Method of Virtual Work for Trusses

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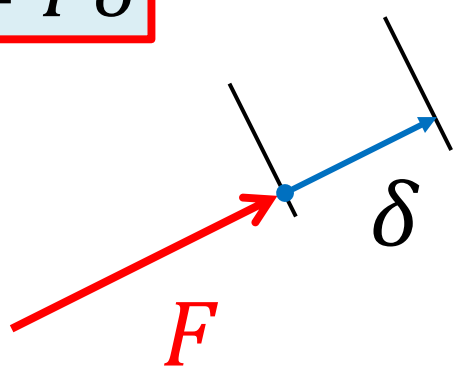
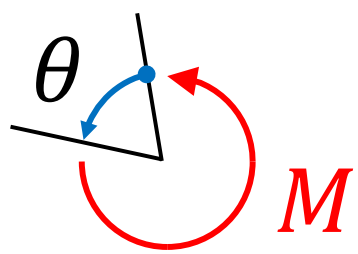
Work Done by Force/Moment

$$W = F\delta$$



Work is done by a force acting through and in-line displacement

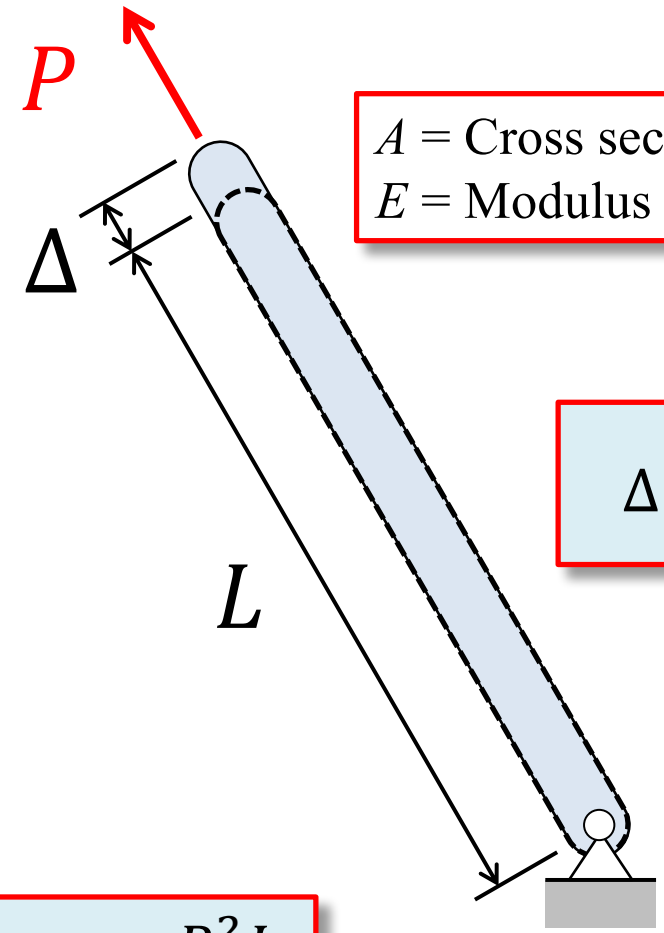
$$W = M\theta$$



Work is done by a moment acting through and in-line rotation

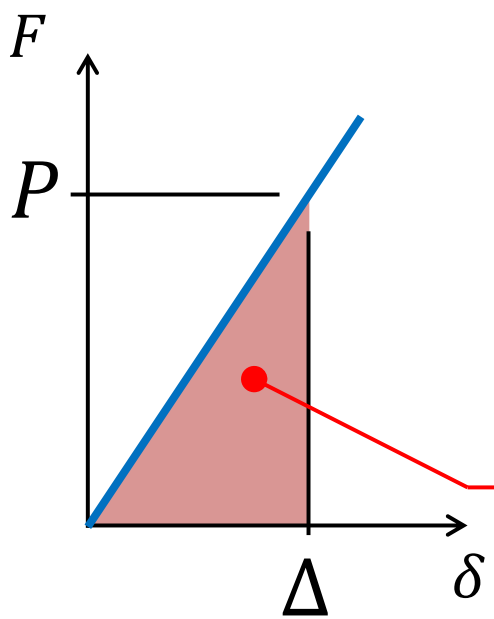
Strain Energy of Deformation of a Truss Member

Internal strain energy stored through axial deformation of the truss member: $U = \frac{1}{2}P\Delta$



