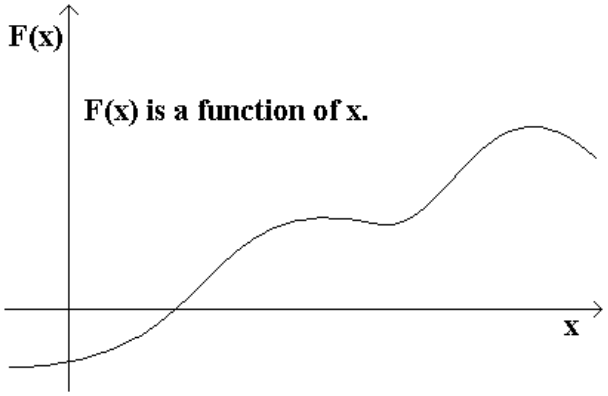
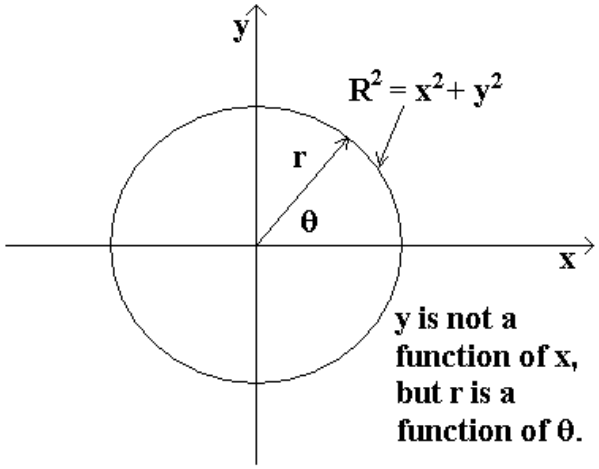
**Calculus for Physics 50 series** – an 8-page guide to concepts important for these courses.
By Todd Sauke

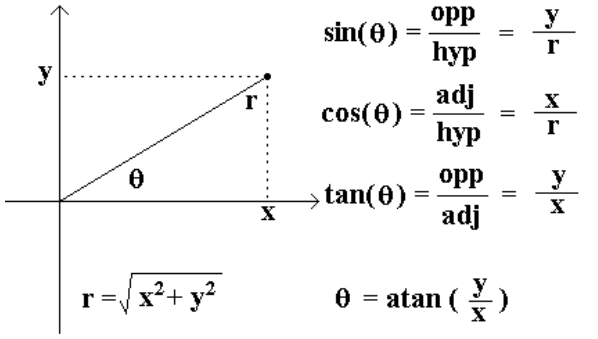
We start with the important concept of a **Function**. A function is a mathematical relation for which, if given an input value (the "argument"), the result (the "output" value of the function for that argument) is given by exactly one number. In the same way that we can refer generally to an arbitrary or unknown quantity as "x", we can refer generally to some arbitrary or unknown function using an abstraction like "F(x)". A function can be thought of as a mathematical **operation** (or "**rule**") that produces a single output number for any input number in the domain of the function. The specific relation, $F(x) = x^2 + 1$ is a function; for *every* value of the input, x, you can compute *exactly one* output value: $F(3) = 3^2 + 1 = 10$, for example. The input value of a function is also called an "**independent variable**". $f(x) = x^{1/2}$ (the square root of x) is *not* a function, since the result, for example, for x=4 gives *plus or minus* 2 (*two* possible results, rather than *exactly one* result. A function can be plotted as a graph showing a curve of *any* shape, as long as there is exactly one and only one value corresponding to each input value. We can also have functions with **multiple arguments** (or multiple independent variables), say a function of x, y, and z such that there is exactly one output result for any combination of input values x, y and z.

For example, $F(x, y, z) = 3 + 2xy^2 - z^4$ is a function of x, y and z. For x=1, y=3 and z=2, F(1, 3, 2) = 3 + 18 – 16 = 5, *exactly one* output number corresponding to the specific input values. Sometimes a curve on a graph, say the circle in the x,y plane given by the equation $x^2 + y^2 = R^2$, does not represent a function y(x). However, this same curve could be considered a function r(θ) by changing from **Cartesian coordinates** to **polar coordinates**. In that case, each value of θ corresponds to exactly one output value r.



