Radiation Heat Transfer in Participating Media

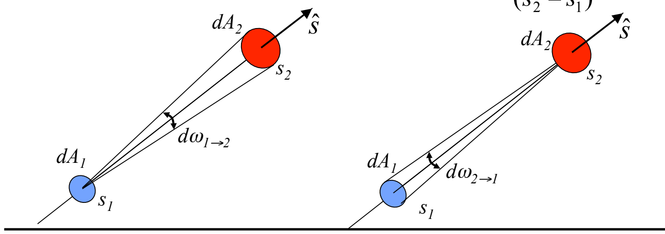
Radiation Intensity in Vacuum

- Radiation energy passing at normal angle through area dA_1 over the time interval dt and wavelength range $d\eta$ that will reach area dA_2 at normal angle is

$$I_\lambda(s_1,t_1)dA_1 dt d\lambda d\omega_{1\to 2} = I_\lambda(s_1,t_1)dA_1 dt d\lambda \frac{dA_2}{(s_2-s_1)^2}$$

- This radiation goes through dA_2 at time $t_2=t_1+(s_2-s_1)/c$ where c is the speed of light.

$$I_\lambda(s_2,t_2)dA_2 dt d\lambda d\omega_{2\to 1} = I_\lambda(s_2,t_2)dA_2 dt d\lambda \frac{dA_1}{(s_2-s_1)^2}$$



Radiation Intensity in Vacuum

- Equating these two equations gives

$$I_\lambda(s_2,t_2) = I_\lambda(s_1,t_1)$$

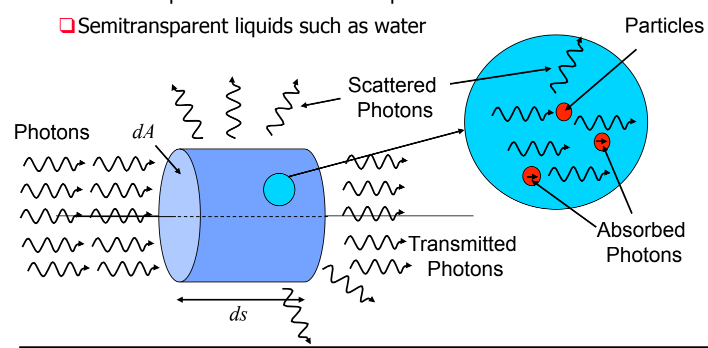
- Since c is very large, $t_1 \approx t_2$ and

$$I_\lambda(s_2) = I_\lambda(s_1)$$

- Radiation intensity in vacuum, along any given direction, is constant along its path.

Participating Medium

- A medium that absorbs, emits and scatters radiation is called a participating medium.
  - Water vapor, CO_2, CO, CH_4, combustion fuels, fog and cloud
  - Semitransparent solids such as doped silicon
  - Semitransparent liquids such as water



Absorption

- The absolute amount of the radiation absorption is directly proportional to the magnitude of the incident energy and the distance the radiation beam travels through the medium;

$$\boxed{(dI_\lambda)_{abs} = -\kappa_\lambda I_\lambda ds.}$$

  - κ_λ is called the absorption coefficient of the medium.
  - It depends on temperature and the number of molecules per unit volume.
- Integrating this equation from 0 to s gives

$$I_\lambda(s) = I_\lambda(0)e^{-\tau_\lambda} \text{ where } \tau_\lambda = \int_0^s \kappa_\lambda ds \quad \text{is called optical thickness.}$$

- Absorptivity of the medium for the path from 0 to s is

$$\alpha_\lambda \equiv \frac{I_\lambda(0) - I_\lambda(s)}{I_\lambda(0)} = 1 - e^{-\tau_\lambda}.$$

Out-Scattering

- A part of the incoming radiation intensity is removed from the direction of propagation;

$$\boxed{(dI_\lambda)_{out-sca} = -\sigma_{s\lambda} I_\lambda ds.}$$

  - $\sigma_{s\lambda}$ is called the scattering coefficient into all directions.
  - It depends on temperature and the number of molecules per unit volume.
- Total attenuation of the radiation intensity by both absorption and scattering is known as extinction.
- The extinction coefficient is defined as

$$\beta_\lambda = \kappa_\lambda + \sigma_{s\lambda}.$$

Emission

- The rate of emission from a volume element is proportional to the magnitude of the volume.
- The emitted intensity along any path is proportional to
  - the length of the path and,
  - the local energy content in the medium.

$$\boxed{(dI_\lambda)_{em} = \kappa_\lambda I_{b\lambda} ds}$$

- Note that the proportionality constant for emission, κ_λ, is the same as for absorption.

