14.3 Partial Derivatives

Given a function \( f(x, y) \)

\[ f_x(x, y) = \text{partial derivative of } f \text{ with respect to } x \]
\[ = \text{derivative of } f \text{ with respect to } x \]
where \( y \) is treated as a constant

\[ f_y(x, y) = \text{partial derivative of } f \text{ with respect to } y \]
\[ = \text{derivative of } f \text{ with respect to } y \]
where \( x \) is treated as a constant

\[ g(u, v, w) \]
\[ g_w(u, v, w) = \text{partial derivative of } g \text{ with respect to } w \]
where \( u, v \) are treated as constants

**Example**

\( f(x, y) = 3x^2 + 2xy \). Find \( f_x, f_y \), and \( f_x(1, 2) \).

\[ f_x(x, y) = 6x + 2y \] (treat \( y \) as constant)

\[ f_y(x, y) = 0 + 2x = 2x \] (treat \( x \) as constant)

\[ f_x(1, 2) = 6(1) + 2(2) = 10 \]

**Limit Definition of Partial Derivatives**

\[ f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} \]

\[ = \lim_{x \to a} \frac{f(x, b) - f(a, b)}{x - a} \]

\[ f_y(a, b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h} \]

\[ = \lim_{y \to b} \frac{f(a, y) - f(a, b)}{y - b} \]
$f_x(a, b) = \text{slope of the tangent to the curve } \ z = f(x, b) \ \text{at} \ x = a$

$\begin{array}{c}
\text{tangent line} \\
\text{curve } z = f(x, b) \\
\text{intersection of surface } z = f(x, y) \text{ and plane } y = b
\end{array}$

$f_x(a, b) = \text{rate of change of } f \text{ w.r.t } x$

at the point $(a, b)$ when $y = b$ is fixed.
For fixed $b$, this is the vertical plane $y = b$ and...

... this is the curve $z = f(x, b)$.

The limit $\lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$ equals...

... the slope of the curve $z = f(x, b)$ at $(a, b, f(a, b))$, which is $f_x(a, b)$.

For fixed $a$, this is the vertical plane $x = a$ and...

... this is the curve $z = f(a, y)$.

The limit $\lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}$ equals...

... the slope of the curve $z = f(a, y)$ at $(a, b, f(a, b))$, which is $f_y(a, b)$.
\[ \text{Other Notation:} \]

Let \( z = f(x,y) \)

\[
\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{dz}{dx} = f_x = D_1 F = D_x f
\]

\[
\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{dz}{dy} = f_y = D_2 F = D_y f
\]

\[\text{Ex} \]

\( f(x,y) = \cos(xy^2) \). Find \( f_y(x,y) \).

\[
\begin{align*}
 f_y(x,y) &= \frac{\partial}{\partial y} \cos(xy^2) \\
 &= -\sin(xy^2) \cdot \frac{\partial}{\partial y} (xy^2) \\
 &= -\sin(xy^2) \cdot 2xy \\
\end{align*}
\]  

(Chain rule)

\[\text{Ex} \]

\( w = \ln(x + 2y + 3z) \). Find \( \frac{\partial w}{\partial y} \) at \( (2,1,0) \).

\[
\begin{align*}
 \frac{\partial w}{\partial y} &= \frac{1}{x + 2y + 3z} \cdot \frac{\partial}{\partial y} (x + 2y + 3z) \\
 &= \frac{2}{x + 2y + 3z} \\
\end{align*}
\]

\[
\begin{align*}
 \frac{\partial w}{\partial y} (2,1,0) &= \frac{2}{(2) + 2(1) + 3(0)} = \frac{2}{4} = \frac{1}{2} \\
\end{align*}
\]
Ex. \( z \) is defined implicitly as a function of \( x \) and \( y \) by
\[
x^3 + y^3 + z^3 + 6xyz = 1
\]
Find \( \frac{dz}{dx} \)

Solution:
\[
\frac{d}{dx} \left( x^3 + y^3 + z^3 + 6xyz \right) = \frac{d}{dx} (1)
\]
\[
3x^2 + 0 + 3z^2 \frac{dz}{dx} + 6yz + 6xy \frac{dz}{dx} = 0
\]

Explanations:
\[
z = z(x, y)
\]
\[
\frac{d}{dx} z^3 = \frac{d}{dx} (z(x, y))^3 = 3(z(x, y))^2 \frac{d}{dx} z(x, y) = 3z^2 \frac{dz}{dx}
\]
\[
\frac{d}{dx} (6xyz) = 6y \frac{d}{dx} (xz) = 6y \left( \frac{dx}{dx} \cdot z + x \cdot \frac{dz}{dx} \right) = 6yz + 6xy \frac{dz}{dx}
\]

Back to the problem:
\[
3x^2 + 0 + 3z^2 \frac{dz}{dx} + 6yz + 6xy \frac{dz}{dx} = 0
\]
\[
\frac{dz}{dx} \cdot (3z^2 + 6xy) = -3x^2 - 6yz
\]
\[
\frac{dz}{dx} = \frac{-3x^2 - 6yz}{3z^2 + 6xy}
\]
A contour map is given for a function $f$. Use it to estimate $f_x(2, 1)$ and $f_y(2, 1)$.

