Characterization of High-$Q$ Resonators for Microwave-Filter Applications

Raymond S. Kwok and Ji-Fuh Liang

Abstract—A one-port reflection technique is developed to measure the unloaded $Q$ and external $Q$ of a microwave resonator. The unique procedure of measuring unloaded $Q$ is outlined in three easy steps. A sample chart is provided to further simplify the process. This method is so simple that even a scalar network analyzer is adequate for the measurement. In addition, a time-delay response around the resonator resonant frequency is also derived and presented. This theoretical result, combined with the advanced capability of modern vector network analyzers, has been proven to be very useful for characterization and tuning of the external $Q$ of a resonator. All the results derived are verified by practical measurement. Finally, this technique is applied to the realization and tuning of a six-pole resonator. All the results derived are verified by practical measurement.

I. INTRODUCTION

Microwave resonators are building blocks [1], [2] for many microwave devices, such as filters, multiplexers, and oscillators. The resonator unloaded $Q$ ($Q_u$) and external $Q$ ($Q_e$) are two fundamental parameters in microwave resonator designs. The former is an important index to determine the limitation of applications, and the latter dictates how the resonator interacts with other microwave devices in the system.

The resonator $Q_u$ measurement has always been an attractive research topic. Excellent literature [3]–[14] is available for most microwave applications. However, some of these methods require sophisticated mathematical treatment or complicated procedures. In this paper, based on a well-known equivalent circuit, we present an original expression for the resonator $Q_u$ as a function of the generalized one-port loaded $Q$ response. Simple procedures and charts are developed and can be directly applied to experiments using either a scalar or vector network analyzer. The advantage of the one-port reflection measurement over the two-port transmission type for resonators used in microwave filters is that the $Q_u$ can usually be measured in the same filter housing so that all the packaging effects are included. It is also more cost effective since it does not require additional fabrication.

The input/output coupling design of a microwave filter is somewhat similar to the resonator $Q_u$ characterization. In this case, the coupling or, equivalently, the $Q_e$ instead of the $Q_u$, is the parameter which not only should be precisely measured, but also should be adjusted to match the required value for a specific filter response. In any case, one can measure the phase or time delay of the input reflection coefficient to evaluate the input/output coupling. Using the same equivalent circuit, a closed-form relationship between the one-port reflection coefficient and external $Q$ is obtained. The measurement reference plane can be properly identified and, thus, the measurement accuracy is guaranteed.

II. EQUIVALENT CIRCUIT FOR A COUPLED RESONATOR

Fig. 1 shows an equivalent circuit for a series resonator coupled to a source impedance $Z_o$. The definition and realization of the

![Fig. 1. Equivalent circuit of a series resonator.](image)

impedance ($K$) inverter can be found in [1] and a generalized concept of the inverters has been suggested by Levy [15]. Alternatively, the inverters can be directly derived from the scattering matrix of a lossless reciprocity of microwave junction [16]. Due to the duality nature of series and parallel resonators, only the former case will be discussed without loss of generality. As shown in Fig. 1, the input resistance presented to the series resonator is defined as $r_A$ and is given by $r_A = K_0 q L / Z_o$, and the reflection coefficient from this resistance looking into the resonator is

$$\rho = (r - r_A) + j(\omega L - 1/\omega C) \left( r + r_A + j(\omega L - 1/\omega C) \right)$$

where $Q = \omega C / \omega_0$, $\omega_0 = \omega r - \omega_0$, $\omega_r = 1 / \sqrt{L C}$, and $\beta$ is the coupling parameter defined as $r_A / r$.

It was pointed out in [1] that $\rho$ is different from the actual measurable reflection coefficient $S_{11}$ by a phase difference of $180^\circ$. This phase difference can be regarded as a shifting of the reference plane in measurement and is not important for characterizing the resonator. Mathematically, $S_{11} = \rho \exp(-j\pi)$ and the measured return loss is $RL = -20 \log |S_{11}| = -20 \log |\rho|$. Here, we have adopted the convention that $RL$ is a positive real number in decibels.