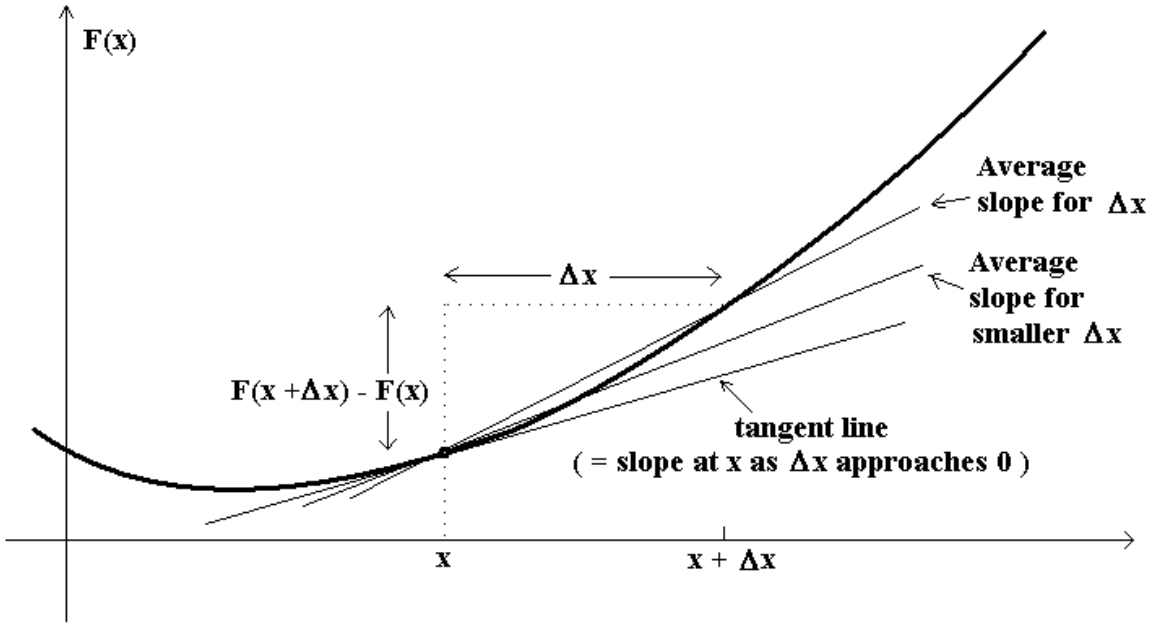

Trigonometry and polar coordinates




**Standard angle θ is measured counterclockwise from the + x axis**

A **vector function** is basically the same as a regular (scalar) function, except that for every input value (or combination of input values, for functions of several variables) the output is *exactly one vector* rather than exactly one number as for scalar functions. A vector, $\bar{\mathbf{v}}$, has a size (magnitude) and a direction and is written as a symbol name in **bold** type, often with a little arrow or line above it. A three-dimensional vector has three **components**, corresponding to the three spatial dimensions x, y, and z; the x, y, and z components of $\bar{\mathbf{v}}$ are written as $v_x$, $v_y$, and $v_z$, respectively. A **unit vector** is a vector of magnitude equal to one, whose only job is to specify a direction for a vector component. The unit vectors for specifying the x, y, and z directions are written as $\hat{\mathbf{i}}$ , $\hat{\mathbf{j}}$ , and $\hat{\mathbf{k}}$, or sometimes as $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ respectively. The vector, $\bar{\mathbf{v}}$, is written in component form as $\bar{\mathbf{v}} = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}}$ . A vector equation is just "shorthand" for three separate equations involving the respective components.

