Method of Virtual Work for Trusses
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Work Done by Force/Moment

\[ W = F\delta \]

Work is done by a force acting through and in-line displacement

\[ W = M\theta \]

Work is done by a moment acting through and in-line rotation
Strain Energy of Deformation of a Truss Member

Internal strain energy stored through axial deformation of the truss member: \( U = \frac{1}{2} P \Delta \)

- \( A \) = Cross sectional area
- \( E \) = Modulus of Elasticity

\[ \Delta = \frac{PL}{AE} \]

\[ U = \frac{1}{2} P \Delta = \frac{P^2 L}{2AE} \]
1. Apply a virtual force $Q$ to the truss member;
2. Apply an additional real force $P$ to the truss member
3. Note that the work done by the virtual force $Q$ only is: \[ W_Q = Q\delta_P \]
Virtual Strain Energy for a Truss Member

\[ \begin{align*}
U_Q &= \int_0^L F_Q \delta_{dLP} \\
\delta_{dLP} &= \frac{F_P dL}{AE} \\
\delta_{dLQ} &= \frac{F_Q dL}{AE} \\
dU_Q &= F_Q \delta_{dLP}
\end{align*} \]
\[ W_Q = U_Q \]

\[ Q \delta_P = \int_0^L F_Q \frac{F_P \, dL}{AE} \]

\( A = \text{Cross sectional area} \)

\( E = \text{Modulus of Elasticity} \)
Principle of Virtual Work for a Prismatic, Elastic Truss member

\[ Q \delta_4 = \frac{F_2 F_4}{AE} dL \]

If the internal axial force is constant and \( A \) and \( E \) are constant:

\[ Q \delta_P = \int_0^L F_Q \frac{F_P dL}{AE} = F_Q \frac{F_P}{AE} \int_0^L dL \]

\[ Q \delta_P = F_Q \frac{F_P L}{AE} \]

\( A \) = Cross sectional area
\( E \) = Modulus of Elasticity
Consider a Truss Structure Subjected To Joint Loads

We want to find the deflection of joint B due to the applied loads

\[ A_i = \text{Cross sectional area} \]
\[ E_i = \text{Modulus of Elasticity} \]
\[ L_i = \text{Length of truss member} \]
\[ n = \text{Total number of truss members} \]
Apply Virtual Force

The virtual force will cause an internal axial force to develop in each truss member, $F_{Qi}$.

Apply a virtual force in-line with the real displacement $\delta_P$.

$W_Q = Q \delta_P$
Apply the Real Loads to the Truss

The real loads cause an axial deformation of each truss member, \( \Delta L_i \).

The real loads cause an internal axial force to develop in each truss member, \( F_{Pi} \).

- \( A_i \) = Cross sectional area
- \( E_i \) = Modulus of Elasticity
- \( L_i \) = Length of truss member
- \( n \) = Total number of truss members

\[
\Delta L_{Pi} = \frac{F_{Pi}L_i}{A_iE_i}
\]
Virtual Strain Energy

Virtual strain energy developed in an individual truss member

\[ U_{Qi} = F_{Qi} \cdot \Delta L_{Pi} = F_{Qi} \frac{F_{Pi}L_i}{A_iE_i} \]

Virtual strain energy for the entire truss

\[ U_Q = \sum_{i=1}^{n} F_{Qi} \frac{F_{Pi}L_i}{A_iE_i} \]

- \( A_i \) = Cross sectional area
- \( E_i \) = Modulus of Elasticity
- \( L_i \) = Length of truss member
- \( n \) = Total number of truss members
Principle of Virtual Work for Truss Deflections

\[ W_Q = U_Q \]

- \( A_i \) = Cross sectional area
- \( E_i \) = Modulus of Elasticity
- \( L_i \) = Length of truss member
- \( n \) = Total number of truss members

\[ Q \delta_P = \sum_{i=1}^{n} F_{Qi} \left( \frac{F_{Pi}L_i}{A_iE_i} \right) \]

Real Deformation

Virtual Loads
We want to find the real deflection of joint B due to the applied loads, $\delta_P$. 

Variables:
- $A_i$ = Cross sectional area
- $E_i$ = Modulus of Elasticity
- $L_i$ = Length of truss member
- $n$ = Total number of truss members
1. Remove all loads from the structure;
2. Apply a unit, dimensionless virtual load in-line with the real displacement, $\delta_P$, that we want to find;
3. Perform a truss analysis to find all truss member virtual axial forces, $F_{Qi}$.
1. Place all of the loads on the structure;
2. Perform a truss analysis to find all truss member real axial forces, $F_{Pi}$
Step 3 – Use the Principle of Virtual Work to Find $\delta_P$

From Step 2 – real analysis

From Step 1 – virtual analysis

$1 \cdot \delta_P = \sum_{i=1}^{n} \frac{F_{Qi}L_i}{A_iE_i}$

$A_i = \text{Cross sectional area}$

$E_i = \text{Modulus of Elasticity}$

$L_i = \text{Length of truss member}$

$n = \text{Total number of truss members}$