III. UNLOAD $Q$ MEASUREMENT

For the following discussions, a generalized loaded $Q$ is introduced as $Q_L(x, \beta) \equiv \omega_0 / (\Delta \omega)_x$, where $(\Delta \omega)_x$ is the bandwidth measured at the $-x$-dB points of the input return loss, as illustrated in Fig. 2. Most modern network analyzers (such as the HP8753C) can measure $Q_L(x, \beta)$ automatically for any chosen value of $x$. This measurable generalized $Q_L(x, \beta)$ is a function of both $x$ and $\beta$, whereas the conventional loaded $Q$ is independent of $x$ and defined by the familiar expression of $Q_L = 1 / (\rho_L + 1) / \omega_0$, where $Q_L = \omega_0 L / r$ and $Q_s = \omega s L / r_A$. Until now, the $Q_u$ measurement using the reflection-coefficient technique requires the following sequence [3], [4]:

1. measure the return loss at resonant frequency;
2. determine the coupling condition (under- or over-coupled);
3. calculate the coupling parameter $\beta$;
4. calculate the correct return loss level $\rho_x$ in which $\Delta f$ is to be measured;
5. go back to the network analyzer and measure the bandwidth $\Delta f$;
6. evaluate $Q_L$ and then calculate $Q_u$ using $Q_u = Q_L(1 + \beta)$.

By introducing the generalized loaded $Q_L(x, \beta)$, the measurement can be easily mapped into the $Q_u$ for any value of $\beta$ with no prior calculation. Derivation of such a technique is presented below.

Around the resonance $\omega_0$, $\Omega = \omega / \omega_0 - \omega_0 / \omega$ can be approximated as $\Delta \omega / \omega_0$. Therefore, the magnitude of the input reflection coefficient measured at $-x$-dB return loss is related to the $Q_L(x, \beta)$.
Fig. 2. A typical one-port $|S_{11}|$ measurement of a microwave resonator. Shown here is the measured $|S_{11}|$ for a dielectric resonator showing a $Q_u$ of 32,810.

by

$$\rho^2 = \frac{[1 - \beta]^2 + [Q_u/Q_L(x, \beta)]^2}{[1 + \beta]^2 + [Q_u/Q_L(x, \beta)]^2}$$

(2)

and $x = -20 \log |\rho|$. Consequently, the unloaded $Q$ can be expressed simply as

$$Q_u = Q_L(x, \beta) F(x, \beta)$$

(3)

where $F(x, \beta)$ is the “mapping function” defined by

$$F(x, \beta) = \frac{(1 + \beta)^2 |\rho|^2 - (1 - \beta)^2}{1 - |\rho|^2}.$$

(4)

Note that both $Q_L(x, \beta)$ and $F(x, \beta)$ are functions of $\beta$ and $x$, but $Q_u$ is, in principle, independent of either. At $\omega_u$, where $\Omega = 0$, the return loss is simply

$$RL_u = -20 \log \frac{1 - \beta}{1 + \beta}.$$

(5)

Alternatively, one can write the mapping function $F(x, \beta)$ as

$$F(x, \beta) = \frac{2}{1 + \rho} \sqrt{|\rho|^2 - \rho_u^2}.$$

(6)

where $\rho_u$ is the magnitude of the input reflection coefficient at $\omega_u$ given by $\rho_u = 10^{-RL_u/20}$. The “−” and “+” signs in (6) correspond to the over-coupled ($\beta > 1$) and under-coupled ($\beta < 1$), respectively. These two solutions can be easily distinguished by inspecting the response circle in the Smith chart from a modern vector network analyzer [3], [4]. Basically, a large response circle enclosing the origin of the Smith chart signifies an over-coupled case; for under-coupling, the response circle is small and excludes the origin.

With only a scalar network analyzer available, one can still employ this technique by observing the change in $RL_u$ while perturbing the input coupling [11]. For example, an increase in $RL_u$ with increasing coupling signifies an over-coupled response. In the worst-case scenario, where the direction of the coupling strength could not be identified, one would simply take additional data points and calculate the $Q_u$ under both assumptions. Only the correct set of calculation would provide a consistent $Q_u$ value. This method is extremely easy to follow because it does not require any curve fitting or complicated mathematics.