A = Cross sectional area
E = Modulus of Elasticity

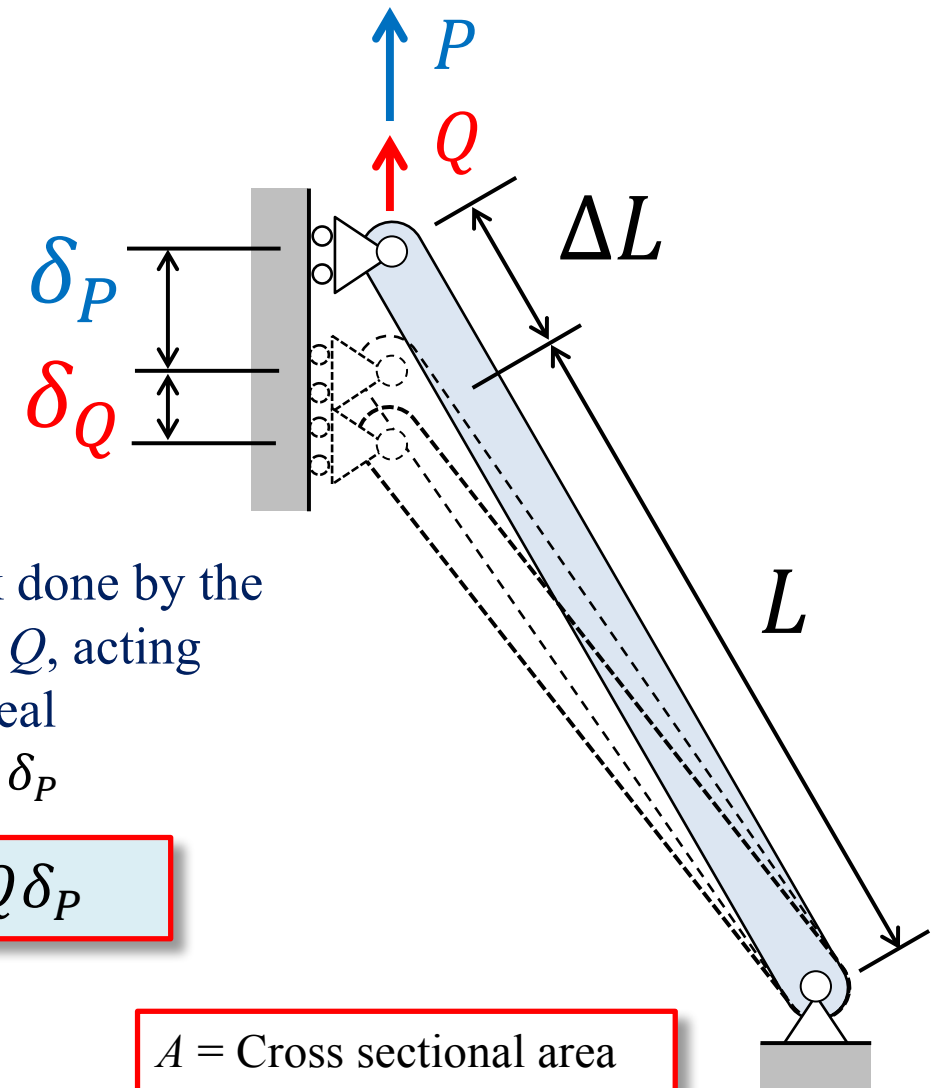
$$\Delta = \frac{PL}{AE}$$

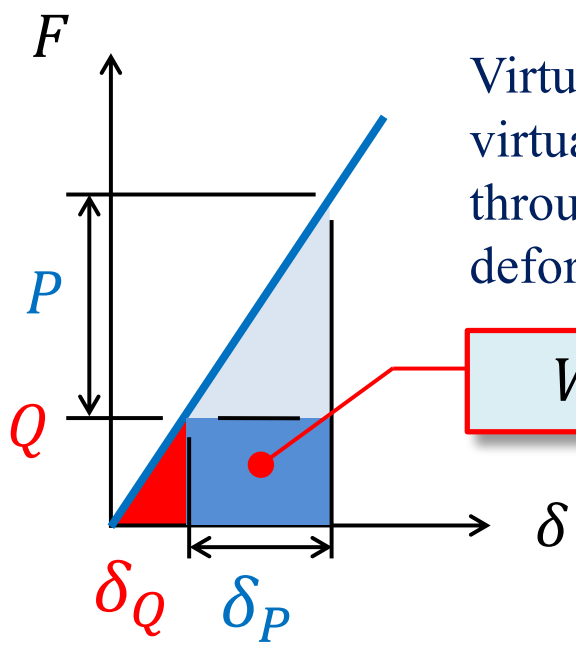


$$U = \frac{1}{2}P\Delta = \frac{P^2L}{2AE}$$

External Virtual Work for a Truss Member

1. Apply a virtual force Q to the truss member;
2. Apply an additional real force P to the truss member
3. Note that the work done by the virtual force Q **only** is: $W_Q = Q\delta_P$



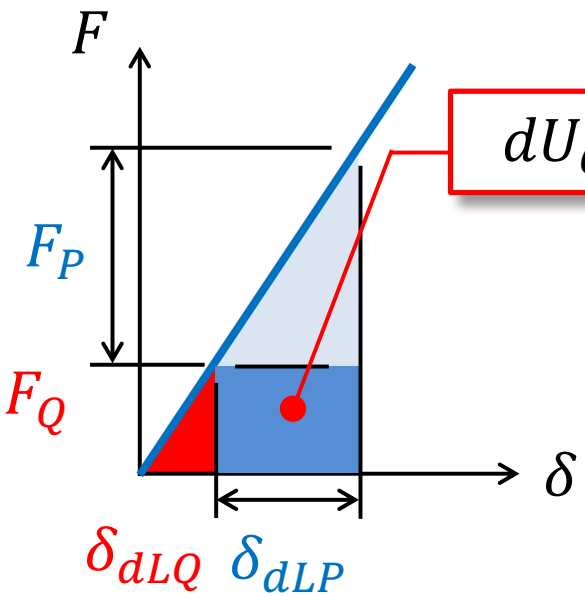


Virtual work done by the virtual force, Q, acting through the real deformation, δ_P