An important characteristic of a function is how much the output changes for a given change of input value. If you change the input from x to x+Δx (where Δx = "**delta-x**" represents some increment in the value of the input, x) what is the change in the output of the function? The change in output value [F(x+Δx) – F(x)] divided by the change in input value, Δx, is called the **average rate of change** of the function with respect to x. We also call this the **average slope** of the function, when drawn as a curve on a graph. If the independent variable is time, t, we can consider the output of the function as the location of an object, say x as a function of t, or x(t). If we consider the change of location corresponding to a change in time (t changing to t+Δt), we refer to the change of location, x(t+Δt) – x(t), divided by the change of time, Δt, as the **average velocity** of the object over the interval from t to t+Δt. If we imagine taking smaller and smaller increments, the average velocity typically gets closer and closer to a specific value. But we can't just let Δt go all the way to zero. Then the average velocity as defined above would be given by zero divided by zero, which is "nonsense". As long as we don't go "all the way" to zero for the increment Δt, we get closer and closer to a fixed value for the average velocity, and we want to introduce the concept of this "limiting value" as Δt gets smaller and smaller, but doesn't become exactly zero. We introduce the concept of a "limit" as a way of avoiding the "nonsense" of zero divided by zero. Zero divided by zero is an example of what is called an "**indeterminate form**". 0 ÷ 0 doesn't have any meaning as such, but we assign a meaning to the "limiting value" of the ratio of two quantities, each of which individually "approaches" zero.



Other indeterminate forms involve a special symbol, "$\infty$", which we call "**infinity**". The concept of infinity is that of a number that is bigger than any specific value you can name. "$\infty$" doesn't correspond to any real number, but we can often treat it as if it did. We say that $1/\infty = 0$, even though we can't really divide by a non-existent number. What we mean by $1/\infty$ is $1/$ (a number that is bigger than any specific value you can name), and this quotient becomes as close to zero as you want, closer to zero than any specific number you can name, however small. If the difference between two values is as small as you want, smaller than any specific number you can name, they are said to be equal "in the limit".

Other indeterminate forms, such as $\infty \div \infty$ , $\infty \cdot 0$, $\infty - \infty$, $0^0$, $1^\infty$, and $\infty^0$, can take on specific values "in the limit" as the individual parts "approach" the given values. We introduce the concept of the **derivative** of a function F(x) with respect to x, as the **limit** as $\Delta x$ approaches zero, of $[F(x+\Delta x) - F(x)] / \Delta x$. The derivative of F(x) with respect to x is the slope of the graph (or the slope of the "**tangent line**") of F(x) at the point F(x). Likewise, the derivative of the function x(t) with respect to t is the rate of change of x at time t (or velocity at time t). The process of taking a derivative is called **Differentiation**. We use a special notation to denote the derivative of a function with respect to an independent variable: "$dF(x)/dx$" means "the limit as $\Delta x$ approaches zero, of $[F(x+\Delta x) - F(x)] / \Delta x$ " or "the derivative of F(x) with respect to x". It is useful to consider the computing of a derivative as a mathematical operation, and the symbol $d/dx\,(\,)$ or $d(\,)/dx$ is said to **operate** on a function F(x) to produce another function, its derivative with respect to x. This notation metaphorically invokes the concept of the indeterminate form $0 \div 0$ involved in the limiting process of taking the derivative. A related notation, dF, refers to the "**differential**" of F, a "tiny" (or "**infinitesimal**") change in a quantity F, smaller than any specific change you could name. Together, these notations allow us to metaphorically divide by "infinitesimal" (differential) quantities. In this notation, we can "multiply and divide" by differential quantities as if they were real values without resulting in nonsensical indeterminate forms. For F(x), whose derivative is the function G(x), we have $dF(x)/dx = G(x)$, and we can symbolically "multiply both sides by" dx to obtain the differential of F(x), $dF(x) = G(x)\,dx$. Sometimes it is convenient to "unclutter" the notation and simply use a capital "D" to denote differentiation. In this notation, D[F(x)] means differentiation of F with respect to x. We also sometimes suppress the explicit functional dependence on the independent variable, and simply write $d/dx\,(F)$ $\left(\text{or } dF/dx \text{ or } D[F]\right)$ for $d/dx\,(F(x))$. Similarly, the differential of F is, $dF = G\,dx$. Some elementary derivatives are listed here, (the letters a and n represent constants).

