Method of Virtual Work
Beam Deflection Example

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Summary of Procedure for Finding Bending Deformation Using Virtual Work

We want to find the deflection at point A and the slope at point B due to the applied loads.

Modulus of Elasticity = $E$
Moment of Inertia = $I$
Step 1 – Remove all loads and apply a virtual force (or moment) to measure the deformation at the point of interest

From an equilibrium analysis, find the internal bending moment function for the virtual system: $M_Q(x)$

Convenient to set $Q = 1$
Step 2 – Replace all of the loads on the structure and perform the real analysis

From an equilibrium analysis, find the internal bending moment function for the real system: $M_P(x)$
Step 3 – Evaluate the virtual work product integrals and solve for the deformation of interest

\[ Q \delta_A = \int_0^L M_Q \frac{M_P}{EI} \, dx \]

If the bending stiffness, \( EI \), is constant:

\[ Q \delta_A = \frac{1}{EI} \int_0^L M_Q M_P \, dx \]

Table in textbook appendix is provided to help evaluate product integrals of this type
### Table to Evaluate Virtual Work Product Integrals

#### Appendix Table.2

<table>
<thead>
<tr>
<th>Shape</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangle</td>
<td>$M_1 M_2 L$</td>
</tr>
<tr>
<td>Trapezium</td>
<td>$\frac{1}{2} M_1 M_2 L$</td>
</tr>
<tr>
<td>Parallelogram</td>
<td>$\frac{1}{2} (M_1 + M_2) M_2 L$</td>
</tr>
<tr>
<td>Triangle</td>
<td>$\frac{1}{6} M_1 M_2 (L + a)$</td>
</tr>
<tr>
<td>Rhombus</td>
<td>$\frac{1}{6} M_1 M_2 (L + b)$</td>
</tr>
<tr>
<td>Parabola</td>
<td>$\frac{1}{3} M_1 M_2 L$</td>
</tr>
<tr>
<td>Parabola</td>
<td>$\frac{1}{12} (M_1 + 3M_2) M_2 L$</td>
</tr>
</tbody>
</table>

Table is as useful tool to evaluate product integrals of the form:

$$ \int_{0}^{L} M_Q M_P dx $$
The overhanging beam shown has a fixed support at A, a roller support at C and an internal hinge at B. $EI_{ABC} = 2,000,000 \text{ k-in}^2$ and $EI_{CDE} = 800,000 \text{ k-in}^2$

For the loads shown, find the following:

1. The vertical deflection at point E;
2. The slope just to the left of the internal hinge at C;
3. The slope just to the right of the internal hinge at C
**Step 1** – Remove all loads and apply a virtual force (or moment) to measure the deformation at the point of interest

From an equilibrium analysis, find the internal bending moment function for the virtual system: $M_Q(x)$
Find the Moment Diagram for the Virtual System

\[ \sum M_A = 0 \Rightarrow M_A = 8 \text{ ft} \]
\[ \sum F_x = 0 \Rightarrow A_x = 0 \]
\[ \sum F_y = 0 \Rightarrow A_y = -0.5 \]

\[ \sum M_C = 0 \Rightarrow D_y = 1.5 \]
\[ \sum F_x = 0 \Rightarrow F_B = 0 \]
\[ \sum F_y = 0 \Rightarrow V_C = -0.5 \]
Support Reactions for the Virtual System
Moment Diagram for the Virtual System

- **8 ft**
- **0.5**
- **1.5**
- **1.0**

- **V_Q**
- **M_Q**

- **+**
- **-**
Step 2 – Replace all of the loads on the structure and perform the real analysis