Absorbing, Emitting and Non-Scattering Medium

- Radiation Emission: $(dI_\lambda)_{em} = \kappa_\lambda I_{b\lambda} ds$
- Radiation Absorption: $(dI_\lambda)_{abs} = -\kappa_\lambda I_\lambda ds$
- Combining the two gives $\dfrac{dI_\lambda}{ds} = \kappa_\lambda (I_{b\lambda} - I_\lambda)$
- The solution of this equation for an isothermal gas layer of thickness s is

$$I_\lambda(s) = I_\lambda(0)e^{-\tau_\lambda} + I_{b\lambda}(1 - e^{-\tau_\lambda})$$

- If only internal emission is considered, $I(0)=0$ and

$$I_\lambda(s) = I_{b\lambda}(1 - e^{-\tau_\lambda}) \quad \Rightarrow \quad \varepsilon_\lambda \equiv \frac{I_\lambda(s)}{I_{b\lambda}} = 1 - e^{-\tau_\lambda}$$

In-Scattering

- Radiation intensity can be augmented by in-scattering from all directions.
- Consider the radiation heat flux impinging on a volume element $dV=dAds$, from direction $\mathbf{s_i}$.
- The total spectral radiation heat flux impinging on area dA from direction $\mathbf{s_i}$ is

$$I_\lambda(\hat{s}_i)(dA\hat{s}_i \cdot \hat{s})d\omega_i d\lambda.$$

- Note that $\hat{s}_i \cdot \hat{s} = \cos\theta$, where θ is the angle between directions $\mathbf{s_i}$ and $\mathbf{s}$.
- The part of this energy which is scattered away from $\mathbf{s_i}$ while passing through the volume element is

$$\sigma_{s\lambda}\left[I_\lambda(\hat{s}_i)(dA\hat{s}_i \cdot \hat{s})d\omega_i d\lambda\right]\frac{ds}{\hat{s}_i \cdot \hat{s}} = \sigma_{s\lambda}I_\lambda(\hat{s}_i)dAd\omega_i d\lambda ds.$$

Scattering Phase Function

- The probability that the radiation ray from one direction, $\mathbf{s_i}$, is scattered into a certain other direction, $\mathbf{s}$, is described by scattering phase function; $\Phi_\lambda(\mathbf{s_i}, \mathbf{s})$.
- The amount of radiation energy from the cone $d\omega_i$ that is scattered into the cone $d\omega$ is

$$\sigma_{s\lambda}I_\lambda(\hat{s}_i)dAd\omega_i d\lambda ds \Phi_\lambda(\hat{s}_i,\hat{s})\frac{d\omega}{4\pi}.$$

- The total spectral radiation energy that is scattered from the cone $d\omega_i$ to all directions is

$$\int_{4\pi}\sigma_{s\lambda}I_\lambda(\hat{s}_i)dAd\omega_i d\lambda ds \Phi_\lambda(\hat{s}_i,\hat{s})\frac{d\omega}{4\pi} = \sigma_{s\lambda}I_\lambda(\hat{s}_i)dAd\omega_i d\lambda ds \int_{4\pi}\Phi_\lambda(\hat{s}_i,\hat{s})\frac{d\omega}{4\pi}.$$

- Therefore, $\quad \dfrac{1}{4\pi}\displaystyle\int_{4\pi}\Phi_\lambda(\hat{s}_i,\hat{s})d\omega = 1$

In-Scattered Radiation

- The total spectral radiation energy that is scattered is direction $\mathbf{s}$ into the cone $d\omega$ from all directions is

$$\int_{4\pi}\sigma_{s\lambda}I_\lambda(\hat{s}_i)dAd\omega_i d\lambda ds \Phi_\lambda(\hat{s}_i,\hat{s})\frac{d\omega}{4\pi}.$$

- This is equal to the scattered radiation flux around the wavelength $d\lambda$ that impinges upon area dA and passes through the solid angle $d\omega$;

$$(dI_\lambda)_{in-sca}(\hat{s})dAd\omega d\lambda$$

- Therefore, the amount of in-scattering into direction $\mathbf{s}$ from all directions is

$$\boxed{(dI_\lambda)_{in-sca}(\hat{s}) = ds\frac{\sigma_{s\lambda}}{4\pi}\int_{4\pi}I_\lambda(\hat{s}_i)\Phi_\lambda(\hat{s}_i,\hat{s})d\omega_i.}$$