$$W_Q = Q\delta_P$$

A = Cross sectional area
E = Modulus of Elasticity

Virtual Strain Energy for a Truss Member



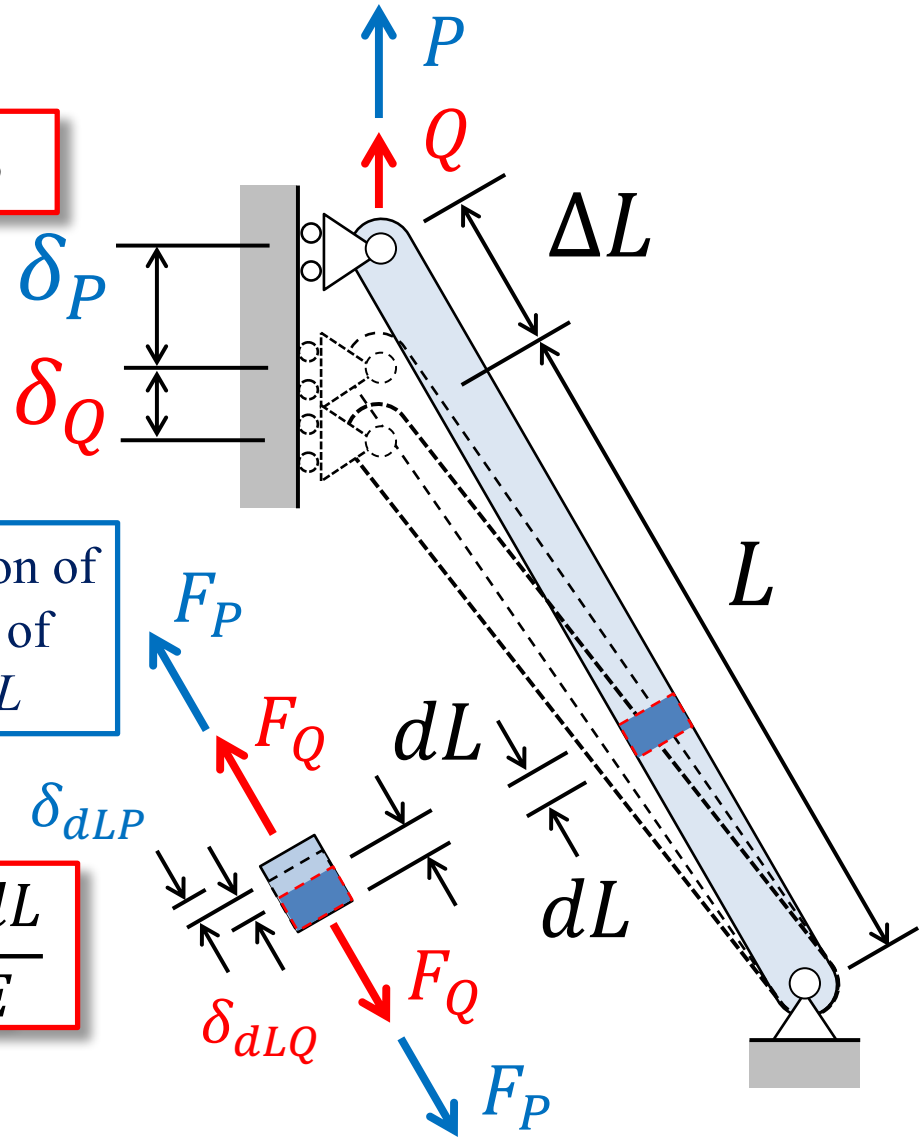
$$dU_Q = F_Q \delta_{dLP}$$

$$U_Q = \int_0^L F_Q \delta_{dLP}$$

$$U_Q = \int_0^L F_Q \frac{F_P dL}{AE}$$

$$\delta_{dLP} = \frac{F_P dL}{AE}$$

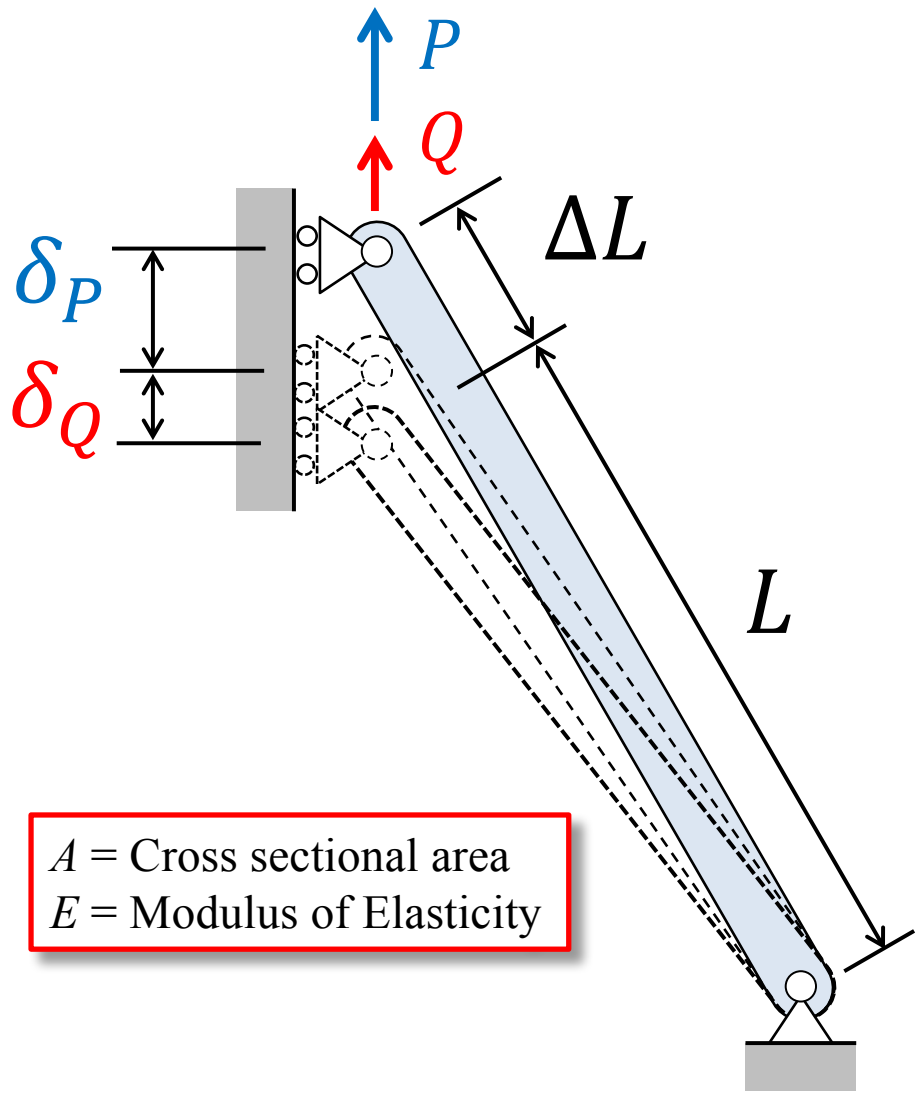
Deformation of small slice of member, dL



