Consider a function $f(x,y)$ at a point $(a,b)$.

- $f$ has a local maximum at $(a,b)$ if $f(x,y) \leq f(a,b)$ for all $(x,y)$ in some disk around $(a,b)$. Call $f(a,b)$ a local maximum value.

- $f$ has a local minimum at $(a,b)$ if $f(x,y) \geq f(a,b)$ for all $(x,y)$ in some disk around $(a,b)$. Call $f(a,b)$ a local minimum value.

- $f$ has an absolute maximum at $(a,b)$ if $f(x,y) \leq f(a,b)$ for all $(x,y)$ in the domain of $f$. Call $f(a,b)$ the absolute maximum value of $f$.

- Similarly for absolute minimum.
Graph of $z = f(x, y)$.

- Absolute maximum
- Local maximum
- Absolute minimum
- Local minimum

Also a local maximum for this graph.
Fact

If \( f(x, y) \) has a local max or local min at \( (a, b) \),
then \( f_x(a, b) = 0 \) and \( f_y(a, b) = 0 \) (if they exist),
and the tangent plane to the graph \( z = f(x, y) \)
at \( (a, b) \) is horizontal.

Critical Points

A point \( (a, b) \) is called a critical point
(or stationary point) if \( f_x(a, b) = f_y(a, b) = 0 \)
or if one or both of \( f_x(a, b) \) and \( f_y(a, b) \)
does not exist.

Restatement of Fact:

If \( f(x, y) \) has a local max or min at \( (a, b) \),
then \( (a, b) \) is a critical point.

The reverse is not necessarily true.
If \( (a, b) \) is a critical point, then \( (a, b) \)
may be a local max, a local min, or neither.

Example

Consider \( f(x, y) = y^2 - x^2 \).

Have \( f_x = -2x \) and \( f_y = 2y \).

Critical Points: \( f_x \) and \( f_y \) exist everywhere
(since no critical points from non-existence of derivatives)
Set \( 0 = f_x \) and \( 0 = f_y \)
\( 0 = -2x \) and \( 0 = 2y \)
\( \therefore (x, y) = (0, 0) \) is a critical point.
Consider \( f(x,y) = y^2 - x^2 \)

Have \( f_x = -2x \) and \( f_y = 2y \)

Critical Points:

- Points where \( f_x(a,b) \) or \( f_y(a,b) \) does not exist?
  - \( f_x \) and \( f_y \) exist everywhere

- Points where \( f_x(a,b) = f_y(a,b) = 0 \)?
  - Set \( 0 = f_x \) and \( 0 = f_y \)
  - \( 0 = -2x \) and \( 0 = 2y \)
  - \((x,y) = (0,0)\) is a critical point

Is \((0,0)\) a local max, local min, or something else?

Note that for points \((x,y)\) near \((0,0)\) on the x-axis \((y = 0)\) we have

\[ f(x,y) = -x^2 < 0 = f(0,0) \]

Note that for points \((x,y)\) near \((0,0)\) on the y-axis \((x = 0)\) we have

\[ f(x,y) = y^2 > 0 = f(0,0) \]

So every disk around \((0,0)\) contains points where \( f(x,y) > f(0,0) \) and where \( f(x,y) < f(0,0) \)

So \((0,0)\) is not a local max and not a local min. The point \((0,0)\) is called a saddle point of \( f \).

See the graph on the next page.
$z = f(x, y) = y^2 - x^2$
Classifying Critical Points

Second Derivative Test
Suppose \( f_x(a,b) = f_y(a,b) = 0 \) and \( f_x, f_y \) are continuous in some disk around \((a,b)\).
Let
\[
D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2
\]
Then
(a) If \( D > 0 \) and \( f_{xx}(a,b) > 0 \), then \((a,b)\) is a local min
(b) If \( D > 0 \) and \( f_{xx}(a,b) < 0 \), then \((a,b)\) is a local max
(c) If \( D < 0 \), then \((a,b)\) is a saddle point
   (hence neither a local max or local min)
(d) If \( D = 0 \), the test gives no information
   (it could have a local max, local min, saddle point or neither at \((a,b)\))

Note
\[
D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2.
\]
Find the local max values, local min values, and saddle points of 
\[ f(x, y) = x^4 + y^4 - 4xy + 1 \]

Solution: Start with critical points.
\[ f_x = 4x^3 - 4y \quad f_y = 4y^3 - 4x \]

Note \( f_x, f_y \) exist everywhere.

