters: $K =$ fixed rate or delivery price; $T_0 =$ start date; $T_n =$ maturity; $T_1, \ldots, T_n =$ payment dates.

We assume $T_{i+1} = T_i + \alpha$ for every $i$ with fixed $\alpha > 0$.

So knowing $n$, $T_0$, and $T_n$ is enough to determine $\alpha$ and every $T_i$: $T_n = T_0 + n\alpha$ and $T_i = T_0 + i\alpha$.

One counterparty (called “buyer”, “payer”, or “long”) agrees at $t$ to pay $\alpha K$ and receive $\alpha L_{T_i}[T_i, T_{i+\alpha}]$ at $T_i + \alpha$ for each $i = 0, \ldots, n-1$. The other counterparty (called “seller”, “receiver”, or “short”) does the opposite.

Therefore a swap is a sequence of forward rate agreements (FRAs).

The floating leg of the swap consists of the payments $\alpha L_{T_i}[T_i, T_{i+\alpha}]$ at times $T_{i+1} = T_i + \alpha$ for $i = 0, \ldots, n-1$.

The fixed leg of the swap consists of the payments $\alpha K$ at times $T_{i+1} = T_i + \alpha$ for $i = 0, \ldots, n-1$.

The buyer pays the fixed leg to and receives the floating leg from the seller. The seller does the opposite.

$\alpha L_{T_i}[T_i, T_{i+\alpha}]$ is the amount of simple interest that would be accrued on notional 1 at floating (unknown, random) rate $L_{T_i}[T_i, T_{i+\alpha}]$ from $T_i$ to $T_i + \alpha$ and paid at $T_i + \alpha$. 
\( \alpha K \) is the amount of simple interest that would be accrued on notional 1 at fixed (known, non-random) rate \( K \) from \( T_i \) to \( T_i + \alpha \) and paid at \( T_i + \alpha \).

\[ \text{Figure 6.1: A swap} \]

**Remark.** This is a swap with notional 1. To treat a swap with notional \( N \), either consider \( N \) swaps of notional 1 or make the payments \( N\alpha L_{T_i, T_i + \alpha} \) and \( N\alpha K \).

**Remark.** There are more general types of swaps. For example, \( \alpha \) can be different for each period, so that \( T_{i+1} = T_i + \alpha_i \). As another example, the payment times for the floating and fixed legs may differ. This will be explored in the exercises.

### 6.2 Value of Swap

Consider a swap from \( T_0 \) to \( T_n \) with fixed rate \( K \).

For \( t \leq T_0 \),

\[
\begin{align*}
V_{SW, K}^t & = \text{value of the swap at } t \\
V_{FL}^t & = \text{value of the floating leg at } t \\
V_{FXD, K}^t & = \text{value of the fixed leg at } t
\end{align*}
\]

They are related by

**Result 6.2.1.**

\[ V_{SW, K}^t = V_{FL}^t - V_{FXD, K}^t. \]

\( \triangle \)

Note that \( V_{SW, K}^t \) is the value to the buyer. The value to the seller is the \( -V_{SW, K}^t \).

The fixed leg is a sequence of ZCBs with maturities at \( T_1, \ldots, T_n \), so

**Result 6.2.2.**

\[ V_{FXD, K}^t = \sum_{i=1}^{n} \alpha K Z(t, T_i) = K P_{t}[T_0, T_n], \]
where we define

$$P_t[T_0, T_n] = \sum_{i=1}^{n} \alpha Z(t, T_i).$$

We call $P_t[T_0, T_n]$ the pv01 of the swap.

The floating leg is a sequence of libor payments at $T_1, \ldots, T_n$, so by Result 5.6.2 we have

**Result 6.2.3.**

$$V^{FL}(t) = \sum_{i=1}^{n} \left( \text{value at } t \text{ of payment } \alpha L_{T_{i-1}}[T_{i-1}, T_i] \text{ at } T_i = T_{i-1} + \alpha \right)$$

$$= \sum_{i=1}^{n} \alpha L_t[T_{i-1}, T_i] Z(t, T_i) = \sum_{i=1}^{n} [Z(t, T_{i-1}) - Z(t, T_i)]$$

$$= Z(t, T_0) - Z(t, T_n)$$

This says the value of receiving a stream of libor interest payments on an investment of 1 is equal to the value of receiving one dollar at the beginning of the stream and paying it back at the end. We simply take the dollar and repeatedly invest in a sequence of libor deposits.

By combining Results 6.2.1, 6.2.2, and 6.2.3 we can write $V^{SW}_K(t)$ as a linear combination of ZCB prices.

**Result 6.2.4.**

$$V^{SW}_K(t) = Z(t, T_0) - Z(t, T_n) - \alpha K \sum_{i=1}^{n} Z(t, T_i).$$

**Example 6.2.5.** Consider a swap starting now with fixed rate 3%, quarterly payment frequency, and ending in 2 years. Suppose the quarterly compounded zero rates for all payment times are 2%. Find the present value of

(a) the floating leg
(b) the swap

(a) Given $t = T_0$, $T_n - t = 2$, $\alpha = 0.25$, $r_4 = 0.02$.

$$V^{FL}(t) = Z(t, T_0) - Z(t, T_n) = 1 - (1 + r_4/4)^{4(T_n-t)}$$

$$= 1 - (1 + 0.02/4)^{-4(2)} = 0.039114 \ldots$$

(b) Since $V^{SW}_K(t) = V^{FL}(t) - V^{FXD}_K(t)$, we just need $V^{FXD}_K(t)$.
Have \( K = 0.03 \). Have \( T_i - t = T_i - T_0 = i\alpha = i0.25 \). Using \( T_n = T_0 + n\alpha \) (or drawing a picture), we get \( n = 8 \). Then

\[
V_K^{FD}(t) = \alpha K \sum_{i=1}^{n} Z(t, T_i) = \alpha K \sum_{i=1}^{n} (1 + r_4/4)^{-4(T_i-t)} = \alpha K \sum_{i=1}^{n} (1 + r_4/4)^{-i}
\]

The sum

\[
\sum_{i=1}^{n} (1 + r_4/4)^{-i} = \sum_{i=1}^{8} (1 + 0.02/4)^{-i}
\]

only has 8 terms, so we could just compute it directly. Or we can use the formula

\[
\sum_{i=1}^{n} (1 + r_4/4)^{-i} = \frac{1 - (1 + r_4/4)^{-n}}{r_4/4}.
\]