Conservation of Energy

$$W_Q = U_Q$$

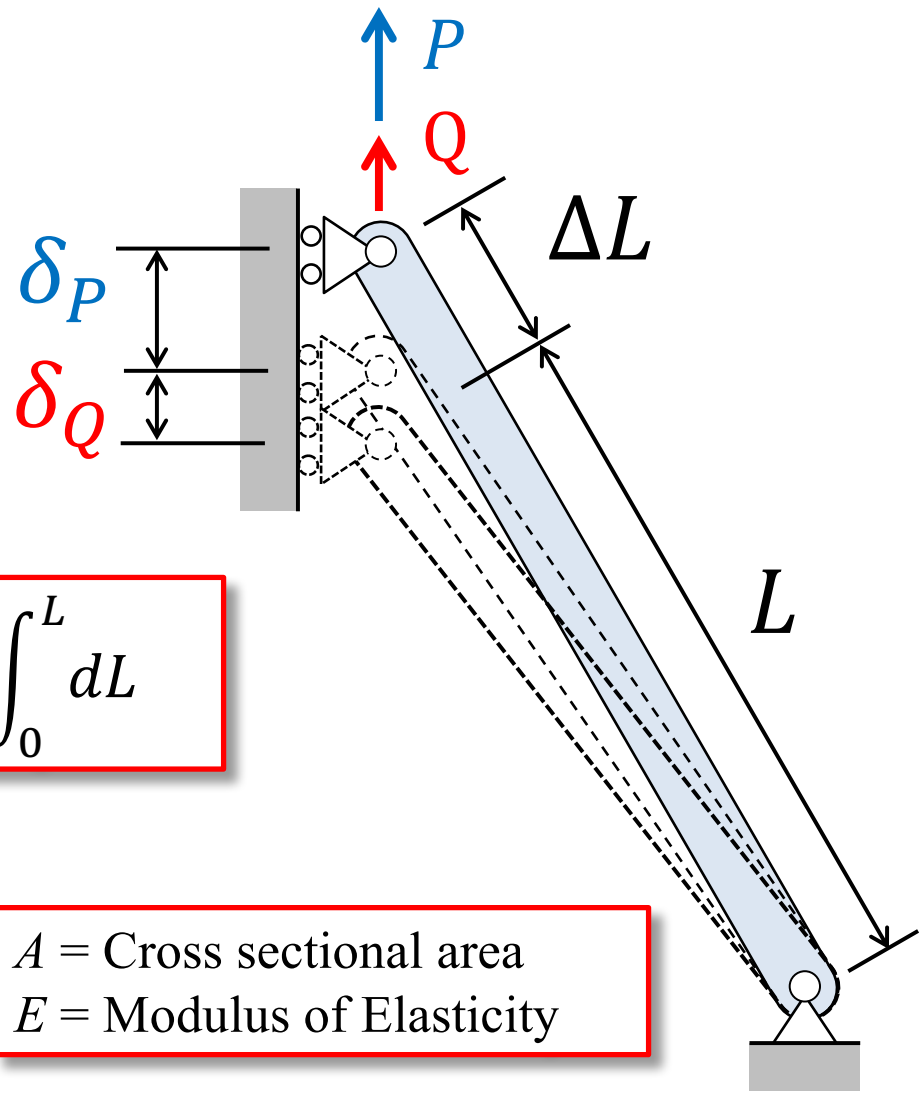
$$Q\delta_P = \int_0^L F_Q \frac{F_P dL}{AE}$$



A = Cross sectional area
E = Modulus of Elasticity

Principle of Virtual Work for a Prismatic, Elastic Truss member

If the internal axial force is constant and A and E are constant:

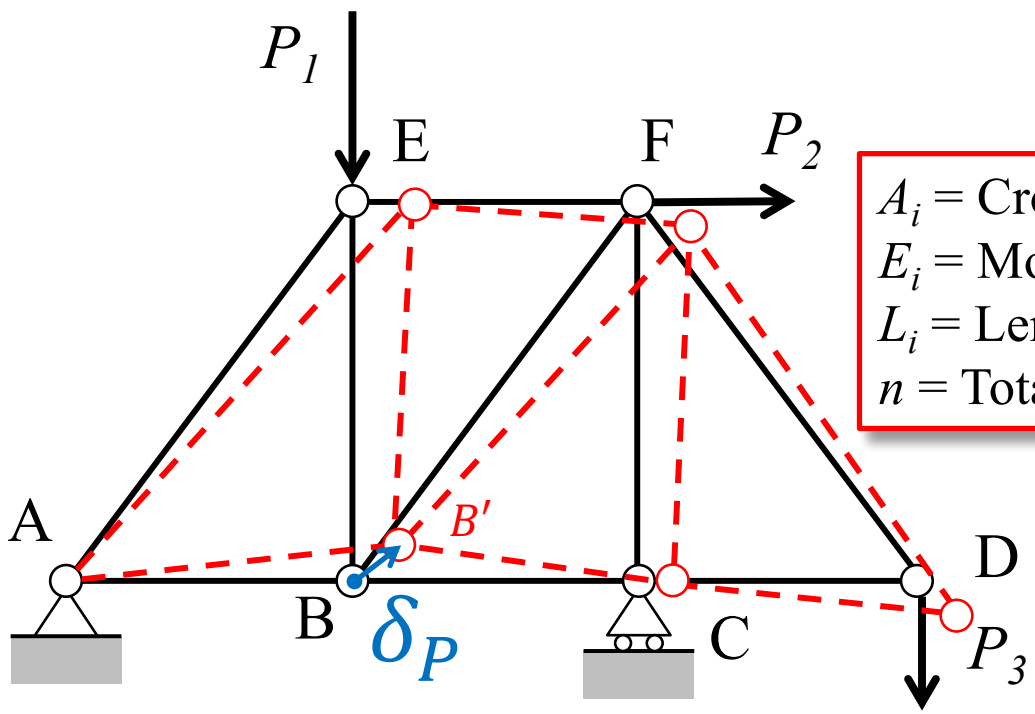


$$Q\delta_P = \int_0^L F_Q \frac{F_P dL}{AE} = F_Q \frac{F_P}{AE} \int_0^L dL$$

$$Q\delta_P = F_Q \frac{F_P L}{AE}$$

A = Cross sectional area
E = Modulus of Elasticity

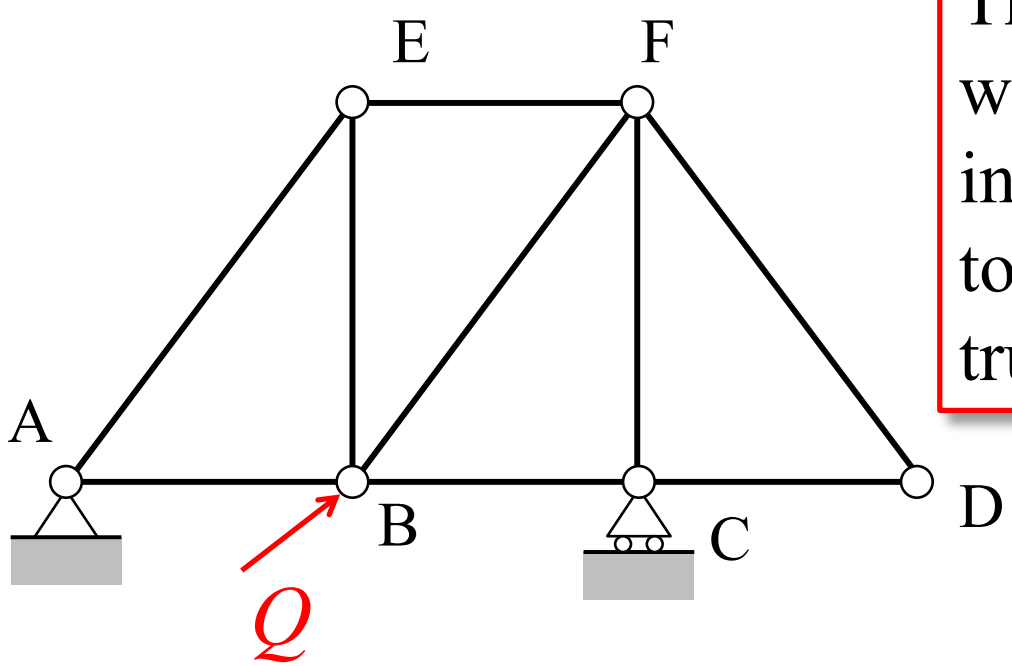
Consider a Truss Structure Subjected To Joint Loads



A_i = Cross sectional area
E_i = Modulus of Elasticity
L_i = Length of truss member
n = Total number of truss members

We want to find the deflection of joint B due to the applied loads

Apply Virtual Force

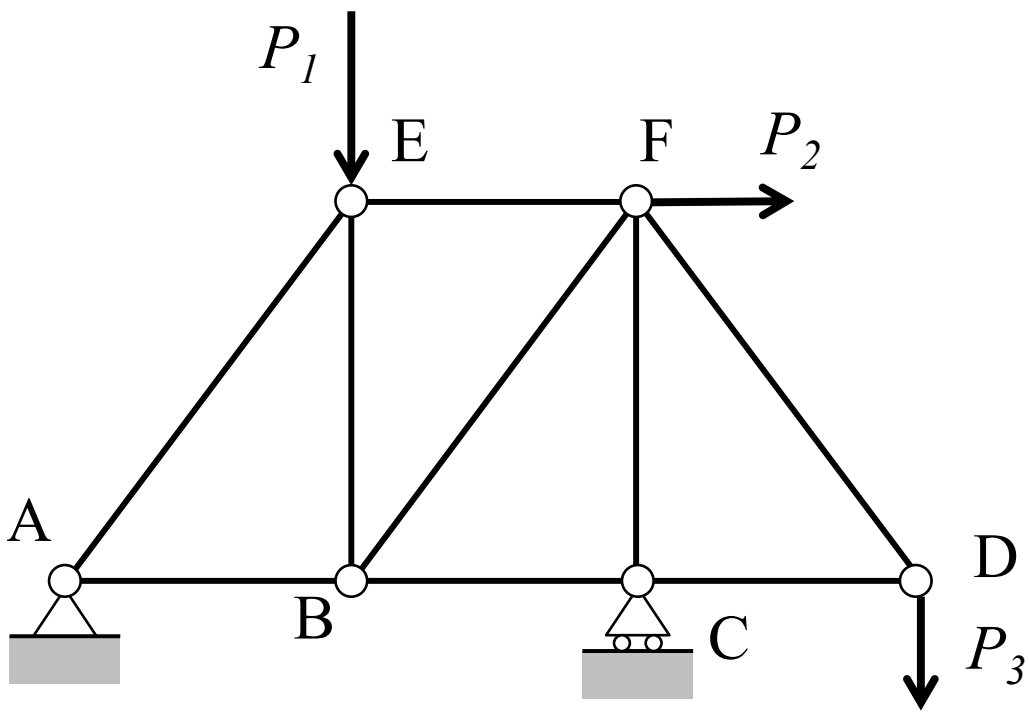


The virtual force will cause an internal axial force to develop in each truss member, F_{Qi}