Set \( f_x = f_y = 0 \).
\[
\begin{align*}
0 &= f_x & 0 &= f_y \\
0 &= 4x^3 - 4y & 0 &= 4y^3 - 4x \\
0 &= x^3 - y & 0 &= y^3 - x \\
y &= x^3 \\
x &= y^3
\end{align*}
\]

Substitute first equation into the second:
\[
\begin{align*}
x &= (x^3)^3 \\
x &= x^9 \\
0 &= x^9 - x \\
0 &= x(x^8 - 1) \\
0 &= x(x^4 - 1)(x^4 + 1) \quad [a^2 - b^2 = (a-b)(a+b)] \\
0 &= x(x^2 - 1)(x^2 + 1)(x^4 + 1) \\
0 &= x(x - 1)(x + 1)(x^2 + 1)(x^4 + 1)
\end{align*}
\]

\( \therefore \) \( x = 0, 1, -1 \)

Now recall \( y = x^3 \)
\[
\begin{align*}
x = 0 & \implies y = (0)^3 = 0 \\
x = 1 & \implies y = (1)^3 = 1 \\
x = -1 & \implies y = (-1)^3 = -1
\end{align*}
\]

Critical Points: \((0, 0), (1, 1), (-1, -1)\)
Now use 2nd Deriv. Test

Compute
\[ f_{xx} = 12x^2 \]
\[ f_{yy} = 12y^2 \]
\[ f_{xy} = -4 \]
\[ D = f_{xx}f_{yy} - (f_{xy})^2 \]
\[ = (12x^2)(12y^2) - (-4)^2 \]
\[ = 144x^2y^2 - 16 \]

Recall:
\[ f(x,y) = x^4 + y^4 - 4xy + 1 \]
\[ f_x = 4x^3 - 4y \]
\[ f_y = 4y^3 - 4x \]

(0,0): \[ D(0,0) = -16 < 0 \]
\[ \therefore (0,0) \text{ is a saddle point of } f \]

(1,1): \[ D(1,1) = 144 - 16 > 0 \]
\[ f_{xx}(1,1) = 12 > 0 \]
\[ \therefore f \text{ has a local min at } (1,1) \]
\[ f(1,1) = (1)^4 + (1)^4 - 4(1)(1) + 1 = -1 \]
\[ \text{is a local minimum value of } F. \]

(-1,-1): \[ D(-1,-1) = 144 - 16 > 0 \]
\[ f_{xx}(-1,-1) = 12(-1)^2 = 12 > 0 \]
\[ \therefore f \text{ has a local min at } (-1,-1) \]
\[ f(-1,-1) = (-1)^4 + (-1)^4 - 4(-1)(-1) + 1 = -1 \]
\[ \text{is a local minimum value of } F. \]
Finding Absolute Maximum and Absolute Minimum Values

1D: \( f(x) \) defined on \([a, b]\)

Step 1: Find the values of \( f \) at the critical points in \((a,b)\).

Step 2: Find the values of \( f \) at the endpoints of \([a, b]\).

Step 3: The largest value from Steps 1 and 2 is the absolute maximum value of \( f \) on \([a, b]\). The smallest value is the absolute minimum value.

2D: \( f(x,y) \) defined on a closed and bounded set \( D \) in \( \mathbb{R}^2 \)

Step 0: Draw \( D \) and determine its boundary.

Step 1: Find the values of \( f \) at the critical points inside \( D \).

Step 2: Find the maximum and minimum values of \( f \) on the boundary of \( D \).

Step 3: The largest value from Steps 1 and 2 is the absolute maximum value of \( f \) on \( D \). The smallest value is the absolute minimum.

Examples of sets \( D \) in \( \mathbb{R}^2 \):

- \( \{(x,y) : x^2 + y^2 \leq 1\} \): Closed, Bounded
- \( \{(x,y) : -4 \leq x \leq 2, 1 \leq y \leq 3\} \): Closed, Bounded
- \( \{(x,y) : x^2 + y^2 < 1\} \): Bounded, Not Closed
Ex. Find the absolute max and absolute min values of 
\[ f(x, y) = x^4 - 8x^2 - 8y^2 + 100 \]
on the region 
\[ D = \{(x, y) : x^2 + y^2 \leq 2 \} \]

Step 0: Draw the region and determine its boundary

\[
\begin{align*}
(0, \sqrt{2}) \\
(0, -\sqrt{2}) \\
(-\sqrt{2}, 0) \\
(\sqrt{2}, 0)
\end{align*}
\]

Boundary: \( x^2 + y^2 = 2 \)

Step 1: Find the values of \( f \) at the critical points inside \( D \).

\[ f_x = 4x^3 - 16x \]
\[ 0 = 4x^3 - 16x \]
\[ 0 = x^3 - 4x \]
\[ 0 = x(x^2 - 4) \]
\[ 0 = x(x-2)(x+2) \]
\[ x = 0 \text{ or } x = 2 \text{ or } x = -2 \]

\[ f_y = -16y \]
\[ 0 = -16y \]
\[ y = 0 \]

Critical Points: \( (0, 0), (2, 0), (-2, 0) \)

Are they inside \( D \)? \( \Rightarrow \) Based on the drawing above, no.