From an equilibrium analysis, find the internal bending moment function for the real system: $M_P(x)$
Find the Moment Diagram for the Real System

\[ \sum M_A = 0 \quad \Rightarrow \quad M_A = -104 \text{ k-ft} \]

\[ \sum F_x = 0 \quad \Rightarrow \quad A_x = 0 \]

\[ \sum F_y = 0 \quad \Rightarrow \quad A_y = 16 \text{ k} \]

\[ \sum M_C = 0 \quad \Rightarrow \quad D_y = 9 \text{ k} \]

\[ \sum F_x = 0 \quad \Rightarrow \quad F_B = 0 \]

\[ \sum F_y = 0 \quad \Rightarrow \quad V_C = -3 \text{ k} \]
Support Reactions for the Real System

- At A: 104 k-ft
- At B: 19 k
- At C: 3 k
- At D: 3 k
- At E: 6 k

Dimensions:
- AB: 8 ft
- BC: 8 ft
- CD: 8 ft
- DE: 4 ft
- AC: 16 k
- BD: 9 k
Moment Diagram for the Real System

- 104 k-ft
- 19 k
- 6 k
- 16 k
- 9 k
- 8 k
- 4 ft
- 8 ft
- 8 ft
- 4 ft
- 3 k
- 24 k-ft
- 24 k-ft
- 104 k-ft

Components:
- \( V_P \): 16 k
- \( V_P \): -3 k
- \( M_P \): 24 k-ft
- \( M_P \): -24 k-ft
- \( V_P \): 6 k
Step 3 – Evaluate the virtual work product integrals and solve for the deformation of interest

\[ 1 \cdot \delta_E = \frac{1}{EI} \int_0^L M_Q M_P dx \]

Use Table to evaluate product integrals

- \( EI_{ABC} = 2,000,000 \text{ k-in}^2 \)
- \( EI_{CDE} = 800,000 \text{ k-in}^2 \)

\[ 8 - 0.5x = 4.75 \text{ ft} \]

\[ \frac{104}{x} = \frac{24}{8 - x} \]

\[ 832 = 128x \]

\[ x = 6.5 \text{ ft} \]
Evaluate Product Integrals

\[ M_Q \]

\[ M_P \]

\[ -104 \text{ k-ft} \]
### Table to Evaluate Virtual Work Product Integrals

#### Appendix Table.2

<table>
<thead>
<tr>
<th>( M_Q )</th>
<th>( M_P )</th>
<th>( M_{1} )</th>
<th>( M_{2} )</th>
<th>( M_{3} )</th>
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<tbody>
<tr>
<td>( M_{L} )</td>
<td>( M_{L} )</td>
<td>( \frac{1}{2} M_{L} M_{L} )</td>
<td>( \frac{1}{6} M_{L} M_{L} )</td>
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<td>( \frac{1}{6} M_{2} M_{L} (L + a) )</td>
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<td>( \frac{1}{6} M_{1} (M_{1} + 2M_{L}) )</td>
<td>( \frac{1}{6} M_{1} (2M_{1} + M_{L}) )</td>
<td>( \frac{1}{6} M_{2} M_{L} (L + b) + \frac{1}{6} M_{2} M_{L} (L + a) )</td>
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<td>( \frac{1}{2} M_{L} M_{L} )</td>
<td>( \frac{1}{6} M_{L} M_{L} )</td>
<td>( \frac{1}{6} M_{L} M_{L} (L + c) + \frac{1}{6} M_{2} M_{L} (L + c) )</td>
<td>( \text{for } c \leq a: \frac{1}{3} - \frac{(a - c)^2}{6a^2 d} M_{1} M_{L} )</td>
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<td>( \frac{2}{3} M_{L} M_{L} )</td>
<td>( \frac{1}{3} M_{1} M_{L} )</td>
<td>( \frac{1}{3} (M_{1} + M_{L}) M_{L} )</td>
<td>( \frac{1}{3} M_{2} M_{L} \left( L + \frac{3b}{L} \right) )</td>
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<td>( \frac{1}{3} M_{L} M_{L} )</td>
<td>( \frac{1}{4} M_{1} M_{L} )</td>
<td>( \frac{1}{12} (M_{1} + 3M_{L}) M_{L} )</td>
<td>( \frac{1}{12} M_{2} M_{L} \left( 3a + \frac{a^3}{L} \right) )</td>
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Table is as useful tool to evaluate product integrals of the form:

\[
\int_{0}^{L} M_{Q} M_{P} dx
\]
Evaluate Product Integrals

\[ M_3 = \frac{1}{6} (M_1 + 2M_2)M_3L \]
\[ = \frac{1}{6} (4.75 + 2(8))(-104)(6.5) \]
\[ = -2337.83 \text{ k-ft}^3 \]

\[ M_2 = 8 \text{ ft} \]
\[ M_1 = 4.75 \text{ ft} \]

\[ c = a = 8 \text{ ft} \]

\[ M_Q \]

\[ M_P \]

\[ c = a = 8 \text{ ft} \]

\[ M_3 = 384 \text{ k-ft}^3 \]

\[ \frac{1}{6}M_1M_3(L + c) \]
\[ = \frac{1}{6}(4.75)(24)(9.5 + 8) \]
\[ = 332.5 \text{ k-ft}^3 \]

\[ \left(\frac{1}{3} - \frac{(a - c)^2}{6ad}\right)M_1M_3L \]

for \( c \leq a \):
Evaluate Product Integrals

\[ EI_{ABC} = 2,000,000 \text{ k-in}^2 \quad EI_{CDE} = 800,000 \text{ k-in}^2 \]

\[
1 \cdot \delta_E = \frac{1}{EI} \int_0^L M_Q M_P \, dx
\]

Segment AX

\[-2337.83 \text{ k-ft}^3\]

Segment XC

\[332.5 \text{ k-ft}^3\]

Segment CDE

\[384 \text{ k-ft}^3\]

\[
\int_0^{L_{ABC}} M_Q M_P \, dx = \left( -2337.83 + 332.5 \text{ k-ft}^3 \right) \left( \frac{12^3 \text{ in}^3}{\text{ft}^3} \right) = -3,465,216.0 \text{ k-in}^3
\]

\[
\int_0^{L_{CDE}} M_Q M_P \, dx = \left( 384 \text{ k-ft}^3 \right) \left( \frac{12^3 \text{ in}^3}{\text{ft}^3} \right) = 663,552 \text{ k-in}^3
\]
Evaluate Product Integrals

\[ \int_{0}^{L_{ABC}} M_Q M_P dx = (-2337.83 + 332.5 k \text{-} ft^3) \left( \frac{12^3 \text{ in}^3}{ft^3} \right) = -3,465,216.0 \text{ k} \text{-} \text{in}^3 \]

\[ \int_{0}^{L_{CDE}} M_Q M_P dx = (384 \text{ k} \text{-} ft^3) \left( \frac{12^3 \text{ in}^3}{ft^3} \right) = 663,552 \text{ k} \text{-} \text{in}^3 \]

\[
\delta_E = \frac{1}{E I_{ABC}} \int_{0}^{L_{ABC}} M_Q M_P dx + \frac{1}{E I_{CDE}} \int_{0}^{L_{CDE}} M_Q M_P dx
\]

\[
\delta_E = \frac{-3,465,216.0 \text{ k} \text{-} \text{in}^3}{2,000,000 \text{ k} \text{-} \text{in}^2} + \frac{663,552 \text{ k} \text{-} \text{in}^3}{800,000 \text{ k} \text{-} \text{in}^2}
\]

\[
\delta_E = -1.733 \text{ in} + 0.8294 \text{ in} = -0.903 \text{ in}
\]

\[
\delta_E = 0.903 \text{ in upward}
\]

Negative result, so deflection is in the opposite direction of the virtual unit load.
Step 1 – Remove all loads and apply a virtual force (or moment) to measure the deformation at the point of interest.