Then

\[
V_K^{FD}(t) = \alpha K \frac{1 - (1 + r_4/4)^{-n}}{r_4/4} = (0.25)(0.03) \frac{1 - (1 + 0.02/4)^{-8}}{0.02/4} = 0.05867219\ldots
\]

Therefore

\[
V_K^{SW}(t) = V_K^{FL}(t) - V_K^{FD}(t)
\]

\[
= 1 - (1 + 0.02/4)^{-4(2)} - (0.25)(0.03) \frac{1 - (1 + 0.02/4)^{-8}}{0.02/4}
\]

\[
= -0.0195573\ldots
\]

\( \triangle \)

### 6.3 Forward Swap Rate

The **forward swap rate** \( y_t[T_0, T_n] \) is the special number such that the value at \( t \) of the swap from \( T_0 \) to \( T_n \) with fixed rate \( K = y_t[T_0, T_n] \) has value \( V_K^{SW}(t) = 0 \).

**Result 6.3.1.** The forward swap rate at \( t \leq T_0 \) for a swap from \( T_0 \) to \( T_n \) is

\[
y_t[T_0, T_n] = \frac{Z(t, T_0) - Z(t, T_n)}{P_t[T_0, T_n]} = \frac{\sum_{i=1}^{n} \alpha L_i[T_{i-1}, T_i] Z(t, T_i)}{\sum_{i=1}^{n} \alpha Z(t, T_i)}.
\]  

(6.3.1)

\( \triangle \)

**Proof.** We use Results 6.2.1, 6.2.2, 6.2.3. Setting \( K = y_t[T_0, T_n] \) gives
\[ V_{K}^{SW}(t) = 0 \iff V_{K}^{FXD}(t) = V_{K}^{FL}(t) \]
\[ \iff KP_{t}[T_0, T_n] = Z(t, T_0) - Z(t, T_n) \]
\[ \iff K \sum_{i=1}^{n} \alpha Z(t, T_i) = \sum_{i=1}^{n} \alpha L_{t}[T_{i-1}, T_i] Z(t, T_i) \]

Solving (6.3.3) for \( K \) gives the first equality in (6.3.1).

Solving (6.3.4) for \( K \) gives the second equality in (6.3.1).  \( \square \)

### 6.4 Value of Swap in Terms of Forward Swap Rate

**Result 6.4.1.** The value at \( t \leq T_0 \) of a swap from \( T_0 \) to \( T_n \) with fixed rate \( K \) is
\[ V_{K}^{SW}(t) = (y_{t}[T_0, T_n] - K)P_{t}[T_0, T_n]. \]

\( \triangle \)

**Remark.** Compare to the value of a forward contract:
\[ V_{K}(t, T) = (F(t, T) - K)Z(t, T). \]

**Proof.** By Results [6.2.1] [6.2.2] and [6.2.3] we have
\[ V_{K}^{SW}(t) = V_{K}^{FL}(t) - V_{K}^{FXD}(t) = Z(t, T_0) - Z(t, T_n) - KP_{t}[T_0, T_n]. \]

By Result [6.3.1] we have
\[ Z(t, T_0) - Z(t, T_n) = y_{t}[T_0, T_n]P_{t}[T_0, T_n]. \]

\( \square \)

### 6.5 Swaps as Difference Between Bonds

A **fixed rate bond** with notional \( N \), coupon \( c \), start date \( T_0 \), maturity \( T_n \), and term length \( \alpha \) is an asset that pays \( N \) at time \( T_n \) and coupon payments \( \alpha N c \) at times \( T_i \) for \( i = 1, \ldots, n \), where \( T_{i+1} = T_i + \alpha \).

If \( N = 1 \), the price at \( t \) of the fixed rate bond is denoted \( B_{c}^{FXD}(t) \).

A **floating rate bond** with notional \( N \), start date \( T_0 \), maturity \( T_n \), and term length \( \alpha \) is an asset that pays \( N \) at time \( T_n \) and coupon payments \( \alpha NL_{T_{i-1}}[T_{i-1}, T_i] \) at times \( T_i \) for \( i = 1, \ldots, n \), where \( T_{i+1} = T_i + \alpha \).
If \( N = 1 \), the price at \( t \) of the floating rate bond is denoted \( B_{FL}^{F}(t) \).

Consider a swap from \( T_0 \) to \( T_n \) with fixed rate \( K \).

**Result 6.5.1.** For \( t \leq T_0 \),

\[
V_K^{SW}(t) = B_{FL}(t) - B_{FXD}^{FXD}(t)
\]

\( \triangle \)

**Proof.** The fixed rate bond with notional 1 and coupon \( K \) equals the fixed leg of the swap plus a payment of 1 at \( T_n \).

The floating rate bond with notional 1 equals the floating leg of the swap plus a payment of 1 at \( T_n \).

Therefore

\[
B_{FL}(t) - B_{FXD}^{FXD}(t) = (V_{FL}(t) + Z(t, T_n)) - (V_{FXD}^{FXD}(t) + Z(t, T_n))
\]

\[
= V_{FL}(t) - V_{FXD}^{FXD}(t)
\]

\[
= V_K^{SW}(t).
\]

\( \square \)

### 6.6 Par or Spot-Starting Swaps

When \( t = T_0 \), we call \( y_{T_0}[T_0, T_n] \) a **par swap rate** or **spot-starting swap rate**.

Given par swap rates \( y_{T_0}[T_0, T_k] \) for every \( T_k \), we can recover the ZCB price \( Z(T_0, T_k) \) for every \( T_k \) by using Result 6.3.1:

\[
y_t[T_0, T_k] = \frac{Z(t, T_0) - Z(t, T_k)}{P_t[T_0, T_k]}
\]

This process is known as bootstrapping and is used frequently in practice. It will be explored in the exercises.

Consider the fixed rate bond with notional 1, coupon \( c \), start date \( T_0 \), maturity \( T_n \), and term length \( \alpha \). It pays \( \alpha N_c \) at times \( T_1, \ldots, T_n \) (where \( T_{i+1} = T_i + \alpha \)) and pays 1 at time \( T_n \). Its value at time \( T_0 \) is \( B_c^{FXD}(T_0) \).