The mapping function $F(x, \beta)$ is plotted in Fig. 3 for the case of $x = 3$ dB. Clearly, similar curves can be constructed easily using (6) for any value of $x$. Once the chart or table is developed (or programmed in a pocket calculator), evaluating $Q_u$ is as easy as: 1) read $Q_L(x, \beta)$ directly from network analyzer for any chosen value of $x$ (or calculate $\omega_u/(\Delta \omega)_{\Delta}$ directly if the network analyzer does not provide an automatic readout); 2) determine whether the resonator is over- or under-coupled by inspecting the Smith Chart (or by observing the changes in return loss while varying the input coupling); and 3) look up the appropriate chart or table for $F(x, \beta)$ at the observed $RL_u$ level (or compute using (6)).

The $Q_u$ is simply the product of $Q_L(x, \beta)$ and $F(x, \beta)$. This procedure is much simpler than other available procedures because no additional mathematical manipulation is required once the function $F(x, \beta)$ is generated as a look-up table or plot. The peak at 6 dB in Fig. 3 is intrinsic to the over-coupled condition of (6). In fact, one can easily show from (6) that there is always a maximum at $RL_u = 2x$. For example, $F(2, \beta > 1)$ has a peak at $RL_u = 4$ dB, $F(5, \beta > 1)$ has a peak at $RL_u = 10$ dB. This suggests that, in practice, one should arrange the input coupling such that $RL_u > 2x$, especially if a scalar network analyzer is to be used.

IV. EXTERNAL $Q$ MEASUREMENT AND TUNING

External $Q (Q_e)$ characterizes the coupling between a microwave resonator and the external circuit. In many practical applications, the absolute values of $L$, $C$, and $r_A$ in Fig. 1 are not important. In those cases, one can normalize $\omega_u L$ to be one (i.e., $L = 1/\omega_u$ and $C = 1/\omega_u$) and write $Q_e = 1/r_A$. For an all-pole microwave filter, $r_A$ can be expressed in terms of the low-pass prototype parameters as

$$r_A = \frac{w}{g_0 g_1}$$

(7)

where $w$ is the filter fractional bandwidth. It is also a common practice to use normalized input resistance or conductance in filter design because they are bandwidth independent, yet can be easily scaled with absolute or fractional bandwidth. The normalized input coupling resistance is defined by

$$R_A = \frac{r_A}{w} = \frac{1}{g_0 g_1}$$

(8)

which is a function of the low-pass prototype parameters only. This normalized resistance has been used in filter design with advanced features for satellite communication applications [17]–[19]. It is
particularly useful for designing quasi-elliptic function filters or filters with asymmetric response. In those cases, analytical expression for the low-pass prototype does not exist. However, the coupling resistance can still be obtained by a rigorous filter-synthesis procedure [20], [21] or simply by filter-response optimization based on properly constructed equivalent networks.

Again, the equivalent circuit of Fig. 1 is used to measure $r_A$. Usually, the $Q_u$ is much greater than the $Q_v$ (or equivalently, $r \ll r_A$) for practical applications. Therefore, the reflection coefficient can be simplified to

$$ \rho = -\frac{r_A - j\Omega}{r_A + j\Omega}. $$

(9)

It is evident from the above equation that at $\Omega = \mp r_A$, the measured phases are $\pm 90^\circ$, respectively. Recall that $\Omega = \frac{\Delta f}{\omega_0} = \frac{\Delta f}{\omega_0}$, the measured bandwidth $\Delta f$ at $\pm 90^\circ$ can be written as

$$ \Delta f_{\pm 90^\circ} = r_A f_v = \frac{f_v}{Q_v} = R_A(\Delta f)_b $$

(10)

where $(\Delta f)_b$ is the absolute bandwidth of the filter. For narrowband filter applications, $R_A(\Delta f)_b$ is the direct measured parameter, as shown in Fig. 4.