The Equation of Transfer

- Combining equations for absorption, emission, in-scattering and out-scattering, the equation of transfer of radiation intensity in as participating medium is

$$dI_\lambda = (dI_\lambda)_{em} + (dI_\lambda)_{abs} + (dI_\lambda)_{out-sca} + (dI_\lambda)_{in-sca}(s).$$

$$\frac{dI_\lambda}{ds} = \kappa_\lambda I_{b\lambda} - (\kappa_\lambda + \sigma_{s\lambda})I_\lambda + \frac{\sigma_{s\lambda}}{4\pi}\int_{4\pi}I_\lambda(\hat{s}_i)\Phi_\lambda(\hat{s}_i,\hat{s})d\omega_i.$$

- This is a first-order integro-differential equation for radiation intensity in direction $\mathbf{s}$.
- A boundary condition, such as radiation intensity in direction $\mathbf{s}$ on a surface of an enclosure that surrounds the medium, is needed for complete solution of this equation.

The Equation of Transfer in Optical Coordinates

- The non-dimensional optical thickness is defined as

$$\tau_\lambda = \int_0^s (\kappa_\lambda + \sigma_{s\lambda}) ds = \int_0^s \beta_\lambda ds.$$

- The single scattering albedo is defined as

$$\omega_\lambda = \frac{\sigma_{s\lambda}}{\kappa_\lambda + \sigma_{s\lambda}} = \frac{\sigma_{s\lambda}}{\beta_\lambda}.$$

- It can be shown that the equation of transfer of radiation intensity in terms on optical thickness and single scattering albedo becomes

$$\frac{dI_\lambda}{d\tau_\lambda} + I_\lambda = (1-\omega_\lambda) I_{b\lambda} + \frac{\omega_\lambda}{4\pi}\int_{4\pi} I_\lambda(\hat{s}_i)\Phi_\lambda(\hat{s}_i,\hat{s})d\omega_i.$$

Non-Scattering Medium

- For a non-scattering medium, $\omega_\lambda=0$, and the equation of transfer of radiation intensity becomes

$$\frac{dI_\lambda}{d\tau_\lambda} + I_\lambda = I_{b\lambda}$$

- The solution of this equation can be written as

$$I_\lambda(\tau_\lambda) = I_\lambda(0)e^{-\tau_\lambda} + e^{-\tau_\lambda}\int_0^{\tau_\lambda} I_{b\lambda}(\tau'_\lambda)e^{\tau'_\lambda} d\tau'_\lambda.$$

Radiation Heat Flux Vector

- The rate of incident and outgoing spectral radiation heat transfer through an area dA_1 is

$$dq_\lambda = I_\lambda(\hat{s})dA_1\cos\theta d\omega$$

where $\cos\theta = \hat{s}\cdot\hat{n}$.

- The spectral radiation heat flux is

$$q''_\lambda = \int_{4\pi} I_\lambda(\hat{s})\hat{s}\cdot\hat{n}\, d\omega.$$

- The total radiation heat flux is

$$q'' = \int_0^\infty\int_{4\pi} I_\lambda(\hat{s})\hat{s}\cdot\hat{n}\, d\omega d\lambda.$$

- The spectral and total radiation heat flux vectors are

$$\overrightarrow{q''_\lambda} = \int_{4\pi} I_\lambda(\hat{s})\hat{s}d\omega. \qquad \overrightarrow{q''} = \int_0^\infty\int_{4\pi} I_\lambda(\hat{s})\hat{s}d\omega d\lambda.$$

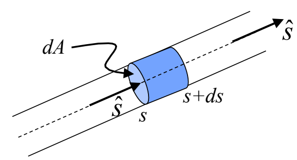
Conservation of Radiation Energy

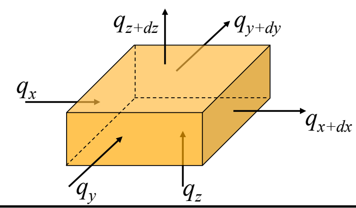
- The equation of transfer is a radiation balance for an infinitesimal pencil of rays:

$$\frac{dI_\lambda}{ds} = \kappa_\lambda I_{b\lambda} - (\kappa_\lambda + \sigma_{s\lambda})I_\lambda + \frac{\sigma_{s\lambda}}{4\pi}\int_{4\pi} I_\lambda(\hat{s}_i)\Phi_\lambda(\hat{s}_i,\hat{s})d\omega_i.$$

- If this equation is integrated over all solid angles, an spectral radiation balance for an infinitesimal volume is obtained:

$$\nabla\cdot q''_\lambda = \kappa_\lambda(4\pi I_{b\lambda} - G_\lambda) \quad \text{where} \quad G_\lambda = \int_{4\pi} I_\lambda d\omega$$