\[ f_x(2, 1) = \lim_{x \to 2} \frac{f(x, 1) - f(2, 1)}{x - 2} \]

\[ \approx \frac{f(x, 1) - f(2, 1)}{x - 2} = \text{change in } f \text{ change in } x \text{ for } x \text{ near } 2 \text{ and } y = 1 \text{ fixed} \]

\[ = \frac{12 - 10}{0.8} = \frac{2}{0.8} = \frac{1}{0.8} \cdot 2 = \frac{10}{8} \cdot 2 = \frac{20}{8} = 2.5 \]

\[ f_y(2, 1) \approx \frac{f(2, y) - f(2, 1)}{y - 1} = \frac{\text{change in } f}{\text{change in } y} \text{ for } y \text{ near } 1 \text{ and } x = 2 \text{ fixed} \]

\[ = \frac{12 - 10}{-1} = \frac{2}{-1} = -2 \]
Higher Order Partial Derivatives

\[ z = f(x, y) \]

\[
(f_x)_x = f_{xx} = \frac{d}{dx} \left( \frac{dF}{dx} \right) = \frac{d^2F}{dx^2} = \frac{d^2z}{dx^2}
\]

\[
(f_x)_y = f_{xy} = \frac{d}{dy} \left( \frac{dF}{dx} \right) = \frac{d^2F}{dy dx} = \frac{d^2z}{dy dx}
\]

\[
(f_y)_x = f_{yx} = \frac{d}{dx} \left( \frac{dF}{dy} \right) = \frac{d^2F}{dx dy} = \frac{d^2z}{dx dy}
\]

\[
(f_y)_y = f_{yy} = \frac{d}{dy} \left( \frac{dF}{dy} \right) = \frac{d^2F}{dy^2} = \frac{d^2z}{dy^2}
\]

2nd Order Partial Derivatives

Example: Find the 2nd order partial derivatives of

\[ f(x, y) = 3x^2 + 2xy^3 + y^6 \]

\[ f_x(x, y) = 6x + 2y^3 \]

\[ f_y(x, y) = 6xy^2 + 6y^5 \]

\[ f_{xx}(x, y) = \frac{d}{dx} \left( \frac{dF}{dx} \right) = \frac{d}{dx} (6x + 2y^3) = 6 \]

\[ f_{yy}(x, y) = \frac{d}{dy} \left( \frac{dF}{dy} \right) = \frac{d}{dy} (6xy^2 + 6y^5) = 12xy + 30y^4 \]

\[ f_{xy}(x, y) = \frac{d}{dy} \left( \frac{dF}{dx} \right) = \frac{d}{dy} (6x + 2y^3) = 6y^2 \]

\[ f_{yx}(x, y) = \frac{d}{dx} \left( \frac{dF}{dy} \right) = \frac{d}{dx} (6xy^2 + 6y^5) = 6y^2 \]
When is $F_{xy} = F_{yx}$?

**Clairaut's Theorem**

If either of $F_{xy}$ or $F_{yx}$ is continuous on some disk around $(a,b)$, then

$$F_{xy}(a,b) = F_{yx}(a,b)$$

For every function $f(x,y)$ in this course, the hypothesis of Clairaut's Theorem will be satisfied.

**Example**: Let $f(x,y) = 2y$ and $g(x,y) = 3x$. Does there exist a function $H(x,y)$ such that $H_x = f$ and $H_y = g$?

**Solution**: If such a function $H$ did exist, then

$$H_{xy} = f_y = 2 \quad \text{and} \quad H_{yx} = g_x = 3$$

But Clairaut's theorem says

$$H_{xy} = H_{yx}$$

Contradiction!

$\therefore$ No such function $H$ exists.
Ex:

Find \( H(x,y) \) s.t. \( H_x = 2xy \) and \( H_y = x^2 + y \)

Solution:

\[
H_x = 2xy
\]

\[
H(x,y) = \int H_x(x,y) \, dx = \int \frac{d(H)}{dx} \, dx = H(x,y) + C(y)
\]

\[
= \int 2xy \, dx = y \int 2x \, dx = y \cdot x^2 + C(y)
\]

\[.\quad \therefore H(x,y) = x^2 y + C(y)\]

Then

\[H_y = \frac{1}{3y} (x^2 y + C(y)) = x^2 + C'(y)\]

But we are given that

\[H_y = x^2 + y\]

\[.\quad \therefore x^2 + C'(y) = x^2 + y\]

\[.\quad \therefore C'(y) = y\]

\[.\quad \therefore C(y) = \int C'(y) \, dy = \int y \, dy = \frac{1}{2} y^2 + C\]

\[.\quad \therefore H(x,y) = x^2 y + C(y)\]

\[= x^2 y + \frac{1}{2} y^2 + C\]