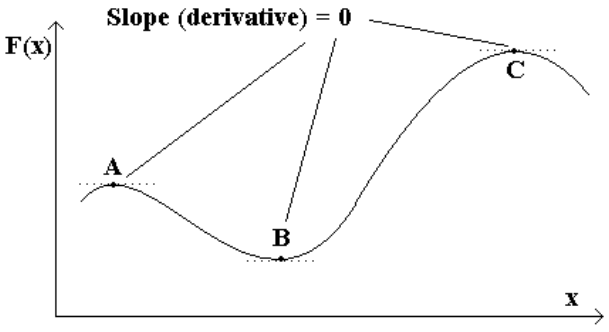
$$d/dx\,(a) = 0 \qquad d/dx\,(x) = 1 \qquad d/dx\,(x^2) = 2\cdot x \qquad d/dx\,(x^n) = n\cdot x^{n-1}$$

$$d/dx\,(\sin(x)) = \cos(x) \qquad d/dx\,(\cos(x)) = -\sin(x) \qquad d/dx\,(e^x) = e^x \qquad d/dx\,(\ln(x)) = 1/x$$

Of course, once we have taken the derivative of a function F(x), to obtain its derivative, the function G(x), we can repeat the process and take the derivative of that function. This is called taking the **second derivative**. The notation we use for the second derivative is:
$d^2(F(x)) / dx^2 = d(G(x)) / dx$. As many higher derivatives as we want could be taken, but in the Physics 50 series we usually limit ourselves to the second derivative.

If the graph of a smooth function F(x) has a peak at some location, we can see that the slope (derivative) of the function is zero there. Similarly, at the bottom of a valley in the function, the derivative is also zero. In an important application of Differential Calculus, we can determine the locations of maxima or minima of F(x), without plotting F, by computing its derivative function and determining where that function is zero. If the derivative of F(x) is G(x), we can set G(x) equal to zero and solve for the value(s) of x that satisfy the equation. We just have to be careful to distinguish the maxima and minima.

Maxima and minima have zero derivatives




In a famous example, an engineer wanted to optimize his design but in solving the equation for G(x) = 0 failed to notice that his selected solution was for the worst possible design (minimum point) rather than the best possible design (maximum point)! If you think about it you can see that even though both maxima and minima have the same zero derivative at their locations, the maximum points have a *negative* second derivative, while the minima have *positive* second derivatives.

Sometimes we have a "chain" of functional dependencies so that the input of one function comes from the output of some other function with respect to the independent variable. So, we could have F(x), a function of x, with F(x) being the input for another function H, so that H(x), the dependence of H on x, is H(x) = H(F(x)). The derivative $^{d}/_{dx}\,(H(x))$ is given by the "**Chain Rule**": $^{d}/_{dx}\,(H(\,F(x))) = {}^{d}/_{dF}\,(H(F)) \cdot {}^{d}/_{dx}\,(F(x))$. Don't let the "$^{d}/_{dF}\,(\,H(F)\,)$" confuse you just because it doesn't involve "x". The function H is just a "rule" for converting inputs to outputs. If the input is denoted as "F" you can take the derivative of H with respect to F just as you would take it with respect to x if the "rule" were operating on the input "x". So if the rule of H is to take the sine of the input, $^{d}/_{dF}\,(\,H(F)\,)$ is equal to $^{d}/_{dF}\,(\,\sin(F)\,) = \cos(F)$. (Just replace x with F in the "recipe" above.) So the chain rule can be used, for example, to obtain the derivative of $\sin(x^2)$. This is just a "nested" pair of functions; first square x (the first function, $F(x) = x^2$), and then take the sine of the result (the second function is H(F) = sin(F)). The chain rule gives:

$$^{d}/_{dx}\,(\,\sin(x^2)\,) = {}^{d}/_{dx}\,(\,H(F(x))\,) = {}^{d}/_{dF}\,(\,H(F)\,) \cdot {}^{d}/_{dx}\,(\,F(x)\,) = \cos(F) \cdot 2{\cdot}x = 2{\cdot}x \cdot \cos(x^2)$$

The chain rule can be extended to any depth of nested functions, so that if 4 function rules, [F, H, J, and K] are combined so that the dependence of F on x is F(x) = F(H(J(K(x)))), the derivative of F(x) with respect to x is:

$$^{d}/_{dx}\,(\,F(G(H(J(x))))\,) = {}^{d}/_{dH}\,(\,F(H)\,) \cdot {}^{d}/_{dJ}\,(\,H(J)\,) \cdot {}^{d}/_{dK}\,(\,J(K)\,) \cdot {}^{d}/_{dx}\,(\,K(x)\,).$$