**Result 6.6.1.**

\[
B_{c}^{FXD}(T_0) = 1 \text{ if and only if } c = y_{T_0}[T_0, T_n] = \text{par swap rate}
\]

\( \triangle \)
Because of this result, the par swap rate $y_{T_0}[T_0, T_n]$ is sometimes called the coupon rate.

In other words, this result says we can invest 1 at time $T_0$, receive 1 back at time $T_n$, and receive fixed payments of $\alpha y_{T_0}[T_0, T_n]$ at times $T_1, \ldots, T_n$ in between. And this isn’t true if $y_{T_0}[T_0, T_n]$ is replaced by any other coupon $c$.

**Proof.** We know

By definition of the par swap rate $y_{T_0}[T_0, T_n]$, we have

$$V_c^{SW}(T_0) = 0 \quad \text{if and only if} \quad c = y_{T_0}[T_0, T_n].$$

But

$$V_c^{SW}(T_0) = B^{FL}(T_0) - B_c^{FXD}(T_0)$$

and

$$B^{FL}(T_0) = V^{FL}(T_0) + Z(T_0, T_n) = Z(T_0, T_0) - Z(T_0, T_n) + Z(T_0, T_n) = Z(T_0, T_0) = 1.$$ 

Therefore

$$1 - B_c^{FXD}(T_0) = 0 \quad \text{if and only if} \quad c = y_{T_0}[T_0, T_n].$$

$\square$
Exercise 1 (Each part 1 mark). The current US Dollar (USD) to Japanese Yen (JPY) exchange rate is 0.0098USD/JPY.
(a) Find the JPY to USD exchange rate.
(b) Find the value in USD of 300,000 JPY

Exercise 2 (Each part 1 mark). Consider a forward with delivery price 200 and maturity $T$. Suppose the underlying asset has price

$$ S_T = \begin{cases} 
150 & \text{with probability } 0.3 \\
200 & \text{with probability } 0.5 \\
250 & \text{with probability } 0.2 
\end{cases} $$

(a) Find the payoff long the forward.
(b) Find the expected value of the forward to the short counterparty at maturity.

Exercise 3 (2 marks). The current time is $t = 0$. Suppose the present value of a forward contract on a certain asset is 10. The delivery price is $K = 100$ and the maturity is $T = 5$. Suppose the forward price on the asset is 110. Suppose the continuous interest rate is 2% for time 0 to $T$. Determine whether there is an arbitrage opportunity. If there is, find an arbitrage portfolio. Verify the portfolio you construct is an arbitrage portfolio.

Exercise 4 (2 marks). At current time $t$, a certain stock paying no income has price 45, the forward price with maturity $T$ on the stock is 40, and the price of a zero coupon bond with maturity $T$ is 0.95. Determine whether there is an arbitrage opportunity. If there is, find an arbitrage portfolio. Verify the portfolio you construct is an arbitrage portfolio.

Exercise 5 (Each part 1 mark). The current price of a certain stock paying no income is 30. Assume the annually compounded zero rate will be 3% for the next 2 years.
(a) Find the current value of a forward contract on the stock if the delivery price is 25 and maturity is in 2 years.
(b) If the stock has price 35 at maturity, find the value of the forward to the long counterparty at maturity.
1. 
(a) $\frac{1}{0.0098} \text{ JPY} \approx 102.040816327 \text{ JPY USD}$
(b) $300,000 \text{ JPY} = 300,000(0.0098) \text{ JPY USD} = 2940 \text{ USD}$

2. 
(a) Delivery price $= K = 200$, so

$$
\text{long payoff} = g(S_T) = S_T - K
$$

$$
= \begin{cases} 
150 - 200 & \text{with probability 0.3} \\
200 - 200 & \text{with probability 0.5} \\
250 - 200 & \text{with probability 0.2} 
\end{cases}
= \begin{cases} 
-50 & \text{with probability 0.3} \\
0 & \text{with probability 0.5} \\
50 & \text{with probability 0.2} 
\end{cases}
$$

(b) value of short forward at maturity $= -$value of long forward maturity $= -V_K(T, T) = -g(S_T)$

expected value of short forward at maturity $= -E(g(S_T)) = -\left[(-50)(0.3) + (0)(0.5) + (50)(0.2)\right] = 5$

3. 
$$
V_K(t, T) = 10 \text{ and } (F(t, T) - K)e^{-r(T-t)} = (110 - 100)e^{-0.02(5-0)} = 9.048 \ldots
$$

So
$$
V_K(t, T) > (F(t, T) - K)e^{-r(T-t)} \quad (0.0.1)
$$

This contradicts the no-arbitrage relationship $V_K(t, T) = (F(t, T) - K)e^{-r(T-t)}$. So there exists an arbitrage portfolio.

Begin Ungraded Part

Here is a description of how to find an arbitrage portfolio. You are not required to put this in your solution. It is included for instruction.

Rewrite (0.0.1) as
$$
V_K(t, T)e^{r(T-t)} > F(t, T) - K.
$$

We plan to build portfolios A and B with $V^A(t) = V^B(t)$, $V^A(T) = V_K(t, T)e^{r(T-t)}$ and $V^B(T) = F(t, T) - K$. Then $C = A - B$ will be an arbitrage portfolio because $V^C(t) = 0$ and $V^C(T) = V_K(t, T)e^{r(T-t)} - (F(t, T) - K) > 0$ with probability one.

Let’s start by building $B$ at time $t$ to get $V^B(T) = F(t, T) - K$. To get the $F(t, T)$ term, it makes sense to add a short forward (on the asset) with delivery price $F(t, T)$ and maturity $T$ because the value at maturity of the short forward will be $F(t, T) - S_T$. Pretty good. But we want to replace the $-S_T$ by $-K$. So we add a long forward (on the asset) with delivery price $K$ and maturity $T$ because its value at maturity will be $S_T - K$. Then $V^B(T) = F(t, T) - S_T + S_T - K = F(t, T) - K$. Perfect. We note that $V^B(t) = -V_F(t, T)(t, T) + V_K(t, T) = V_K(t, T)$.