From an equilibrium analysis, find the internal bending moment function for the virtual system: 

\[ M_Q(x) \]
Find the Moment Diagram for the Virtual System

\[ \sum_{\text{about } A} M_A = 0 \rightarrow M_A = 1 \]

\[ \sum_{\text{along } x} F_x = 0 \rightarrow A_x = 0 \]

\[ \sum_{\text{along } y} F_y = 0 \rightarrow A_y = 0 \]

\[ \sum_{\text{about } C} M_C = 0 \rightarrow D_y = 0 \]

\[ \sum_{\text{along } x} F_x = 0 \rightarrow F_B = 0 \]

\[ \sum_{\text{along } y} F_y = 0 \rightarrow V_C = 0 \]
Support Reactions for the Virtual System

1

8 ft

8 ft
Moment Diagram for the Virtual System

1

V_Q

0

1

M_Q

0
Evaluate the Virtual Work Product Integrals

\[ 1 \cdot \theta_C = \frac{1}{EI} \int_0^L M_Q M_P dx \]

Use Table to evaluate product integrals

\[ EI_{ABC} = 2,000,000 \text{ k-in}^2 \]
\[ EI_{CDE} = 800,000 \text{ k-in}^2 \]

\[ M_Q \]

\[ 8 \text{ ft} \quad 8 \text{ ft} \quad 8 \text{ ft} \quad 4 \text{ ft} \]

\[ M_P \]

\[ 24 \text{ k-ft} \quad -24 \text{ k-ft} \]

\[ -104 \text{ k-ft} \quad 6.5 \text{ ft} \]
Table to Evaluate Virtual Work Product Integrals

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<thead>
<tr>
<th>Appendix Table.2</th>
<th>Table is as useful tool to evaluate product integrals of the form:</th>
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</thead>
</table>

\[
\int_{0}^{L} M_Q M_P dx
\]

### Table

<table>
<thead>
<tr>
<th>Shape</th>
<th>Relationship</th>
<th>Formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangle</td>
<td>( M_0 )</td>
<td>( M_0 )</td>
</tr>
<tr>
<td>Triangle</td>
<td>( M_1 )</td>
<td>( \frac{1}{2} M_1 M_0 )</td>
</tr>
<tr>
<td>Triangle</td>
<td>( M_2 )</td>
<td>( \frac{1}{2} (M_1 + M_0) M_0 )</td>
</tr>
<tr>
<td>Triangle</td>
<td>( M_3 )</td>
<td>( \frac{1}{2} M_3 M_0 )</td>
</tr>
<tr>
<td>Triangle</td>
<td>( M_4 )</td>
<td>( \frac{1}{6} (M_1 + 2M_0) M_0 )</td>
</tr>
<tr>
<td>Triangle</td>
<td>( M_5 )</td>
<td>( \frac{1}{6} M_5 (L + a) )</td>
</tr>
<tr>
<td>Triangle</td>
<td>( M_6 )</td>
<td>( \frac{1}{6} M_6 (L + b) )</td>
</tr>
<tr>
<td>Parabola</td>
<td>( M_7 )</td>
<td>( \frac{1}{2} M_7 M_0 )</td>
</tr>
<tr>
<td>Parabola</td>
<td>( M_8 )</td>
<td>( \frac{1}{3} M_8 M_0 )</td>
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<tr>
<td>Parabola</td>
<td>( M_9 )</td>
<td>( \frac{1}{4} (M_1 + M_0) M_0 )</td>
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<tr>
<td>Parabola</td>
<td>( M_{10} )</td>
<td>( \frac{1}{12} (M_1 + 3M_0) M_0 )</td>
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<tr>
<td>Parabola</td>
<td>( M_{11} )</td>
<td>( \frac{1}{12} M_1 (3M_0 \frac{a^2}{L}) )</td>
</tr>
</tbody>
</table>
Evaluate Product Integrals

\[
\begin{align*}
\text{Evaluate Product Integrals} \\
\begin{array}{c}
\text{Evaluate Product Integrals} \\
\text{Evaluate Product Integrals} \\
\end{array}
\end{align*}
\]
Evaluate Product Integrals

\[
1 \cdot \theta_{c}^- = \frac{1}{EI} \int_{0}^{L} M_Q M_P \, dx
\]

