“An Introduction to Spectral Analysis and Linear Systems”

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This proposed book has come from the cumulative experience of teaching EE112 and EE 200, Undergraduate and Graduate Linear Systems courses at San Jose State University, over an extended period of time. It represents the way that I have come to tell the story of linear systems and spectral analysis to students who have had a course in differential equations and an introductory course in circuit analysis in their second year and to graduate students with some background in linear systems.

There are a variety of ways to tell this story. I have come to the conclusion that the story is most easily and effectively told by dealing with continuous time analysis first, followed by discrete time analysis.

Electrical Engineering students enter their third year with some knowledge of differential equations, Laplace Transforms, and circuit analysis. My book takes that experience as a given and begins with a review and extension of those ideas. Concentrating on the Theorems of Laplace Transforms, the stage is set for extending the very same Theorems in subsequent chapters to Fourier Series and then to Fourier Transforms. With this approach, the internal unity of these three analytical tools is established and reestablished. Having accomplished this, it is straightforward to extend these ideas to discrete – time systems via the z – transform and to the Discrete Time Fourier Transform.

The relationship of duality between the time domain and the frequency domain is a core issue for both linear systems and for spectral analysis. That relationship is first asserted by establishing that the transfer function H(s) and impulse response of a linear system h(t) are a Laplace Transform pair and that multiplication and convolution are dual operations in both time and frequency. This central idea is reinforced in the subsequent discussions of Fourier Series and Transforms. Similarly, the core ideas of linearity and superposition are not simply asserted; they arise from the discussion of the Laplace Transform of linear differential equations.

This book attempts to deal with the question of the “meaning” of the spectrum of a signal. The “classic” approaches of Fourier Series and Transforms yield results that, under close scrutiny, can be very ambiguous and elusive. The Fourier Series analysis of a periodic signal yields a definitive analysis of an idealized signal that lasts forever. The Fourier Transform of a real signal requires processing the signal for its entire duration before obtaining a result and that result is global rather than local.

The STFT (Short Time Fourier Transform) examines successive finite time intervals of a signal, but then runs into the problem that the finite processing time inherently produces ambiguity or fuzziness in the result. This is where spectral analysis runs headlong into the Heisenberg Uncertainty Principle.

At this point wavelets and MRA (Multiresolution Analysis) come into the picture as an alternative approach to analyzing signals. The objective of the book is to introduce some of the core issues of how continuous and discrete wavelets work and how they arise from Fourier Analysis. A deeper discussion of their structure and applications is more properly left to a more focused course.

I also think that it is important to provide many substantive applications and examples that illustrate and extend the analytic skeleton of the mathematics. Although the core material of a Linear Systems course is directly connected to communications, control, and filter design, existing textbooks too often do not make those connections in a substantive manner. Although the mathematical core of this course is relatively dense and the instructor will typically not have the time to explore a lot of applications, I think that it is important that these extensions be presented so that some of them can become part of the curriculum. I have come to the conclusion that many of the mathematical operations of signals and linear systems will be opaque to students unless they are reinforced by applications oriented examples.

In my experience, Electrical Engineering students in this introductory Linear Systems course best learn the mathematical abstractions when these abstractions are firmly tied to a discussion of how they extend to the workings of actual systems.

Similarly, I value the use of MATLAB in this course but I do not accept building the course around the many operations and functions of MATLAB. I recognize that acquisition of MATLAB skills are increasingly important for Electrical Engineers, but I think that it is a mistake to have that goal take the place of dealing with the very important mathematics of spectral analysis.

The accompanying Table of Contents represents a completed draft of each of the chapters except for the last.

I think that any textbook should provide some degree of flexibility about designing a course. Chapters 2 and 3 on Laplace Transforms and their applications can be made more or less selective depending upon the background of the students and the interests of the instructor. Chapter 7 on State Space approaches may be skipped without loss of continuity and Chapter 9 on Discrete Wavelets may be postponed to another course.

The extension to the Discrete Fourier Transform not only develops the Sampling Theorem and helps to clarify the relationship between the Fourier Series and Transform, it also provides some basic information that students need to know in order to get along in the world of digital signal processing.

I will freely admit to the common practice of including far more material in a textbook than can be accommodated in a single course. In part, this can be explained by the desire to provide flexibility. But it is necessary to concede that part of the process is the ubiquitous desire of professors to "get it all in".

Either way, this is my proposal and I welcome criticism and suggestion for improvement.