Now let’s build $A$ at time $t$ to get $V^A(t) = V^B(t) = V_K(t, T)$ and $V^A(T) = V_K(t, T)e^{r(T-t)}$. The most obvious thing to try is $V_K(t, T)$ cash. It works perfectly. Alternatively, we can use $V_K(t, T)/Z(t, T)$ ZCBs with maturity $T$. 
In other words, \( C = A - B : V_K(t, T) \) cash; -1 short forward with delivery price \( F(t, T) \) and maturity \( T \); 1 long forward with delivery price \( K \) and maturity \( T \).

An arbitrage portfolio is
\[
C = A - B : V_K(t, T) \text{ cash; } -1 \text{ short forward with delivery price } F(t, T) \text{ and maturity } T; 
1 \text{ long forward with delivery price } K \text{ and maturity } T.
\]

In other words,
\[
C = A - B : V_K(t, T) \text{ cash; } 1 \text{ long forward with delivery price } F(t, T) \text{ and maturity } T; 
1 \text{ short forward with delivery price } K \text{ and maturity } T.
\]

Here is the verification that \( C \) is an arbitrage portfolio:
\[
V^C(t) = V_K(t, T) + V_{F(t,T)}(t, T) - V_K(t, T) = 0
\]
\[
V^C(T) = V_K(t, T)e^{r(T-t)} + S_T - F(t, T) + K - S_T = V_K(t, T)e^{r(T-t)} - (F(t, T) - K) > 0.
\]

Other arbitrage portfolios are possible.

4. \( F(t, T) = 40 \) and \( S_t/Z(t, T) = 45/0.95 = 47.368 \ldots \) So
\[
F(t, T) < \frac{S_t}{Z(t, T)} \quad (0.0.2)
\]

This contradicts the no-arbitrage relationship \( F(t, T) = S_t/Z(t, T) \). So there exists an arbitrage portfolio.

An arbitrage portfolio is
\[
C = B - A : S_t/Z(t, T) \text{ ZCBs with maturity } T; -1 \text{ short forward with delivery price } F(t, T) \text{ and maturity } T; 
1 \text{ long forward with delivery price } F(t, T)
\]

In other words,
\[
C = B - A : S_t/Z(t, T) \text{ ZCBs with maturity } T; 1 \text{ long forward with delivery price } F(t, T)
\]

An arbitrage portfolio is
\[
C = A - B : V_K(t, T) \text{ cash; } -1 \text{ short forward with delivery price } F(t, T) \text{ and maturity } T; 
-1 \text{ long forward with delivery price } K \text{ and maturity } T.
\]

In other words,
\[
C = A - B : V_K(t, T) \text{ cash; } 1 \text{ long forward with delivery price } F(t, T) \text{ and maturity } T; 
-1 \text{ short forward with delivery price } K \text{ and maturity } T.
\]
and maturity $T$; -1 stock. Here is the verification that $C$ is an arbitrage portfolio:

$$V^C(t) = (S_t/Z(t, T))Z(t, T) + V_{F(t,T)}(t, T) - S_t = 0$$

$$V^C(T) = S_t/Z(t, T) + (S_T - F(t, T)) - S_T = S_t/Z(t, T) - F(t, T) > 0.$$  

Other arbitrage portfolios are possible.

Recall that we assume we can hold any positive, negative, or fractional amount of an asset. But you may ask what it means practically to hold $-1$ of an asset. This is accomplished by short selling. See Section 2.5 of the textbook (An Introduction to Quantitative Finance by Stephen Blyth) for a brief discussion.

5. Given $S_t = 30 =$ current stock price of stock paying no income, $r = 0.03 =$ annually compounded zero rate for $t$ to $t + 2$.

(a) Forward contract with $K = 25 =$ delivery price and $T = t + 2 =$ maturity.

$$V_K(t, T) = (F(t, T) - K)(1 + r)^{-(T-t)} = \left(\frac{S_t}{(1 + r)^{-(T-t)}} - K\right)(1 + r)^{-(T-t)}$$

$$= S_t - K(1 + r)^{-(T-t)} = 30 - 25(1 + 0.03)^{-2} = 6.435 \ldots$$

(b) Given $S_T = 35$ and (from (a)) $K = 25$. Then

Value of long forward at maturity $= V_K(T, T) = S_T - K = 35 - 25 = 10.$
Interest Rates, Forward Rates, and Forward Rate Agreements

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October 10, 2016

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1 Preface

The purpose of this note is to describe interest rates, forward rates, and forward rate agreements in a different way. You may find this additional point of view helpful.

We will use notation here that is more explicit and informative than the notation originally used in class.

2 Rates and Forward Rates with Continuous Compounding

Let $t \leq u \leq v$ be times, with $t$ the current time. We let $r(t, u, v, \infty)$ denote the continuously compounded interest rate at $t$ for period $u$ to $v$. We agree at time $t$ to lend/borrow $N$ at time $u$ and receive/pay back

$$Ne^{r(t,u,v,\infty)\cdot(v-u)}$$

at time $v$. When $t < u$, we call $r(t, u, v, \infty)$ a forward rate.

Let $t < T_1 < T_2$.

$r(t, t, T_1, \infty) = \text{continuously compounded interest rate at } t \text{ for period } t \text{ to } T_1$. We agree at time $t$ to lend/borrow $N$ at time $t$ and receive/pay back

$$Ne^{r(t,t,T_1,\infty)\cdot(T_1-t)}$$

at time $T_1$. Note this rate is known at time $t$. It is non-random.

$r(T_1, T_1, T_2, \infty) = \text{continuously compounded interest rate at } T_1 \text{ for period } T_1 \text{ to } T_2$. We agree at time $T_1$ to lend/borrow $N$ at time $T_1$ and receive/pay back

$$Ne^{r(T_1,T_1,T_2,\infty)\cdot(T_2-T_1)}$$

at time $T_2$. Note this rate is unknown at time $t$. It is random.

$r(t, T_1, T_2, \infty) = \text{continuously compounded forward rate at } t \text{ for period } T_1 \text{ to } T_2$. It is really just the continuously compounded interest rate at $t$ for period $T_1$ to $T_2$. We agree at time $t$ to lend/borrow $N$ at time $T_1$ and receive/pay back

$$Ne^{r(t,T_1,T_2,\infty)\cdot(T_2-T_1)}$$

at time $T_2$. Note this rate is known at time $t$. It is non-random.
Result 2.1.

\[ e^{r(t, t, T_2, \infty) - (T_2 - t)} = e^{r(t, t, T_1, \infty) - (T_1 - t)} e^{r(t, T_1, T_2, \infty) - (T_2 - T_1)} \]  

(1)

Proof. Consider two portfolios following two different strategies decided upon at time \( t \).