Conservation of Radiation Energy

- A total radiation balance is obtained by integrating this over all wavelengths:

$$\nabla \cdot q'' = \int_0^\infty \kappa_\lambda \left(4\pi I_{b\lambda} - G_\lambda\right) d\lambda$$

- For the special case of a gray medium where $\kappa_\lambda = \kappa$,

$$\nabla \cdot q'' = \kappa\left(4\sigma T^4 - G\right)$$

Molecular Internal Energy

- The internal energy of atoms and molecules consists of
  - Energy of electrons spinning around the nucleus.
  - Energy of atoms spinning around one another.
  - Energy of atoms within a molecule vibrating against each other.
- Energy levels for electron orbits and molecular rotation and vibration are quantized.
- The energy of a photon is proportional to its frequency:
  - It must have a certain frequency to be captured or released.
  - Spectral lines for absorption and emission are discrete.

Change in Molecular Internal Energy

- Changing the orbit of an electron requires a relatively large amount of energy on the order of several eV (1.602×10^{-19} J).
  - Absorption and emission of high frequency photons
  - Short wavelength spectral lines between UV and near IR (10^{-2} – 1.5 μm)
- Vibrational energy level changes require energy levels on the order of 0.1 eV.
  - Spectral lines in the IR (1.5 – 20 μm)
- Changes in rotational energy levels require small amount of energy on the order of 10^{-3} eV.
  - Spectral lines in the far IR (larger than 20 μm)

Rotational Spectra

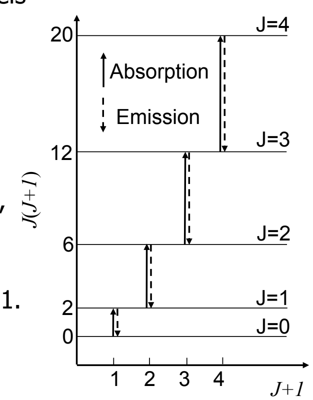
- The possible rotational energy levels for a diatomic molecule, based on the rigid-rotator model, are

$$E_J = \frac{h^2}{8\pi^2 I} J(J+1)$$

where h is Plank constant, I is the moment of inertia of the molecule, and J is the rotational quantum number.

- The allowed transitions are $\Delta J = \pm1$.
- The frequency of the absorbed or emitted photon is

$$\nu_{J\to J+1} = \frac{E_{J+1} - E_J}{h} = \frac{h}{4\pi^2 I}(J+1)$$



Vibrational Spectra

- The possible vibrational energy levels for a diatomic molecule, based on the harmonic oscillator model, are

$$E_\upsilon = (\upsilon + \frac{1}{2}) h\nu_0$$

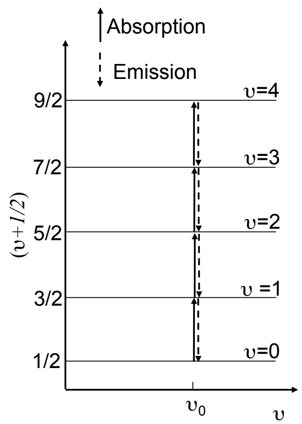
where υ is the vibrational quantum number and

$$\nu_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m'}} \quad \text{where} \quad m' = \frac{m_1 m_2}{m_1 + m_2}$$

is the equilibrium frequency of oscillation of such a two-body oscillator.

- The allowed transitions are $\Delta\upsilon = \pm 1$.
- The frequency of the absorbed or emitted photon is

$$\nu_\upsilon = \frac{E_{\upsilon+1} - E_\upsilon}{h} = \nu_0$$



Vibrational-Rotational Spectra

- Vibrational energy changes always happen together with many simultaneous rotational energy changes.
- The combined rotational and vibrational energy level of a diatomic molecule is

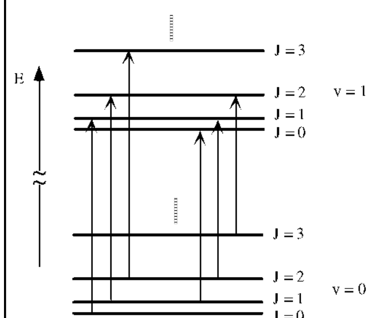
$$E_J = \frac{h^2}{8\pi^2 I} J(J+1) + (\upsilon + \frac{1}{2}) h\nu_0$$