Segment AX

\[
\int_{0}^{L_{ABC}} M_Q M_P \, dx = (-338 + 114 \text{ k-ft}^2) \left( \frac{12^2 \text{ in}^2}{\text{ft}^2} \right) = -32,256 \text{ k-in}^2
\]

Segment XC

\[
\int_{0}^{L_{CDE}} M_Q M_P \, dx = 0
\]

\[
\int_{0}^{L_{ABC}} M_Q M_P \, dx = (-338 + 114 \text{ k-ft}^2) \left( \frac{12^2 \text{ in}^2}{\text{ft}^2} \right) = -32,256 \text{ k-in}^2
\]
Evaluate Product Integrals

\[
\int_{0}^{L_{ABC}} M_Q M_P dx = (-338 + 114 \text{ k-} \text{ft}^2) \left( \frac{12^2 \text{ in}^2}{\text{ft}^2} \right) = -32,256 \text{ k-} \text{in}^2
\]

\[
\int_{0}^{L_{CDE}} M_Q M_P dx = 0
\]

\[
1 \cdot \theta_C^- = \frac{1}{EI_{ABC}} \int_{0}^{L_{ABC}} M_Q M_P dx + \frac{1}{EI_{CDE}} \int_{0}^{L_{CDE}} M_Q M_P dx
\]

\[
\theta_C^- = \frac{-32,256 \text{ k-} \text{in}^2}{2,000,000 \text{ k-} \text{in}^2} + \frac{0}{800,000 \text{ k-} \text{in}^2}
\]

\[
\theta_C^- = -0.0161 \text{ rad} + 0 = -0.0161 \text{ rad}
\]

\[
\theta_C^- = 0.0161 \text{ radians clockwise}
\]

Negative result, so rotation is in the opposite direction of the virtual unit moment.
Step 1 – Remove all loads and apply a virtual force (or moment) to measure the deformation at the point of interest

From an equilibrium analysis, find the internal bending moment function for the virtual system:

\[ M_Q(x) \]
Find the Moment Diagram for the Virtual System

\[ \sum M_A = 0 \implies M_A = -2 \]
\[ \sum F_x = 0 \implies A_x = 0 \]
\[ \sum F_y = 0 \implies A_y = 0.125 \text{ /ft} \]
\[ \sum M_C = 0 \implies D_y = -0.125 \text{ /ft} \]
\[ \sum F_x = 0 \implies F_B = 0 \]
\[ \sum F_y = 0 \implies V_C = 0.125 \text{ /ft} \]
Support Reactions for the Virtual System

2

A | B | C

0.125/ft

8 ft | 8 ft

0.125/ft

1

C | D | E

0.125/ft

8 ft | 4 ft
Moment Diagram for the Virtual System

- \( V_Q \): 0.125 /ft
- \( M_Q \): -2 to -1
- \( 0.125 /ft \)
- 8 ft sections
- Points A, B, C, D, E

+ forces and moments indicated
Moment Diagram for the Real System

104 k-ft

19 k

6 k

16 k

9 k

8 ft

8 ft

8 ft

4 ft

16 k

V_p

16 k

− 3 k

− 104 k-ft

24 k-ft

M_p

24 k-ft

− 24 k-ft

− 104 k-ft

+
Evaluate the Virtual Work Product Integrals

\[ 1 \cdot \theta_{c+} = \frac{1}{EI} \int_{0}^{L} M_Q M_P \, dx \]

Use Table to evaluate product integrals

- \( EI_{ABC} = 2,000,000 \text{ k-in}^2 \)
- \( EI_{CDE} = 800,000 \text{ k-in}^2 \)

- \( M_Q \) changes from -2 to -1.1875 to -1
- \( M_P \) changes from -104 k-ft to 24 k-ft to -24 k-ft