Apply a virtual force **in-line** with the real displacement δ_P

$$W_Q = Q\delta_P$$

Apply the Real Loads to the Truss

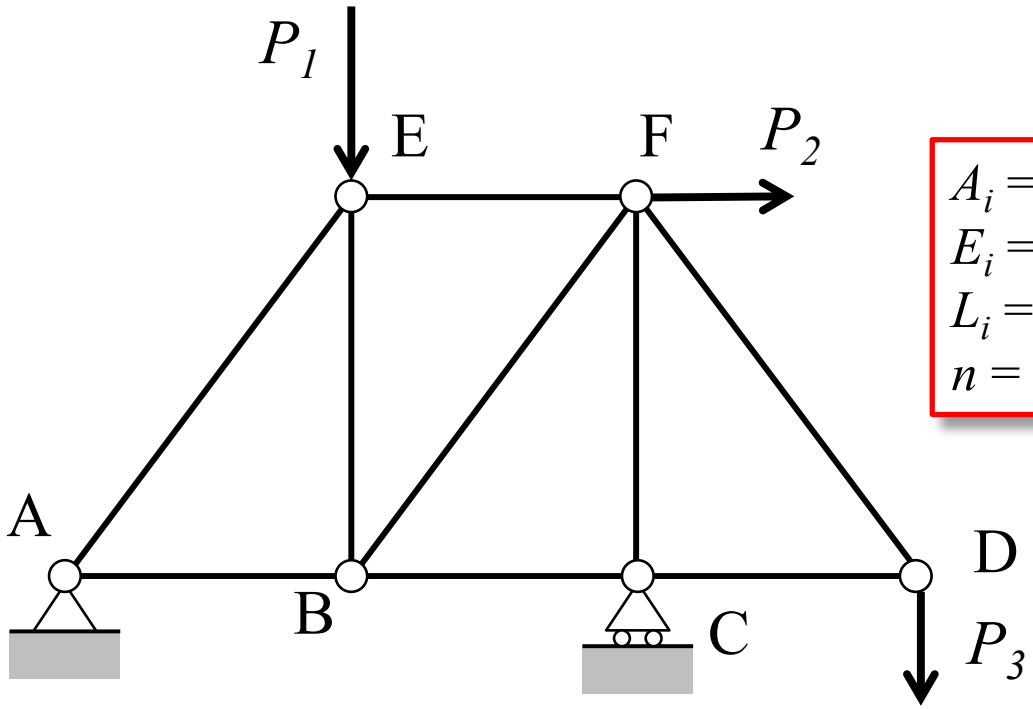


The real loads will cause an internal axial force to develop in each truss member, F_{Pi}

A_i = Cross sectional area
E_i = Modulus of Elasticity
L_i = Length of truss member
n = Total number of truss members

The real loads cause an axial deformation of each truss member, $\Delta L_{Pi} = \frac{F_{Pi} L_i}{A_i E_i}$

Virtual Strain Energy



A_i = Cross sectional area
E_i = Modulus of Elasticity
L_i = Length of truss member
n = Total number of truss members

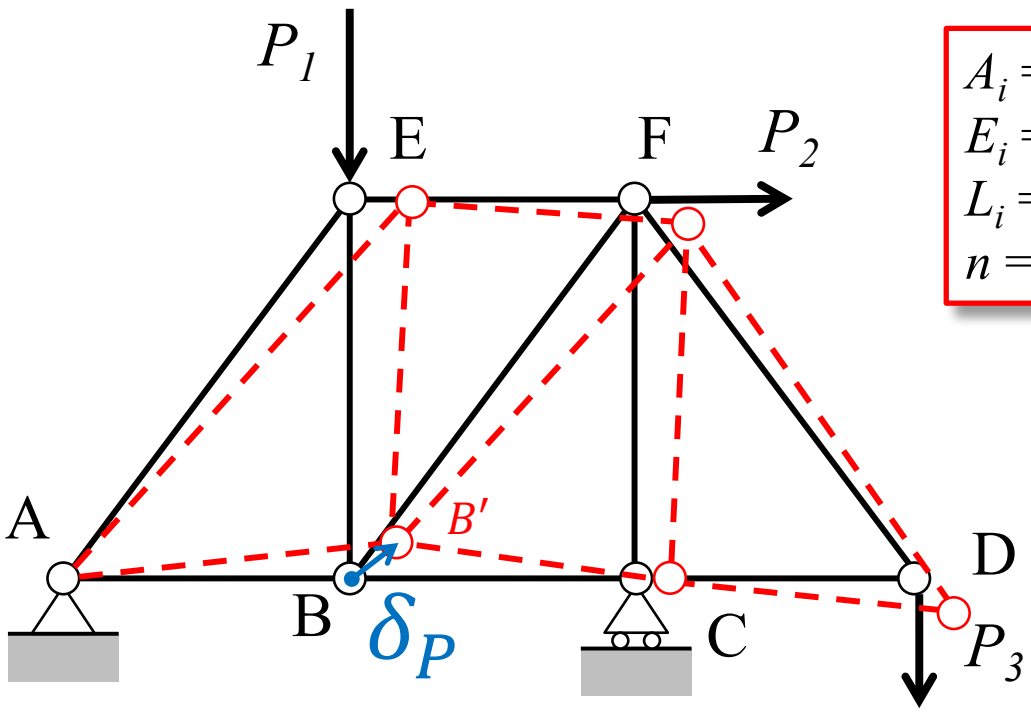
Virtual strain energy developed in an individual truss member

$$U_{Qi} = F_{Qi} \cdot \Delta L_{Pi} = F_{Qi} \frac{F_{Pi} L_i}{A_i E_i}$$

Virtual strain energy for the entire truss

$$U_Q = \sum_{i=1}^{n} F_{Qi} \frac{F_{Pi} L_i}{A_i E_i}$$

Principle of Virtual Work for Truss Deflections



A_i = Cross sectional area
E_i = Modulus of Elasticity
L_i = Length of truss member
n = Total number of truss members