A: Deposit 1 from \( t \) to \( T_2 \) at rate \( r(t, t, T_2, \infty) \).

B: Deposit 1 from \( t \) to \( T_1 \) at rate \( r(t, t, T_1, \infty) \), then take the amount accumulated and deposit it from \( T_1 \) to \( T_2 \) at rate \( r(t, T_1, T_2, \infty) \).

We have

\[ V^A(t) = V^B(t) = 1 \]
\[ V^A(T_2) = e^{r(t, t, T_2, \infty) - (T_2 - t)} \]
\[ V^B(T_2) = e^{r(t, t, T_1, \infty) - (T_1 - t)} e^{r(t, T_1, T_2, \infty) - (T_2 - T_1)} \]

If (1) is not true, then either

\[ V^A(T_2) > V^B(T_2) \quad \text{or} \quad V^A(T_2) < V^B(T_2). \]

In the first case, \( C = A - B \) is an arbitrage portfolio because

\[ V^C(t) = V^A(t) - V^B(t) = 0 \quad \text{and} \quad V^C(T_2) = V^A(T_2) - V^B(T_2) > 0. \]

In the second case, \( C = B - A \) is an arbitrage portfolio. Either way, we get an arbitrage portfolio, which contradicts the no-arbitrage assumption. Therefore (1) must be true.  \( \square \)
3 Rates and Forward Rates with Finite Compounding Frequency

Let \( t \leq u \leq v \) be times, with \( t \) the current time. We let \( r(t, u, v, m) \) denote the interest rate at \( t \) for period \( u \) to \( v \) with compounding \( m \)-times per year. We agree at time \( t \) to lend/borrow \( N \) at time \( u \) and receive/pay back

\[
N \left( 1 + \frac{r(t, u, v, m)}{m} \right)^{m(v-u)}
\]

at time \( v \). When \( t < u \), we call \( r(t, u, v, m) \) a forward rate.

Let \( t < T_1 < T_2 \).

\( r(t, t, T_1, m) = \) interest rate at \( t \) for period \( t \) to \( T \) with compounding \( m \) times per year. We agree at time \( t \) to lend/borrow \( N \) at time \( t \) and receive/pay back

\[
N \left( 1 + \frac{r(t, t, T_1, m)}{m} \right)^{m(T_1-t)}
\]

at time \( T_1 \). Note this rate is known at time \( t \). It is non-random.

\( r(T_1, T_1, T_2, m) = \) interest rate at \( T_1 \) for period \( T_1 \) to \( T_2 \) with compounding \( m \) times per year. We agree at time \( T_1 \) to lend/borrow \( N \) at time \( T_1 \) and receive/pay back

\[
N \left( 1 + \frac{r(T_1, T_1, T_2, m)}{m} \right)^{m(T_2-T_1)}
\]

at time \( T_2 \). Note this rate is unknown at time \( t \). It is random.

\( r(t, T_1, T_2, m) = \) forward rate at \( t \) for period \( T_1 \) to \( T_2 \) with compounding \( m \) times per year. It is just the continuously compounded interest rate at \( t \) for period \( T_1 \) to \( T_2 \). We agree at time \( t \) to lend/borrow \( N \) at time \( T_1 \) and receive/pay back

\[
N \left( 1 + \frac{r(t, T_1, T_2, m)}{m} \right)^{m(T_2-T_1)}
\]

at time \( T_2 \). Note this rate is known at time \( t \). It is non-random.
Result 3.1.

\[
\left(1 + \frac{r(t, t, T_2, m)}{m}\right)^{m(T_2-t)} = \left(1 + \frac{r(t, T_1, m)}{m}\right)^{m(T_1-t)} \left(1 + \frac{r(t, T_1, T_2, m)}{m}\right)^{m(T_2-T_1)}
\]
4 Libor/Simple Rates and Forward Libor/Simple Rates

Let $t \leq u \leq v$ be times, with $t$ the current time. We let $r(t, u, v, S)$ denote the simple interest rate at $t$ for period $u$ to $v$. We agree at time $t$ to lend/borrow $N$ at time $u$ and receive/pay back

$$N(1 + (v - u) \cdot r(t, u, v, S))$$

at time $v$. When $t < u$, we call $r(t, u, v, S)$ a forward rate.

The libor rate $t$ for period $u$ to $v$ is exactly the simple interest rate at $t$ for period $u$ to $v$. Since we have used it in class, we will use the alternate notation

$$L_t[T_1, T_2] = r(t, u, v, S)$$

Let $t < T_1 < T_2$.

$L_t[t, T_1]$ = the libor rate (simple interest rate) at $t$ for period $t$ to $T_1$. We agree at time $t$ to lend/borrow $N$ at time $t$ and receive/pay back

$$N(1 + (T_1 - t) \cdot L_t[t, T_1])$$

at time $T_1$. Note this rate is known at time $t$. It is non-random.

$L_{T_1}[T_1, T_2]$ = the simple interest rate at $T_1$ for period $T_1$ to $T_2$. We agree at time $T_1$ to lend/borrow $N$ at time $T_1$ and receive/pay back

$$N(1 + (T_2 - T_1) \cdot L_{T_1}[T_1, T_2])$$

at time $T_2$. Note this rate is unknown at time $t$. It is random.

$L_t[T_1, T_2]$ = the forward libor rate (simple forward rate) at $t$ for period $T_1$ to $T_2$. It is just the simple interest rate at $t$ for period $T_1$ to $T_2$. We agree at time $t$ to lend/borrow $N$ at time $T_1$ and receive/pay back

$$N(1 + (T_2 - T_1) \cdot L_t[T_1, T_2])$$

at time $T_2$. Note this rate is known at time $t$. It is non-random.
Result 4.1.

\[(1 + (T_2 - t) \cdot L_\xi[t, T_2]) = (1 + (T_1 - t) \cdot L_\xi[t, T_1])(1 + (T_2 - T_1) \cdot L_\xi[T_1, T_2])\]
5 Zero Coupon Bond Prices and Forward Zero Coupon Bond Prices

A zero coupon bond (ZCB) with maturity $T$ is an asset that pays 1 at time $T$ (and nothing else). It is the promise of a dollar at time $T$. The value (price) at time $t \leq T$ of a ZCB with maturity $T$ is denoted $Z(t, T)$.