- Displacements:
  - A to B: 8 ft
  - B to C: 8 ft
  - C to D: 8 ft
  - D to E: 4 ft

- Use Table to evaluate product integrals.
### Table to Evaluate Virtual Work Product Integrals

**Appendix Table.2**

<table>
<thead>
<tr>
<th>$M_Q$</th>
<th>$M_I$</th>
<th>$M_1$</th>
<th>$M_2$</th>
<th>$M_L$</th>
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</thead>
<tbody>
<tr>
<td>$M_Q$</td>
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</tr>
<tr>
<td>$M_1$</td>
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<tr>
<td>$L$</td>
<td>$L$</td>
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</tr>
</tbody>
</table>

Table is as useful tool to evaluate product integrals of the form:

$$\int_0^L M_Q M_P dx$$
Evaluate Product Integrals Using the Table

\[ \frac{1}{6} (M_1 + 2M_2)M_3L \]

\[ \frac{1}{6} (-1.1875)(24)(9.5 + 8) \]

\[ \frac{1}{6} (-1.1875 + 2(-2))(-104)(6.5) \]

584.458 k-ft²

-83.125 k-ft²

32 k-ft²

0
Evaluate the Virtual Work Product Integrals

\[ 1 \cdot \theta_{c^+} = \frac{1}{EI} \int_0^L M_Q M_P \, dx \]

Segment AX
- 584.458 k-ft²

Segment XC
- 83.125 k-ft²

Segment CD
32 k-ft²

Segment DE
0

\[ \int_0^{L_{ABC}} M_Q M_P \, dx = (584.458 - 83.125 \text{ k-ft}^2) \left( \frac{12^2 \text{ in}^2}{\text{ft}^2} \right) = 72,191.95 \text{ k-in}^2 \]

\[ \int_0^{L_{CDE}} M_Q M_P \, dx = (32 \text{ k-ft}^2) \left( \frac{12^2 \text{ in}^2}{\text{ft}^2} \right) = 4608 \text{ k-in}^2 \]
Evaluate Product Integrals

\[
\int_{0}^{L_{ABC}} M_Q M_P \, dx = (584.458 - 83.125 \text{ k-ft}^2) \left(\frac{12^2 \text{ in}^2}{\text{ft}^2}\right) = 72,191.95 \text{ k-in}^2
\]

\[
\int_{0}^{L_{CDE}} M_Q M_P \, dx = (32 \text{ k-ft}^2) \left(\frac{12^2 \text{ in}^2}{\text{ft}^2}\right) = 4608 \text{ k-in}^2
\]

\[
1 \cdot \theta_{C^+} = \frac{1}{EI_{ABC}} \int_{0}^{L_{ABC}} M_Q M_P \, dx + \frac{1}{EI_{CDE}} \int_{0}^{L_{CDE}} M_Q M_P \, dx
\]

\[
\theta_{C^+} = \frac{72,191.95 \text{ k-in}^2}{2,000,000 \text{ k-in}^2} + \frac{4608 \text{ k-in}^2}{800,000 \text{ k-in}^2}
\]

\[
\theta_{C^+} = 0.0361 + 0.00576 \text{ rad} = 0.0419 \text{ rad}
\]

Positive result, so rotation is in the same direction of the virtual unit moment

\[
\theta_{C^+} = 0.0419 \text{ radians counter-clockwise}
\]
The overhanging beam shown has a fixed support at A, a roller support at C and an internal hinge at B. $EI_{ABC} = 2,000,000 \text{ k-in}^2$ and $EI_{CDE} = 800,000 \text{ k-in}^2$

For the loads shown, find the following:

1. The vertical deflection at point E;
2. The slope just to the left of the internal hinge at C;
3. The slope just to the right of the internal hinge at C

The beam deflection example results:

- $0.0419$ radians
- $0.903$ inches
- $0.0161$ radians