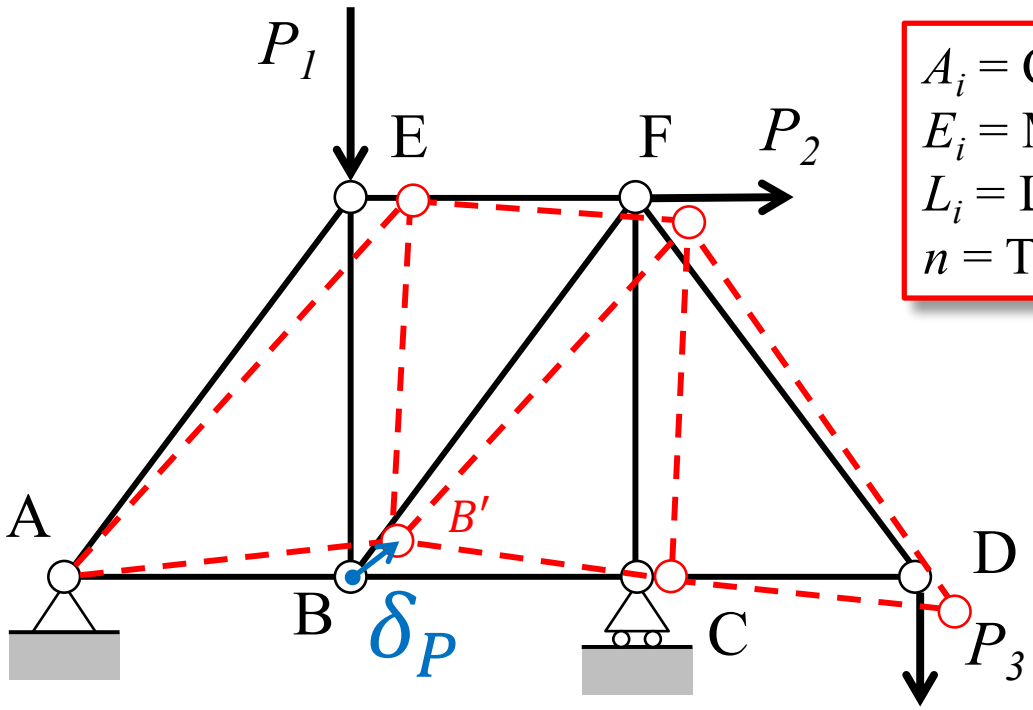
$$W_Q = U_Q$$

Real Deformation

$$Q\delta_P = \sum_{i=1}^{n} F_{Qi} \frac{F_{Pi} L_i}{A_i E_i}$$

Virtual Loads

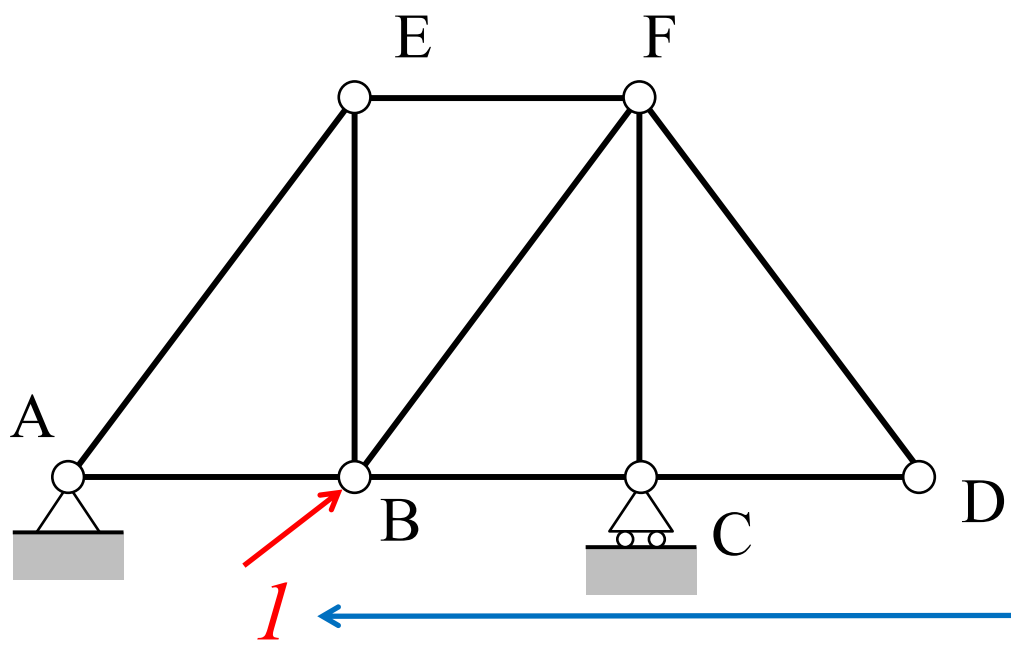
Procedure For Virtual Work Deflection Analysis



A_i = Cross sectional area
E_i = Modulus of Elasticity
L_i = Length of truss member
n = Total number of truss members