Fix $T_1 \leq T_2$. Consider a forward contract with maturity $T_1$ where the underlying asset is a ZCB with maturity $T_2$. The forward price of the ZCB is denoted $F(t, T_1, T_2)$ for $t \leq T_1$.

Result 5.1.

$$Z(t, T_2) = Z(t, T_1)F(t, T_1, T_2)$$

Proof. Consider two portfolios.

A: 1 ZCB with maturity $T_2$

B: 1 long forward contract with delivery price $K$ and maturity $T_1$ on a ZCB with maturity $T_2$; $K$ ZCBs with maturity $T_1$

We have $V^A(T_1) = Z(T_1, T_2)$ and $V^B(T_1) = (Z(T_1, T_2) - K) + K = Z(T_1, T_2)$. Therefore $V^A(T_1) = V^B(T_2)$ with probability one. By the replication theorem, $V^A(t) = V^B(t)$, which means

$$Z(t, T_2) = V_K(t, T_1) + KZ(t, T_1). \quad (2)$$

The forward price $F(t, T_1, T_2)$ is the value of $K$ such that $V_K(t, T) = 0$. Setting $K = F(t, T_1, T_2)$ and $V_K(t, T) = 0$ in (2) leads to

$$F(t, T_1, T_2) = \frac{Z(t, T_2)}{Z(t, T_1)}.$$
6 Relating Different Rates and Zero Coupon Bond Prices

**Result 6.1.** For any \( t \leq u \leq v \),
\[
e^{r(t,u,v,\infty):(v-u)} = \left(1 + \frac{r(t,u,v,m)}{m}\right)^{m(v-u)} = (1 + (v-u) \cdot L_t[u,v])
\]
\[
\triangle
\]

**Proof.** We will prove the second equality. The first can be proved in the same way.

Seeking a contradiction, suppose
\[
\left(1 + \frac{r(t,u,v,m)}{m}\right)^{m(v-u)} > (1 + (v-u) \cdot L_t[u,v])
\]

A: At time \( t \), invest 1 at \( m \)-times-per-year compounding rate \( r(t,u,v,m) \) from \( u \) to \( v \) and borrow 1 at libor (simple) rate \( L_t[u,v] \) from \( u \) to \( v \). Then \( V^A(t) = 0 \) and
\[
V^A(v) = \left(1 + \frac{r(t,u,v,m)}{m}\right)^{m(v-u)} - (1 + (v-u) \cdot L_t[u,v]) > 0.
\]

So \( A \) is an arbitrage portfolio. This contradicts the no-arbitrage assumption. A similar argument shows
\[
\left(1 + \frac{r(t,u,v,m)}{m}\right)^{m(v-u)} < (1 + (v-u) \cdot L_t[u,v])
\]
also leads to a contradiction. Therefore the second equality is proved. \( \square \)

**Result 6.2.** For \( t \leq T \),
\[
Z(t,T) = e^{-r(t,T,\infty):(T-t)} = \left(1 + \frac{r(t,t,T,m)}{m}\right)^{-m(T-t)} = (1 + (T-t) \cdot L_t[t,T])^{-1}.
\]
\[
\triangle
\]

**Proof.** We prove
\[
Z(t,T) = (1 + (T-t) \cdot L_t[t,T])^{-1}.
\]

The other equalities will then follow from Result 6.1.

A: At time \( t \), 1 ZCB.

B: At time \( t \), \( N = (1 + (T-t) \cdot L_t[t,T])^{-1} \) cash invested at libor rate \( L_t[t,T] \) until \( T \).

Then \( V^A(T) = 1 \) and \( V^B(T) = N \cdot (1 + (T-t) \cdot L_t[t,T]) = 1 \). By replication,
\[
V^A(t) = V^B(t),
\]
i.e.,
\[
Z(t,T) = (1 + (T-t) \cdot L_t[t,T])^{-1}.
\]
\( \square \)
7 Putting It All Together

Let us consolidate our results.

**Result 7.1.** Let $t < T_1 < T_2$.

\[
e^{r(t,t,T_1,\infty)-(T_1-t)} e^{r(t,T_1,T_2,\infty)-(T_2-T_1)}
\]

\[
\left(1 + \frac{r(t, t, T_2, m)}{m}\right)^{m(T_2-t)} = \left(1 + \frac{r(t, t, T_1, m)}{m}\right)^{m(T_1-t)} \left(1 + \frac{r(t, T_1, T_2, m)}{m}\right)^{m(T_2-T_1)}
\]

\[
(1 + (T_2 - t) \cdot L_t[t, T_2]) = (1 + (T_1 - t) \cdot L_t[t, T_1]) \cdot (1 + (T_2 - T_1) \cdot L_t[T_1, T_2])
\]

\[
Z(t, T_2) = Z(t, T_1) F(t, T_1, T_2).
\]

All of these formulas are saying the same thing because

**Result 7.2.** For $t \leq T$,

\[
Z(t, T) = e^{-r(t,t,T,\infty)-(T-t)} = \left(1 + \frac{r(t, t, T, m)}{m}\right)^{-m(T-t)} = (1 + (T - t) \cdot L_t[t, T])^{-1}.
\]

By the previous two results, we clearly must have

**Result 7.3.** Let $t < T_1 < T_2$.

\[
F(t, T_1, T_2) = e^{-r(t,T_1,T_2,\infty)-(T_2-T_1)} = \left(1 + \frac{r(t, T_1, T_2, m)}{m}\right)^{-m(T_1-T_2)} = (1+(T-t) \cdot L_t[T_1, T_2])^{-1}.
\]
8 An Example

The following example illustrates how we can put everything together.

Example 8.1. The two-year forward one-year libor rate is 3%. The price of a ZCB maturing in 3 years is 0.7. Find the continuous two-year zero rate.

We are given

\[ L_t[T_1, T_2] = 0.03 \quad \text{and} \quad Z(t, T_2) = 0.7, \]

where \( t \) is the current time, \( T_1 = t + 2 \), and \( T_2 = T_1 + 1 = t + 3 \). We want \( r(t, t, T_1, \infty) \).