Another important rule for differentiation is the **Product Rule**, which holds that the derivative of a product of two functions $^{d}/_{dx}\,(\,F \cdot H\,) = H \cdot {}^{dF}/_{dx} + F \cdot {}^{dH}/_{dx}$. This rule will be used repeatedly in any real-world application of taking derivatives. The Chain Rule and the Product Rule can be combined to prove the **Quotient Rule** of differentiation:

$$^{d}/_{dx}\,(\,F \div H\,) = (H \cdot {}^{dF}/_{dx} - F \cdot {}^{dH}/_{dx}\,) / H^2.$$

Differentiation is a **linear operation**. This means that all of the normal arithmetic and algebraic rules of Distributivity, Commutativity and Associativity apply. Therefore (taking a and b to be constants, and using the D[ ] notation for differentiation) we have that:

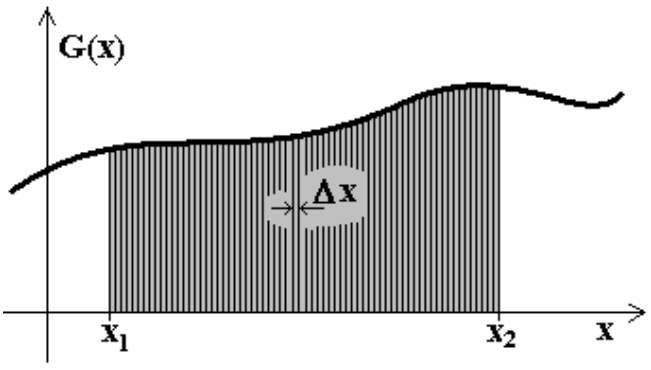
$D[FH] = D[HF]$ $D[F+H] = D[H+F]$ $D[a\cdot(F+H)] = a\cdot D[F+H] = a\cdot D[F] + a\cdot D[H]$
$D[a\cdot F + b\cdot H] = a\cdot D[F] + b\cdot D[H]$ (etc.)

So we can extend, or generalize, our list of elementary derivatives to include:

$$\frac{d}{dx}\left( a\cdot F(x)^n \right) = a\cdot n\cdot F(x)^{n-1}\cdot \frac{d}{dx}\left( F(x) \right) \quad \text{and} \quad \frac{d}{dx}\left( a\cdot e^{F(x)} \right) = a\cdot e^{F(x)}\frac{d}{dx}\left( F(x) \right) \quad \text{etc.}$$

Any time we have a mathematical operation, we can consider the "inverse" of that operation, an operation that "undoes" what the first operation did. The inverse of taking a derivative is often referred to as an "anti-derivative". If the derivative of $x^2$ is [2 times x], then the anti-derivative of [2 times x] is equal to $x^2$. But this anti-derivative is not unique. An equally good anti-derivative of [2 times x] could be $x^2 + 2$, or $x^2$ + any constant, since $x^2$ plus any constant has the same derivative, namely, [2 times x].

An important thing to do in math and science is to add up individual things to find a total. We denote the taking of this sum with the upper case Greek letter sigma, Σ. The sum of several items is written as $\Sigma_i\, y_i$. The letter i is the "index" distinguishing and enumerating the individual items to be summed. But a difficulty arises if the number of things that need to be added together is infinite. The **Integral Calculus** was invented to allow us to add together an infinite number of tiny (infinitesimal) pieces of something to get a correct final answer. This is another of our dreaded "indeterminate forms": $\infty \cdot 0$ . But we get around the "nonsense" of the indeterminate form in the same way we did in the definition of the derivative, using the same concept of the limit. If we consider the graph of a function G(x) and want to determine the total area under the curve of the plot, we conceptually "chop up" the area into a large number of tiny slices of width Δx and height G(x) and add up the areas of the individual slices to obtain an approximation of the total area. In the limit as Δx goes to zero we have an infinite sum of infinitesimal areas, and the limit of this sum is the area under the curve. The word we use for the process of adding together an infinite number of infinitesimal pieces is "**integration**", and this is the essence of the Integral Calculus. The notation for this integration is: $\int_{x_1}^{x_2} G(x)\ dx$ . Compare this to the notation for the discrete sum $\Sigma_i\, y_i$ . Instead of the Σ, we use the symbol $\int$ to indicate an infinite sum of infinitesimals. The independent variable, x, is used in place of i to distinguish and enumerate the infinite number of slices to be summed. The individual slices have a tiny (differential) width dx and a height given by the value of the function at x, G(x). The product $G(x)\cdot dx$ is the area of one such slice, and the integration process takes the sum of the infinitely many pieces, starting at $x_1$ on the left and ending at $x_2$ on the right. The values $x_1$ and $x_2$ are called the "**limits of integration**". The function G(x) is called the **integrand**.