Check \( x^2 + y^2 < 2 \)

\( (0, 0) : (0)^2 + (0)^2 < 2 \)
\[ \therefore (0, 0) \text{ inside } D \]

\( (2, 0) : (2)^2 + (0)^2 = 4 > 2 \)
\[ \therefore (2, 0) \text{ not inside } D \]

\( (-2, 0) : (-2)^2 + (0)^2 = 4 > 2 \)
\[ \therefore (-2, 0) \text{ not inside } D \]

Can also check algebraically.
Critical Points in \( D \): \((0,0)\).

Evaluate \( f \) at critical points in \( D \):

\[
f(0,0) = (0)^4 - 8(0)^2 - 8(0)^2 + 100 = 100
\]

\[
f(0,0) = 100
\]

Step 2: Find the maximum and minimum values of \( f \) on the boundary of \( D \).

\[
f(x,y) = x^4 - 8x^2 - 8y^2 + 100.
\]

Boundary of \( D \):

\[
x^2 + y^2 = 2
\]

\[
y^2 = 2 - x^2
\]

Substitute into formula for \( f(x,y) \):

\[
f(x,y) = x^4 - 8x^2 - 8(2 - x^2) + 100
\]

\[
= x^4 - 16 + 100
\]

\[
= x^4 + 84
\]

\[
F(x) = x^4 + 84
\]

On the boundary of \( D \), \( x^2 + y^2 = 2 \), \( x \) varies from \(-\sqrt{2}\) to \(\sqrt{2}\).

\[
\therefore \text{Need to find absolute max and min of } f(x) = x^4 + 84 \text{ on } [-\sqrt{2}, \sqrt{2}].
\]

For this particular function, calculus is not needed.

Abs. Max of \( F \) at \( x = \pm \sqrt{2} \):

\[
F(\pm \sqrt{2}) = (\pm \sqrt{2})^4 + 84 = 88
\]

At \( x = \pm \sqrt{2} \), using \( x^2 + y^2 = 2 \) gives \( y = 0 \).

Points: \((-\sqrt{2},0), (\sqrt{2},0)\)

\[
F(\pm \sqrt{2},0) = F(\pm \sqrt{2}) = 88
\]

Abs. Min of \( F \) at \( x = 0 \):

\[
F(0) = (0)^4 + 84 = 84
\]

At \( x = 0 \), using \( x^2 + y^2 = 2 \) gives \( y = \pm \sqrt{2} \).

Points: \((0,-\sqrt{2}), (0,\sqrt{2})\)

\[
F(0,\pm \sqrt{2}) = F(0) = 84
\]
Step 3: Largest $f$ value above is absolute max.
Smallest $f$ value above is absolute min.

Absolute Max = $f(0, 0) = 100$
Absolute Min = $f(0, 12) = 84$
Find the absolute maximum and minimum values of 
\[ f(x, y) = x^2 - 4xy + 8y \]
on the closed triangular region with vertices 
(0,0), (3,3), (3,0).

Solution:

Step 0:

Critical Points: (2,1)

Is it inside the region? Yes (see the diagram above).

\[ f(2,1) = (2)^2 - 4(2)(1) + 8(1) = 4 \]

Step 2:

Boundary consists of line segments \( L_1, L_2, L_3 \)

\[ L_1: \quad y = 0, \quad 0 \leq x \leq 3 \]
\[ F(x) = f(x, 0) = x^2 \]
\[ \text{Max: } f(3,0) = 9 \]
\[ \text{Min: } f(0,0) = 0 \]

\[ L_2: \quad x = 3, \quad 0 \leq y \leq 3 \]
\[ F(y) = f(3, y) = 9 - 4y \]
\[ \text{Max: } f(3,0) = 9 \]
\[ \text{Min: } f(3,3) = -3 \]

\[ L_3: \quad x = y, \quad 0 \leq x \leq 3 \]
\[ F(x) = f(x, x) = x^2 - 4x^2 + 8x \]
\[ \text{Max: } f(3,3) = -3 \]
\[ \text{Min: } f(3,3) = -3 \]

Endpoints: 
\[ f(0,0) = 0 \]
\[ f(3,3) = -3 \]
Step 3:
Abs. Max: \( f(3,0) = 9 \)
Abs. Min: \( f(3,3) = -3 \)