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Sections marked with an asterisk (*) can be skipped with no loss of continuity

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Chapter 1

Introduction

1.1 The Idea of Spectral Analysis

The title of this book, “An Introduction to Spectral Analysis and Linear Systems”, was not chosen at random. It was chosen in order to convey the unity of a class of signals, sinusoids and sums of sinusoids, and the class of systems that are represented by linear differential equations having constant coefficients (abbreviated as LDE’s). This unity expresses itself mathematically in many ways and all of these ways are significant in a wide variety of engineering applications.

As we shall see in the text, this material is central to all of communications engineering (radio, television, audio, radar, and even smartphones) as well as electronic instrumentation of all sorts. It has application in areas as diverse as geology, specch recognition and synthesis, vibration analysis, and image processing.

We could start this discussion by defining spectral analysis as the decomposition of an arbitrary signal – subject to a collection of seemingly strange mathematical conditions – into a summation of scaled orthogonal basis functions. But we won’t do that. Instead we will start some ideas about spectral analysis that are within our experience. In general, the intention of this text is to go from the specific to the abstract and then, using insights from the abstract, back to the specific. Although the subject will be introduced with a relatively narrow range of examples, we shall see that “arbitrary signals” to be analyzed can come from a wide variety of phenomena.

Let’s start with some music.

It is part of our common cultural knowledge that sound is composed of “frequencies”, that frequencies are measured in hertz (Hz), and that good reproduction of musical sound requires a “bandwidth” of 15-20 thousand Hz (or 15-20khz). Low notes correspond to lower frequencies and higher notes correspond to higher frequencies.

Typically, when we speak of frequencies, we are referring to a sinusoidal function, **fig. 1.1-1**,

$$\mathbf{f_1(t) = 2|f_1|Cos(\omega_0 t + \theta_1); \quad where\ T = {}^{2\pi}\!/_{\omega_0}} \tag{1.1-1}$$

where T is called the period of the sinusoid and $\mathbf{f = {}^{1}\!/_{T}}$ is the frequency measured in Hz (or cycles per second). The particular labling of the amplitude and phase in eq. 1.1-1 is in anticipation of subsequent elaboration. By calling T the period of the sinusoid, we are observing that it is periodic; it repeats every T seconds. Functions that are periodic are very important in this text. Some Problems at the end of this Chapter will explore the properties of periodicity a bit further.

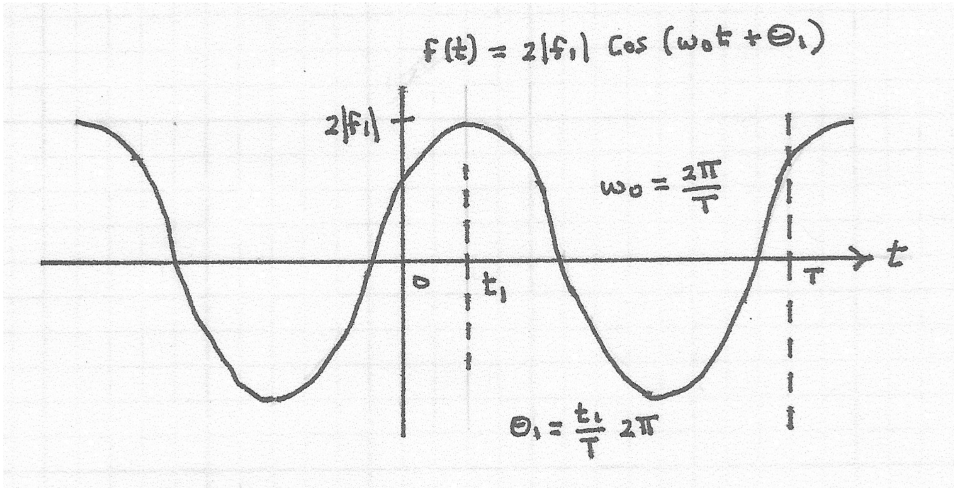


fig. 1.1-1

Everyone who has been to a music venue (one hopes that this includes engineering students) has seen the equalizer board, **fig. 1.1-2**, and the sound engineer manipulating that board. The board

fig. 1.1-2

consists of parallel slides that we understand to correspond to adjacent frequency bands in the music. A column of light bulbs or light – emitting diodes often accompanies these slides as a display. As the volume of the sound in a frequency band increases, either by moving the slide or by the music itself, more of this display is activated. By manipulating the equalizer, the sound engineer can amplify or attenuate different frequency bands. An example of these different adjacent frequency bands is shown in **fig. 1.1-3**