The area under the curve of G(x), between the limits $x_1$ and $x_2$ is called the integral of G(x), and is equal, in the limit as Δx goes to zero, to the sum of the areas of all the little slivers of width Δx and height G(x). This sum is equal to the anti-derivative of G(x), evaluated between the limits as discussed in the main text.

In fact, the process of integration is the inverse of differentiation, and the integral of a function G(x) is computed by the use of the anti-derivative of G(x). How can you be convinced that this summation, or integration, is the same as the anti-derivative? Consider a simple case of a person saving money. Each week Bob puts some amount, say $7, into his piggy bank. The rate of increase of money in the bank in dollars per week is equal to the amount put in weekly, $7. The slope of the graph of the total dollars in the bank as a function of time is $7 per week. The total amount in the bank and the amount put in per week are related in the same way as the integrals and derivatives of functions. If the amount put in per week (which need not be constant) is a function called G(t), and the total amount in the bank is another function F(t), F(t) is the total (integral) of the deposits described by G(t). And the rate of increase of the total, the slope (derivative) of F(t), is the function G(t). G is the derivative of F, and F is the sum (integral) of G, so the integral must be the same as the anti-derivative. The final detail to take care of is to be careful about the starting value of the money in the bank. The summation only gives you the increment of the money in the bank, the difference between the amount at the end of some period compared to the amount at the beginning. We evaluate the anti-derivative at the final time (or state) minus the anti-derivative at the initial time (or state) to obtain the increment over the period. We have already seen that the anti-derivative of a function is not unique; any arbitrary constant can be added to it and it will still have the same derivative. (This arbitrary constant is called the "**constant of integration**".) But when we subtract the starting value of the anti-derivative from the ending value to compute the integral, this constant will subtract out, regardless of its value. So we don't have to worry at all about that little problem!

It's now time to dispense with the cumbersome term "anti-derivative" and start calling it by the conventional name, the "**indefinite integral**". If the indefinite integral of G(x) is the function F(x) (plus any arbitrary constant), the result of integrating G(x) (or computing the area under the curve of G) is indicated in this way: $\int_{x_1}^{x_2} G(x)\,dx = F(x)\Big]_{x_1}^{x_2} = F(x_2) - F(x_1)$. The "$\Big]$" is used to indicate that the function F(x) is to be evaluated at the indicated values of the independent variable, $x_1$ and $x_2$ with the difference taken as shown. This relationship between the integral of a function G(x) (the area under its curve) and its anti-derivative (indefinite integral) F(x) evaluated at the beginning and ending as shown, is called the **Fundamental Theorem of Integral Calculus**. The process of evaluating the indefinite integral at the specific beginning and ending points and taking the difference is what turns the indefinite integral into a **definite integral**.

Extensive tables of various functions and their indefinite integrals have been compiled, but it still frequently happens that the function we need to integrate in a specific application is not in the table. We need to develop various "tricks" and methods to help us put an integrand into a form that appears in the table. One such method is the **method of substitution**. I won't go into great detail other than to say that it is sometimes useful to make a substitution of one function for another to rewrite the integrand. For example, if we want to integrate $\int \tan(x)\,dx$ and have no idea where to start, we can rewrite tan(x) in the integrand as sin(x) / cos(x) and note that the product of sin(x) dx is minus the differential of cos(x), –d[cos(x)]. If we substitute the function y = cos(x) we see that $\int \tan(x)\,dx = \int [\sin(x) / \cos(x)]\,dx = \int -d[\cos(x)] / \cos(x) = -\int 1/y\,dy$. The indefinite integral of $1/y$ is known to be ln(y), so we have transformed a "hard" integral into an "easy" one by way of substitution. Another method is called "**integration by parts**". Without going into great detail, the basic result is that for functions u & v (suppressing the explicit dependence on x) $\int u\,dv = uv - \int v\,du + C$ for the indefinite integral (with arbitrary constant C