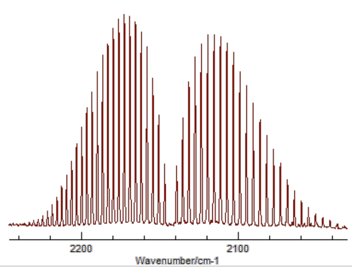
- The allowed transitions are $\Delta\upsilon = \pm 1$ and $\Delta J = \pm 1$ and 0.
- The frequency of the spectral lines for $\upsilon = 0$ to $\upsilon = 1$ is

$$\nu = \nu_0 - \frac{h}{4\pi^2 I} J \quad \text{if} \quad \Delta J = -1$$

$$\nu = \nu_0 + \frac{h}{4\pi^2 I}(J+1) \quad \text{if} \quad \Delta J = 1$$

Vibrational-Rotational Spectra





Vibrational-Rotational Spectra of CO

Total Emissivity and Absorptivity of CO_2 and H_2O

- Total emissivity of CO_2 or H_2O at very low partial pressure p_a , total pressure of p = 1 bar, and length L is

$$\varepsilon_0(p_a L, p = 1\,\text{bar}, T_g) = \exp\left[\sum_{i=0}^{M}\sum_{j=0}^{N} c_{ji}\left(\frac{T_g}{T_0}\right)^j \left(\log_{10}\frac{p_a L}{(p_a L)_0}\right)^i\right]$$

- The emissivity at different pressure p is

$$\frac{\varepsilon(p_a L, p, T_g)}{\varepsilon_0(p_a L, p = 1\,\text{bar}, T_g)} = 1 - \frac{(a-1)(1-P_E)}{a+b-1+P_E}\exp\left[-c\left(\log_{10}\frac{(p_a L)_m}{p_a L}\right)^2\right]$$

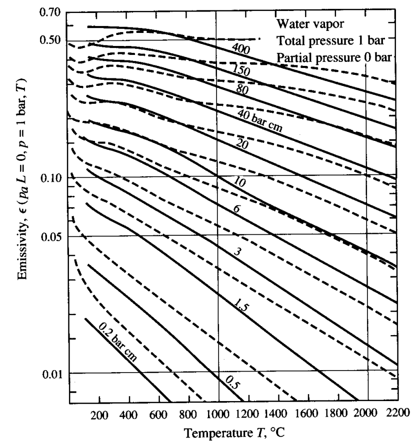
- The absorptivity at pressure p is

$$\alpha = \left(\frac{T_g}{T_s}\right)^{1/2} \varepsilon_0(p_a L \frac{T_g}{T_s}, p = 1\,\text{bar}, T_s)\left(\frac{\varepsilon}{\varepsilon_0}\right)$$

Correlation Constants for Water Vapor

M, N	2,2
$c_{00} \ \cdots \ c_{0M}$ $\vdots \ \ddots \ \vdots$ $c_{N0} \ \cdots \ c_{NM}$	$-2.2118 \quad -1.1987 \quad 0.035596$ $0.85667 \quad 0.93048 \quad -0.14391$ $-0.10838 \quad -0.17156 \quad 0.045915$
P_E	$(p + 2.56 p_a/\sqrt{t})/p_0$
$(p_aL)_m/(p_aL)_0$	$13.2t^2$
a	$2.479, \quad t < 0.75$ $1.888 - 2.053 \log_{10} t, \quad t > 0.75$
b	$1.10/t^{1.4}$
c	0.5
$T_0 = 1000\,\text{K}$,	$p_0 = 1\,\text{bar},\ t = T/T_0,\ (p_aL)_0 = 1\,\text{bar cm}$

Total Emissivity of Water Vapor



Correlation Constants for Carbon Dioxide

M, N	2,3
$c_{00} \ \cdots \ c_{0M}$ $\vdots \ \ddots \ \vdots$ $c_{N0} \ \cdots \ c_{NM}$	$-3.9893 \quad 2.7669 \quad -2.1081 \quad 0.39163$ $1.2710 \quad -1.1090 \quad 1.0195 \quad -0.21897$ $-0.23678 \quad 0.19731 \quad -0.19544 \quad 0.044644$
P_E	$(p + 0.28 p_a)/p_0$
$(p_aL)_m/(p_aL)_0$	$0.054/t^2, \quad t < 0.7$ $0.225t^2, \quad t > 0.7$
a	$1 + 0.1/t^{1.45}$
b	0.23
c	1.47
$T_0 = 1000\,\text{K}$,	$p_0 = 1\,\text{bar},\ t = T/T_0,\ (p_aL)_0 = 1\,\text{bar cm}$

Total Emissivity of Carbon Dioxide