The quantities are related by

\[ (Z(t, T_2))^{-1} = e^{r(t,t,T_1,\infty)\cdot(T_1-t)}(1 + (T_2 - T_1)L_t[T_1, T_2]). \]

Solving for \( r(t, t, T_1, \infty) \) and substituting gives

\[ r(t, t, T_1, \infty) = \frac{1}{T_1-t} \ln \left( (Z(t, T_2))^{-1}(1 + (T_2 - T_1)L_t[T_1, T_2])^{-1} \right) \]

\[ = \frac{1}{2} \ln \left( ((0.7)^{-1}(1 + (1)(0.03)))^{-1} \right) \]

\[ = 0.163558 \ldots \]

\[ \triangle \]
9 Fixed and Floating Payments

Let $t \leq T \leq T + \alpha$, where $t$ is the current time.

If notional 1 is invested at simple interest rate $r$ from time $T$ until time $T + \alpha$, then it will grow to $1 + \alpha r$. The amount of interest accrued is $\alpha r$. If $r$ is a fixed (known, non-random) rate $K$, then the interest accrued is $\alpha K$. If $r$ is the floating (unknown, random) rate $L_T[T, T + \alpha]$, then the interest accrued is $L_T[T, T + \alpha]$.

**Result 9.1.** Fix a constant $K > 0$. The value at $t$ of an agreement to receive the fixed (i.e., known, non-random) payment $\alpha K$ at time $T + \alpha$ is

$$\alpha K Z(t, T + \alpha)$$

$\triangle$

**Proof.** The payment is equivalent to $\alpha K ZCBs$. $\square$

**Result 9.2.** The value at $t$ of an agreement to receive the floating (i.e., unknown, random) payment $\alpha L_T[T, T + \alpha]$ at time $T + \alpha$ is

$$Z(t, T) - Z(t, T + \alpha),$$

which is the same value as an agreement to receive 1 at time $T$ and pay back 1 at time $T + \alpha$. $\triangle$

**Proof.** Consider two portfolios.

A: An agreement to receive $\alpha L_T[T, T + \alpha]$ at time $T + \alpha$.

B: An agreement to receive 1 at time $T$ and pay back 1 at time $T + \alpha$.

Clearly $V^B(t) = Z(t, T) - Z(t, T + \alpha)$.

In each portfolio, we are permitted at any fixed time to take actions that have no total cost (i.e., are self-financing).

In portfolio B at time $T$, we take the following action that has no cost. We deposit the 1 received from the agreement at random libor rate $L_T[T, T + \alpha]$ until time $T + \alpha$.

In portfolio B at time $T + \alpha$, the deposit gives $1 + \alpha L_T[T, T + \alpha]$ and we must pay 1 for the agreement.

At time $T + \alpha$:

$$V^A(T + \alpha) = \alpha L_T[T, T + \alpha],$$

$$V^A(T + \alpha) = 1 + \alpha L_T[T, T + \alpha] - 1 = \alpha L_T[T, T + \alpha].$$

By replication, $V^A(t) = V^B(t)$. $\square$
10 Forward Rate Agreements

A forward rate agreement (FRA) is a derivative contract to exchange two cashflows. It has three parameters: \( K \) = fixed rate or delivery price; \( T \) = maturity; \( \alpha \) = term length or daycount fraction or accrual factor.

The buyer (long counterparty) of the FRA agrees at time \( t \leq T \) to

\[
\begin{align*}
\text{receive } & \alpha L_T[T, T + \alpha] \\
\text{pay } & \alpha K
\end{align*}
\]

at time \( T + \alpha \).

The seller (short counterparty) agrees to do the opposite.

The value or payout at time \( T + \alpha \) of the FRA for the buyer is

\[
\alpha L_T[T, T + \alpha] - \alpha K.
\]

The value of the FRA at time \( t \leq T \) is

\[
V_K(t, T) = Z(t, T) - Z(t, T + \alpha) - \alpha K Z(t, T + \alpha).
\]

\( \alpha L_T[T, T + \alpha] \) is the amount of simple interest that would be accrued on notional 1 at floating (unknown, random) rate \( L_T[T, T + \alpha] \) over period \( T \) to \( T + \alpha \).

\( \alpha K \) is the amount of simple interest that would be accrued on notional 1 at fixed (known, non-random) rate \( K \) over period \( T \) to \( T + \alpha \).

A FRA is an agreement to exchange the floating (unknown, random) amount \( \alpha L_T[T, T + \alpha] \) for the fixed (known, non-random) amount \( \alpha K \).

FRAs are similar to a forward contracts, but \( \alpha L_T[T, T + \alpha] \) is not the price of an asset that is being traded.

11 Forward Libor Rate

**Result 11.1.** The forward libor rate \( L_t[T, T + \alpha] \) is the number such that

\[
V_K(t, T) = 0 \text{ if and only if } K = L_t[T, T + \alpha].
\]

**Proof.**

\[
V_K(t, T) = 0 \iff Z(t, T) - Z(t, T + \alpha) - \alpha K Z(t, T + \alpha) = 0
\]

\[
\iff K = \frac{Z(t, T) - Z(t, T + \alpha)}{\alpha Z(t, T + \alpha)}
\]

\[
\iff K = L_t[T, T + \alpha]
\]
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Determinants in 2D

The determinant of the 2x2 matrix \[ \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \] is

\[ \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1 \]

\[ \left| \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \right| = \left| a_1 b_2 - a_2 b_1 \right| \]

is the area of the parallelogram formed by \( \mathbf{a} = [a_1, a_2] \) and \( \mathbf{b} = [b_1, b_2] \)

(Not obvious! See proof on p. 25-27)

Consequence

\[ \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = 0 \] if and only if the area of the parallelogram is zero

if and only if \( \mathbf{a} \) and \( \mathbf{b} \) are parallel (lie on the same line)

\[ \text{Geometry} \]

\[ \text{sign of } \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \text{ determined by right hand rule.} \]

\[ \det \begin{bmatrix} -a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \text{ positive if thumb points out of the page when curling fingers of right hand from } \mathbf{a} \text{ to } \mathbf{b} \text{ (short way).} \]

\[ \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \text{ negative if thumb points into the page when curling fingers of right hand from } \mathbf{a} \text{ to } \mathbf{b} \text{ (short way).} \]

Consequence

\[ \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = -\det \begin{bmatrix} b_1 & b_2 \\ a_1 & a_2 \end{bmatrix} \]
Curl: Fingers from top row vector to bottom row vector to determine sign.