added) and $\int_{x_1}^{x_2} u\ dv = uv \Big]_{x_1}^{x_2} - \int_{x_1}^{x_2} v\ du$ for the definite integral, where du and dv are the differentials of the functions u and v, as explained above. This can be very helpful if $\int v\ du$ is in your table of integrals, but $\int u\ dv$ is not. For example, we may want to integrate $\int \ln(x)\ dx$ but have no idea how. If we choose functions u=ln(x) and v=x we have (taking the differentials of u and v) that du = d[ln(x)] = $^1/_x\ dx$ and dv = dx. Our desired integral has the form $\int u\ dv$ so we see that we can rewrite it as uv – $\int v\ du = x\cdot\ln(x) - \int {}^x/_x\ dx =$ $x\cdot\ln(x) - \int dx = x\cdot\ln(x) - x + C$ which was easy to solve. Again, all of the normal arithmetic and algebraic rules of Distributivity, etc., apply, and we have:

$$\int_{x_1}^{x_2} [\ F_1(x) + F_2(x)\ ]\ dx = \int_{x_1}^{x_2} F_1(x)\ dx + \int_{x_1}^{x_2} F_2(x)\ dx \qquad \& \qquad \int_{x_1}^{x_2} a \cdot F(x)\ dx = a \cdot \int_{x_1}^{x_2} F(x)\ dx$$

etc., and we say that "we can integrate the terms of an integrand separately" and "we can take any constant out of the integral".

Now, how do we deal with differentiation and integration for functions of more than one independent variable (**Multivariate Calculus**)? Consider some scalar function of independent variables x, y, and z, F(x,y,z). We want to know how the function changes as we let one or the other of the independent variables change. The principle is straightforward. To find out how much a multivariate function changes as we change *one* of the independent variables, we don't allow the others to vary. This is called taking a **partial derivative**, and we use a special notation. $\partial F/\partial x$ denotes the partial derivative of F with respect to x. The partial derivative with respect to x is the same as the regular derivative, but we "pretend" for the time being that the other independent variables are constants. So the partial derivative of $F(x,y,z) = 2xy - 3\ z\ x^2$ with respect to x is $\partial F/\partial x = 2y - 6\ z\ x$. The partial derivative of F(x,y,z) with respect to y is $\partial F/\partial y = 2x - 0 = 2x$, and the partial derivative of F(x,y,z) with respect to z is $\partial F/\partial z = -3\ x^2$. If you follow the details, you will see that in each case we temporarily treated the "other" (non-subject) independent variables as constants for the purpose of taking the indicated partial derivative. Each partial derivative is the corresponding component of a vector function called the **gradient** of the function F. The gradient of F is written as:

Gradient(F) = $\bar{\nabla}(F) = \left( {}^{\partial F}/_{\partial x}\ \hat{\imath} + {}^{\partial F}/_{\partial y}\ \hat{\jmath} + {}^{\partial F}/_{\partial z}\ \hat{k} \right)$ (the $\bar{\nabla}$ symbol is called the "**del**" operator).

We also can integrate functions of multiple independent variables. Again, the basic principle is easy, and the difficulties are only in the details. In the same way that an integral of a function of one variable is interpreted as the area under the curve, the integral of a function of two variables, say x and y, is interpreted as the volume under a two-dimensional surface. We just have to "chop up" the volume in both the x and y dimensions into little (infinitesimally thin) "matchsticks" of width dx, breadth dy, and height F(x,y) and add up all of the little (infinitely many) pieces to get the final answer. The notation is similar and looks as follows:

$\int_{y_1}^{y_2} \int_{x_1}^{x_2} F(x,y)\ dx\ dy$ and the interpretation is similar. The volume of the individual matchsticks is the product of F(x,y) (the height), multiplied by the width, dx, and the breadth dy.