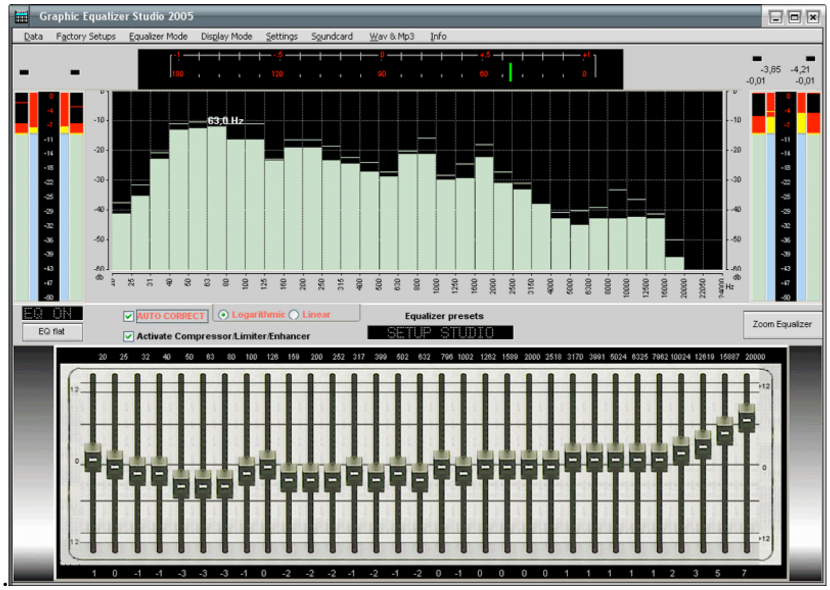


fig. 1.1-3

In this instance, we understand that a frequency refers to a sinusoid and that a band of frequencies refers to a collection of sinusoids having frequencies that are relatively close to each other. **Fig. 1.1-3** raises some interesting issues that go to the heart of this book. First, the music is to be separated into different bands of frequencies. Second, those individual frequency bands are to be amplified or attenuated in magnitude. Third, those frequency bands are to be reassembled into music that is not quite the same as the music that constituted the input to the equalizer.

Although these operations appear to be simple enough, perhaps because they are so common and almost intuitive, the ability to perform them actually is the consequence of the unique properties of linear systems and the unique relationship between these systems and sinusoidal functions. In a most elemental sense, exploring those properties is the subject of this text.

We have more or less established that since music applied to an equalizer board results in indications of output from each of the frequency bands, that this music can reasonably be thought of as consisting of bands of sinusoids. By observation, we can see that these bands are there in the music. But the magnitude of the energy in each of these bands, collectively called the spectrum of the signal, is constantly changing. Consequently, it becomes impossible to characterize the entire spectrum of this music for all time with a single measurement. Addressing this issue is a central problem of spectral analysis. In order to address this issue properly, we shall have to establish a conceptual and mathematical framework.

1.2 A First Look at Fourier Series and Spectra

The mathematical means of characterizing a spectrum, collectively known as **Fourier Analysis,** comprises a collection of mathematically related techniques, an introduction to which is outlined below.

As a first step, referring to eq. 1.1-1, and using the Euler equations:

$$\mathbf{Cos(x) = \frac{1}{2}\left[e^{jx} + e^{-jx}\right]} \tag{1.2-1a}$$

and

$$\mathbf{Sin(x) = \frac{1}{2j}\left[e^{jx} - e^{-jx}\right]} \tag{1.2-1b}$$

we can see that the sinusoidal function of eq. 1.1-1 can be expressed as

$$\mathbf{f_1(t) = 2|f_1|Cos(\omega_0 t + \theta_1) = f_1 e^{j\omega_0 t} + f_1^* e^{-j\omega_0 t};\ where\ f_1 = |f_1|e^{j\theta_1}} \tag{1.2-2a}$$

and further as:

$$\mathbf{f_1(t) = f_1 e^{j\omega_0 t} + f_{-1} e^{-j\omega_0 t};\ where\ |f_1| = |f_{-1}|\ and\ \theta_{-1} = -\theta_1} \tag{1.2-2b}$$

The plot of the coefficients $\mathbf{f_1}$ **and** $\mathbf{f_{-1}}$ vs. frequency is shown in **fig. 1.2-1a**. This plot is called **the spectrum** of $\mathbf{f_1(t)}$.