We want to find the real deflection of joint B due to the applied loads, δ_P

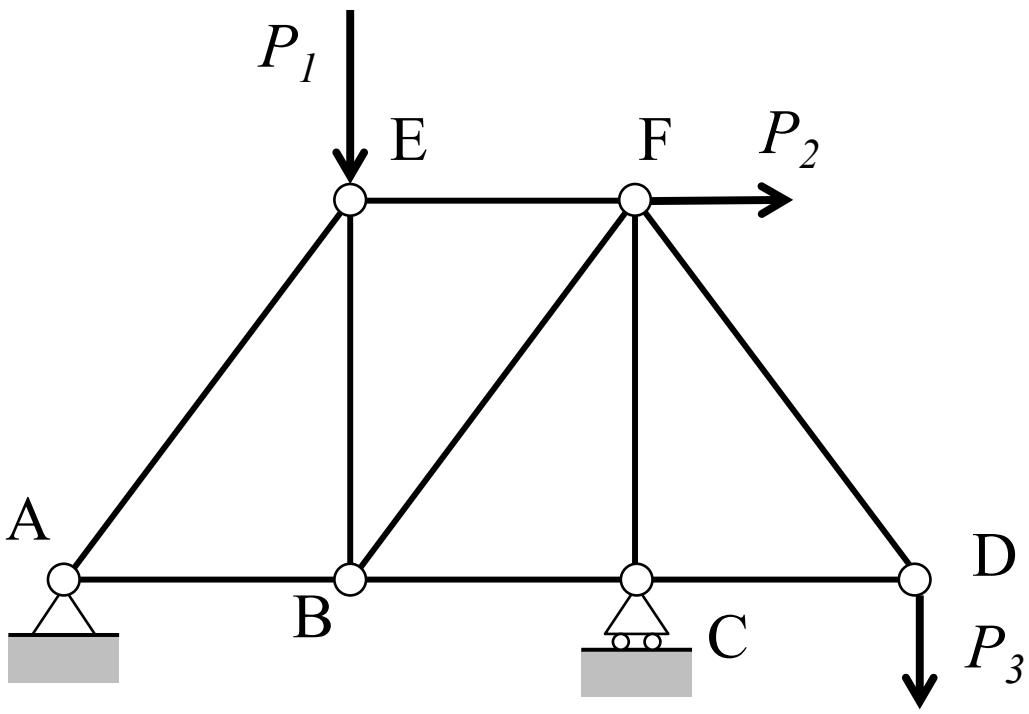
Step 1 –Virtual Analysis



Convenient to set $Q = 1$

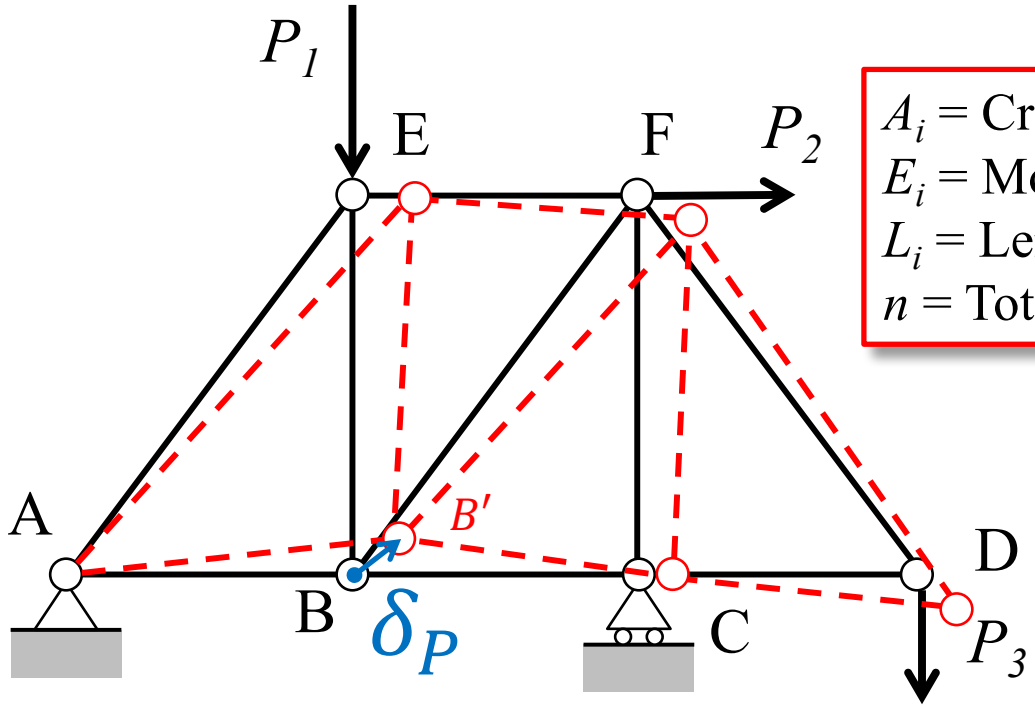
1. Remove all loads from the structure;
2. Apply a unit, dimensionless virtual load **in-line** with the real displacement, δ_P, that we want to find;
3. Perform a truss analysis to find all truss member virtual axial forces, F_{Qi}

Step 2 –Real Analysis



1. Place all of the loads on the structure;
2. Perform a truss analysis to find all truss member real axial forces, F_{Pi}

Step 3 – Use the Principle of Virtual Work to Find δ_P



A_i = Cross sectional area
E_i = Modulus of Elasticity
L_i = Length of truss member
n = Total number of truss members

$$1 \cdot \delta_P = \sum_{i=1}^{n} F_{Qi} \frac{F_{Pi} L_i}{A_i E_i}$$

From Step 2 – real analysis

From Step 1 –virtual analysis