We do each integral sequentially in nested fashion: $\int_{y_1}^{y_2} \int_{x_1}^{x_2} F(x,y)\ dx\ dy = \int_{y_1}^{y_2} \left( \int_{x_1}^{x_2} F(x,y)\ dx \right) dy$

where the integral in parentheses is done just as before with the only change being that we

temporarily treat "y" as a constant for the time being. In fact, the limits $x_1$ and $x_2$ can be explicit functions of y, and we just treat them symbolically as if y is a constant. When we're done with the integral in parentheses, we proceed to do the next integral with respect to the next independent variable. The process can be extended indefinitely with as many repeated integrals of as many independent variables as we want. Each integral just treats the other variables as constants for the time being.

The final subject is that of **Differential Equations**. A differential equation is an equation involving the value of a function, say F(x), in addition to one or more of its derivatives. The simplest differential equation of them all is $F(x) = {}^{dF(x)}/_{dx}$ . This equation can't be true for just any old function F(x). If $F(x) = 2x^2$, for example, we have the derivative of F(x) = 4x. The equation says that the function is equal to its derivative, but $2x^2$ is not equal to 4x. Whenever we have a differential equation, we will want to *find* some function that satisfies the equation. We will also need to satisfy one or more "**boundary conditions**" which specify what the function is doing at some boundary, say $x_0$. In the example above, the boundary condition might be that the function must have a slope of – 3 at x=0. How do we go about this? The uncomfortable situation is that we must *guess* the answer. But the good news is that there are many ways to be smart about how to guess an answer. Once we guess an answer, it is easy to plug it into the differential equation to test if the equation and boundary condition(s) are satisfied. The even better news is that once we verify that a guess satisfies the equation, we can be sure that we have found the *one and only* true solution. This is because of a neat little theorem called the **Uniqueness Theorem**, which states that there can be only one solution to a differential equation. For our example above, since we know that the derivative of $e^x$ is equal to the same function $e^x$, we will guess a solution of the form $F(x) = c \cdot e^x$ . We know that ${}^{dF(x)}/_{dx} = c \cdot e^x$ . In order for the slope, ${}^{dF(x)}/_{dx} = c \cdot e^x$ to equal –3 at x=0, we must have that c = –3. Now we're done. Our guess, with c set equal to –3 as we have determined is necessary, is that $F(x) = -3\, e^x$ and it is verified that this solution satisfies the differential equation as well as the boundary condition, so it is the one true answer. We have solved this simplest of differential equations.

The primary differential equation we will encounter in Physics 50 is a second order differential equation with respect to time, t: $d^2( x(t) ) / dt^2 = -({}^{k}/_{m})\, x(t)$ . This equation describes the position as a function of time, x(t), of a mass m on a spring with spring constant k, and we see that the second derivative of x with respect to t is proportional to the negative of x. We know that the derivative of sin(t) is cos(t) and the derivative of cos(t) is –sin(t), which gets us back to the same function, sin(t) with the required minus sign, so we can guess a solution of the form $x(t) = A \sin(\omega t + c)$. We can use the chain rule to find that the second derivative of x(t) is $-A\, \omega^2 \sin(\omega t + c)$. We plug these functions into the equation and see that the only way for the equation to be satisfied is if $\omega^2 = {}^{k}/_{m}$, or $\omega = \sqrt{{}^{k}/_{m}}$ . A and c can be anything, but some boundary conditions on the situation, such as the initial position and velocity of the motion will uniquely determine the solution. We are done. (You may object that you can find a function involving $\cos(\omega t + c_2)$ that also solves the equation, so the solution must not be unique. But the value of $c_2$ you find that satisfies the initial condition will be such that the cosine solution and the sine solution are exactly equivalent.) It is an important and interesting fact that the angular frequency, ω, of the motion is completely determined by m and k, and has nothing to do with the boundary (initial) conditions. In the physics 50 series, we will only encounter differential equations for which the form of the answer (the correct "guess" of the solution) is provided. We can then do the easy part, plugging it in to verify the equation and relate the coefficients to the boundary (or initial) conditions.