With a modern vector network analyzer, one could also measure the input coupling through the time delay of the reflection coefficient. Explicitly, the time delay of the one-port measurement at $\omega_0$ can be written as

$$ \tau_{\omega_0} = -\frac{\partial \phi}{\partial \omega} \bigg|_{\omega_0} = \frac{2}{\pi R_A(\Delta f)_b} $$

(11)

In special cases where Chebyshev prototypes are used and $g_v = 1$, (11) is reduced to the familiar expression of

$$ \tau_{\omega_0} = \frac{2g_v}{\pi(\Delta f)_b}. $$

(12)

Measurement of the time delay might seem redundant because the coupling resistance can be evaluated from (10) with a phase measurement. However, in a practical environment, the reference plane $A' - A'$ of the coupling structure corresponding to the ideal circuit in Fig. 1 is not always precisely identified. To locate the correct reference plane, the delay and phase of the network analyzer should be readjusted until a consistent value of the measured input coupling from (10) and (11) is obtained, as illustrated in Fig. 4.

![Fig. 4. Measured input coupling in terms of $S_{11}$ phase and time delay for a 0.5% fractional bandwidth dielectrically loaded cavity filter. The electrical delay is adjusted so that (12) is valid.](image)

### Table I

<table>
<thead>
<tr>
<th>$f_v$ (MHz)</th>
<th>$RL_v$ (dB)</th>
<th>$x$ (dB)</th>
<th>$Q_u$ ($x_0$)</th>
<th>$\beta$ (coupling)</th>
<th>$Q_v$</th>
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<td>4.26</td>
<td>1</td>
<td>18,499</td>
<td>0.24 (&lt;1, under)</td>
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### Table II

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<td>636</td>
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<td>11 (&gt;1, over)</td>
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### V. Measurement Results

The accuracy of the procedure in evaluating $Q_u$ is tested with various resonators for $Q_u$ ranging from orders of $10^2$ to $10^4$. Table I summarizes the result for a high-$Q_u$ dielectric resonator tested under various couplings. The value of $\beta$ is not required for the $Q_u$ determination, only provided here as a reference. Most data points in Table I agree with the critically coupled $Q_u$ value of 32 810 to within 3%. Errors are expected to be lower (less than 3%) if one follows the recommendations of choosing only $RL_v > 2x$ for the measurement.

The same procedure has also been applied to a relatively low $Q_u$ miniaturized coaxial resonator. The average $Q_u$ was evaluated to be 643, ±3% as summarized in Table II. Data point with the smallest value of $x$ (1.51 dB) is provided as a reference only. In practice, one should always use larger values of $RL_v$ and $x$ to evaluate the $Q_u$. Nonetheless, the $Q_u$ estimated from this level of $RL_v$ (1.51 dB) is still quite acceptable (about 6% error).

Fig. 5 shows the measured input coupling for a 0.5% fractional bandwidth quasi-elliptical filter at $f_v = 19.46$ MHz. The electrical delay of the network analyzer is adjusted so that the correct reference plane is located and the measured $\Delta f_{\pm 90^\circ}$ and time delay follow (11). Using the described techniques, a six-pole TE01 mode dielectrically loaded cavity filter is built and the final performance is shown in Fig. 5.

### VI. Summary

A simple empirical technique for characterizing the unloaded $Q$ and external $Q$ of a high $Q_u$ microwave resonator is presented. The
method we proposed to evaluate the unloaded $Q$ of the microwave resonator is based on a one-port measurement of the generalized loaded $Q$. By far, this is the simplest method which involves only a look-up table or a chart, as demonstrated. The time delay of the one-port reflection coefficient at the resonant frequency is derived and related to the coupling resistance or the external $Q$. Combined with the measured phase of the reflection coefficient, one can adjust the reference plane of the network analyzer correctly such that the measured response coincides with the ideal and simulated counterpart. Consequently, the accuracy of the measured coupling is guaranteed. The techniques are proven to be very useful through various experiments.

REFERENCES


