4.4 Tangent Planes and Linear Approximations

Review: Tangent Lines

\[ y = f(x) \]

Tangent line to \( y = f(x) \) at \( x = a \)

\[ y = f(a)(x-a) + f(x) \]

\[ y = f(x) \]

Near \((a, f(a))\), tangent line approximates graph

Linear Approximation or Tangent Line Approximation to \( f \) at \( a \):

\[ f(x) \approx f(a)(x-a) + f(a) \quad \text{for } x \text{ near } a \]

Linearization of \( f \) at \( a \):

\[ L(x) = f(a) + \frac{dx}{dx} \]

\[ y = L(x) \text{ is the tangent line} \]

Tangent Plane

\[ z = f(x, y) \]

\[ C_1: z = f(x, y) \text{ curve} \]

\[ C_2: z = f(x, y) \text{ curve} \]

\[ T_1: \text{tangent to } C_1 \text{ at } x = a \]

\[ \text{slope } = f_x(a, b) \]

\[ T_2: \text{tangent to } C_2 \text{ at } y = b \]

\[ \text{slope } = f_y(a, b) \]

Tangent plane to \( z = f(x, y) \)
at \((x, y) = (a, b)\)

It's the plane through point \((a, b, f(a, b))\) and parallel to tangent lines \(T_1\) and \(T_2\)

Equation of Tangent Plane:

\[ z = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) \]

\[ \mathbf{n} = \langle f_x(a, b), f_y(a, b), -1 \rangle \]

Near \((a, b, f(a, b))\), tangent plane approximates graph

Linear/Tangent Plane Approximation to \( f \) at \((a, b)\):

\[ f(x, y) \approx f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) \]

Linearization of \( f \) at \((a, b)\):

\[ L(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) \]
Ex. Find the tangent plane to the elliptic paraboloid
\[ z = 2x^2 + y^2 \] at the point \((1,1,3)\).

\[
\begin{align*}
    f(x,y) &= z = 2x^2 + y^2 \\
    f_x(x,y) &= 4x \\
    f_y(x,y) &= 2y \\
    f_x(1,1) &= 4(1) = 4 \\
    f_y(1,1) &= 2(1) = 2 \\

    \text{Tangent plane:} \\
    z &= f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) \\
    &= 3 + 4(x-1) + 2(y-1)
\end{align*}
\]

Ex. Continuing the previous example, find the linearization at \((1,1,3)\) and use it to approximate \(f(1.1,0.9)\).

\[
L(x,y) = 3 + 4(x-1) + 2(y-1)
\]

\[
\begin{align*}
    f(1.1,0.9) &\approx L(1.1,0.9) \\
    &= 3 + 4(1.1-1) + 2(0.9-1) \\
    &= 3 + 4(0.1) + 2(-0.1) \\
    &= 3 + 0.4 - 0.2 \\
    &= 3.2
\end{align*}
\]

Compare to \(f(1.1,0.9)\): Recall \(f(x,y) = 2x^2 + y^2\)

\[
\begin{align*}
    f(1.1,0.9) &= 2(1.1)^2 + (0.9)^2 = 3.23
\end{align*}
\]

Pretty close!
Differentials: Single Variable

Tangent line at \( x = a \)

\[
y = b(x) = f(a) + f'(a)(x-a)
\]

\[
dx = \Delta x = \text{change in } x
\]

\[
\Delta y = \text{change in height of curve (graph)} \ y = f(x)
\]

\[
= f(a + \Delta x) - f(a)
\]

\[
dy = \text{change in height of tangent plane}
\]

\[
= f'(a)\,dx
\]

Differential of \( y \): \( dy = f'(x)\,dx \)
Differentials: Multiple Variables

\[ z = f(x, y) \text{ with } (x, y) \text{ changing from } (a, b) \text{ to } (a + \Delta x, b + \Delta y) \]

\[ dx = \Delta x = \text{change in } x \]
\[ dy = \Delta y = \text{change in } y \]

\[ \Delta z = \text{change in height of surface (graph)} \]
\[ = f(a + \Delta x, b + \Delta y) - f(a, b) \]

\[ dy = \text{change in height of tangent plane} \]
\[ = f_x(a, b) \, dx + f_y(a, b) \, dy \]

Total Differential of \( z \): \( dz = \frac{\partial z}{\partial x} \, dx + \frac{\partial z}{\partial y} \, dy \)
Example

(a) If \( z = f(x, y) = x^2 + 3xy - y^2 \), find the differential \( dz \).

Solution:

\[
dz = \frac{dz}{dx} \, dx + \frac{dz}{dy} \, dy
\]

\[
= (2x + 3y) \, dx + (3x - 2y) \, dy
\]

(b) If \( x \) changes from 2 to 2.05 and \( y \) changes from 3 to 2.96, find and compare the values of \( dz \) and \( \Delta z \).

Solution:

\[
a = 2, \quad a + \Delta x = 2.05, \quad b = 3, \quad b + \Delta y = 2.96
\]

\[
dx = \Delta x = 2.05 - 2 = 0.05
\]

\[
dy = \Delta y = 2.96 - 3 = -0.04
\]

\[
dz = \frac{dz}{dx} \bigg|_{(2,3)} \, dx + \frac{dz}{dy} \bigg|_{(2,3)} \cdot dy
\]

\[
= (2(2) + 3(3))(0.05) + (3(2) - 2(3))(-0.04)
\]

\[
= 0.65
\]

\[
\Delta z = f(2.05, 2.96) - f(2, 3)
\]

\[
= \left( (2.05)^2 + 3(2.05)(2.96) - (2.96)^2 \right) - \left( (2)^2 + 3(2)(3) - (3)^2 \right)
\]

\[
= 0.6449
\]

Note: \( \Delta z \approx dz \)
Example: The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm. Each measurement is correct to within 0.2 cm. Use differentials to estimate the maximum possible error in the volume of the box.

Solution: Let $x, y, z$ be the dimensions of the box. The volume is

$$V = xyz$$

Therefore

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

$$\therefore dV = yzdx + xzdy + yxdz \quad (\times)$$

Given

$$|dx| = |\Delta x| \leq 0.2, \quad |dy| = |\Delta y| \leq 0.2, \quad |dz| = |\Delta z| \leq 0.2$$

For the maximum error we take

$$dx = 0.2, \quad dy = 0.2, \quad dz = 0.2$$

and $x = 75, \quad y = 60, \quad z = 40$

in the formula $(\times)$

$$\Delta V \approx dV = (60)(40)(0.2) + (75)(40)(0.2) + (75)(60)(0.2)$$

$$= 1980$$

So an error of 0.2 cm in measuring each dimension can result in an error of approximately 1980 cm$^3$ in the volume.