Determinants in 3D:
\[
\begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix} = a_1 \begin{vmatrix}
  b_2 & b_3 \\
  c_2 & c_3 \\
\end{vmatrix} - a_2 \begin{vmatrix}
  b_1 & b_3 \\
  c_1 & c_3 \\
\end{vmatrix} + a_3 \begin{vmatrix}
  b_1 & b_2 \\
  c_1 & c_2 \\
\end{vmatrix}
\]

\[
= a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)
\]

Example:
\[
\begin{vmatrix}
  1 & 2 & 3 \\
  1 & 0 & 1 \\
  5 & 3 & 1 \\
\end{vmatrix}
\]

\[
= -3 + 8 + 9 = 14
\]

Geometric Meaning:
\[
\begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix}
\]

is the volume of the parallelepiped formed by \(\vec{a} = [a_1, a_2, a_3]\), \(\vec{b} = [b_1, b_2, b_3]\), \(\vec{c} = [c_1, c_2, c_3]\).
\[
det \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0 \quad \text{if and only if} \quad \text{volume of parallelepiped} \\
\text{is zero} \quad \text{if and only if} \quad \vec{a}, \vec{b}, \vec{c} \quad \text{lie on a plane}
\]

Sign of \( \det \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \) determined by relative orientation of \( \vec{a}, \vec{b}, \vec{c} \) (see page 30 of textbook)

Interchanging two rows changes the sign

For example,

\[
\det \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = - \det \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}
\]

\[
\det \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \det \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}
\]

Cross Product

**Input:** Two vectors. **Output:** A vector

Only defined in \( \mathbb{R}^3 \)

\[\vec{a} = [a_1, a_2, a_3] \]

\[\vec{b} = [b_1, b_2, b_3] \]

\[\vec{a} \times \vec{b} = [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1] \]

\[\vec{a} \times \vec{b} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i} \det \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j} \det \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \det \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = [a_2b_3 - a_3b_2, -(a_1b_3 - a_3b_1), a_1b_2 - a_2b_1] \]
Example
\[ \vec{a} = [1, 1, 1] \]
\[ \vec{b} = [1, 2, 3] \]
\[ \vec{a} \times \vec{b} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} \]
\[ = \hat{i}(2-2) - \hat{j}(3-1) + \hat{k}(2-1) \]
\[ = \hat{i} - 2\hat{j} + \hat{k} \]
\[ = [1, 0, 0] - 2[0, 1, 0] + [0, 0, 1] \]
\[ = [1, -2, 1] \]

\[ \vec{b} \times \vec{a} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} \]
\[ = \hat{i}(2-3) - \hat{j}(1-3) + \hat{k}(1-2) \]
\[ = [-1, 2, -1] \]
\[ = -[1, -2, 1] \]

\[ \vec{a} \times \vec{b} = - (\vec{b} \times \vec{a}) \quad \text{Always!} \]

Triple Product
\( (\text{Relationship between Dot Product, Cross Product, Determinant}) \)
\[ \vec{a} \cdot (\vec{b} \times \vec{c}) = \det \begin{vmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{vmatrix} = \det \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \]

Proof
\[ \vec{a} \cdot (\vec{b} \times \vec{c}) = 5a_1, a_2, a_3 \cdot \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \]
\[ \langle a_1, a_2, a_3 \rangle \cdot \left( \hat{i} \det \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - \hat{j} \det \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + \hat{k} \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \right) \]

\[ = \langle a_1, a_2, a_3 \rangle \cdot \left[ \det \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix}, - \det \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix}, \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \right] \]

\[ = a_1 \det \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \det \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \]

\[ = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \]

**Properties of Cross Product**

0. \( \vec{a} \times \vec{b} = - (\vec{b} \times \vec{a}) \)

1. \( \vec{a} \times \vec{b} \) is orthogonal to \( \vec{a} \) and \( \vec{b} \)

2. Direction of \( \vec{a} \times \vec{b} \) is given by the Right Hand Rule. When curling the fingers of your right hand from \( \vec{a} \) to \( \vec{b} \), your thumb points in the direction of \( \vec{a} \times \vec{b} \).

3. \( || \vec{a} \times \vec{b} || = || \vec{a} || || \vec{b} || \sin \theta \) \( (\theta = \text{angle between } \vec{a} \text{ and } \vec{b}, \ 0 \leq \theta \leq \pi) \)

\[ \text{area of the parallelogram formed by } \vec{a} \text{ and } \vec{b} \]

\[ \sqrt{\text{volume of the parallelepiped formed by } \vec{a}, \vec{b}, \vec{a} \times \vec{b}} \]
Proof of 3. 

$||\mathbf{a} \times \mathbf{b}||^2 = (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b})$ 

$= \det \begin{bmatrix} \mathbf{a} \times \mathbf{b} \\ \mathbf{a} \\ \mathbf{b} \end{bmatrix}$ 

$= \text{volume of parallelepiped formed by } \mathbf{a}, \mathbf{b}, \mathbf{c}$ 

$= \left( \text{area of base } \right) \cdot \left( \text{length of } \mathbf{a} \times \mathbf{b} \right)$ 

$\mathbf{a} \times \mathbf{b}$ is orthog. to $\mathbf{a}$ and $\mathbf{b}$ 

$\text{area of parallelogram formed by } \mathbf{a}, \mathbf{b} = ||\mathbf{a}|| ||\mathbf{b}|| \sin \theta$ 

$||\mathbf{a} \times \mathbf{b}||^2 = ||\mathbf{a}|| ||\mathbf{b}|| \sin \theta \cdot ||\mathbf{a} \times \mathbf{b}||$ 

$||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}|| ||\mathbf{b}|| \sin \theta$ 

More Properties 

(4) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ 

(5) CAB - BAC Rule 

$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{b} \cdot \mathbf{c}) \mathbf{a} \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ (scalar - vector - (scalar) vector) 

(6) $s(\mathbf{a} \times \mathbf{b}) = (s \mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (s \mathbf{b})$, $s$ is a scalar 

(7) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$