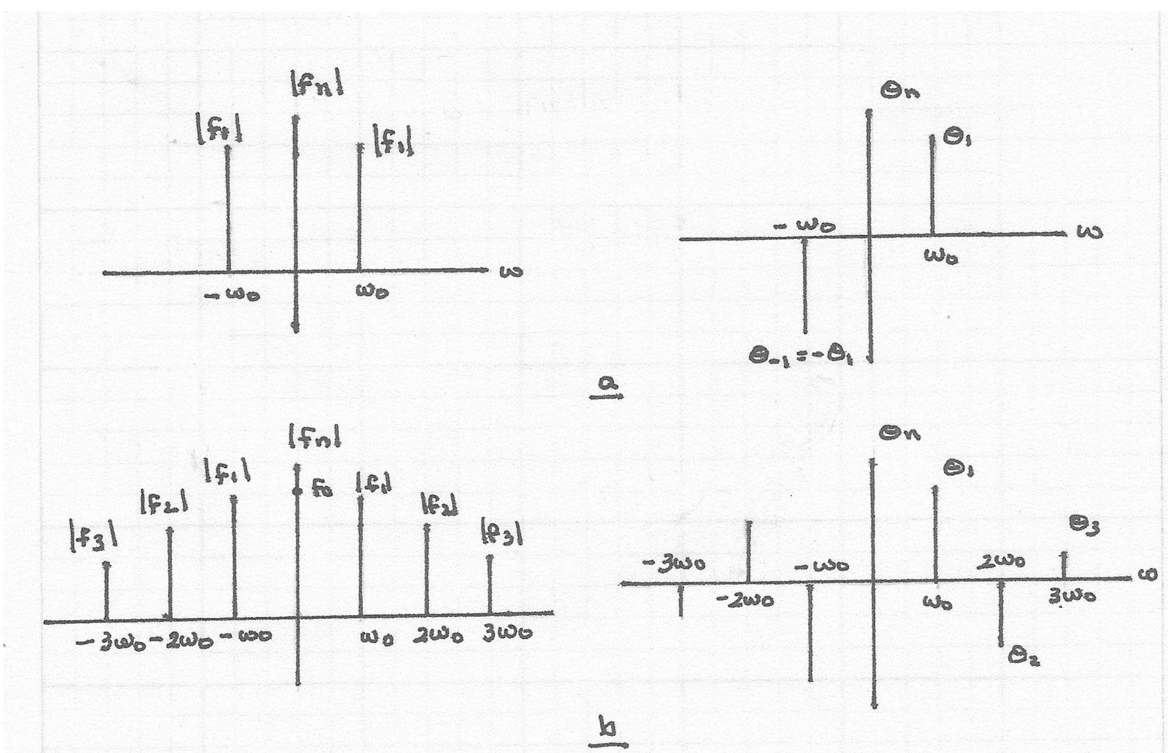


fig. 1.2-1

The sinusoidal function

$$f_n(t) = 2|f_n|\,\mathrm{Cos}(n\omega_0 t + \theta_n) \qquad (1.2\text{-}3a)$$

has period $T_n = T/n = \frac{1}{n} 2\pi/\omega_0$. But it also repeats every T sec as well, so it can be said to have period equal to T as well.

Consequently, the more general function:

$$f_N(t) = f_0 + \sum_{n=1}^{N} 2|f_n|\,\mathrm{Cos}(n\omega_0 t + \theta_n) = \sum_{n=-N}^{N} f_n e^{jn\omega_0 t} \qquad (1.2\text{-}3b)$$

$$\text{where } |f_n| = |f_{-n}| \text{ and } \theta_n = -\theta_{-n}$$

also has period equal to T.

Eq. 1.2-3b is the structure of a **Fourier Series**. The coefficient f_0 is the dc component, ω_0 is called the fundamental frequency, and $n\omega_0$ is the n^{th} harmonic. The **spectrum** of these coefficients is shown in **fig. 1.2-1b**. Despite the formidable designation of "spectrum", this is a simple graphical representation of the amplitudes and phases of the constituent sinusoids.

According to equation 1.2-3b, the plot of $|f_n|$ **vs.** $n\omega_0$, the **amplitude spectrum,** is an even function and the plot of θ_n **vs.** $n\omega_0$, the **phase spectrum**, is an odd function.

In Chapter 4, Fourier Series, we shall see that with the appropriate choice of the f_n coefficients, any periodic function having period T can be represented by eq. 1.3-3b. Finding those Fourier Series Coefficents for a particular periodic waveform will require some analytical effort. Two different functions $f_1(t)$ and $f_2(t)$ having the same period T will differ in their Fourier Series representations only by their Fourier Series coefficients, the values of the amplitudes and phases of the sinusoids, and not in the frequencies of the sinusoids contained in the series. Note that if $\theta_n = 0$ the terms of the series are all cosines but if $\theta_n = 90^0$ then these terms become sines.

As an illustration, consider the periodic square wave $f_1(t)$ of **fig 1.2-2** and the periodic triangular wave $f_2(t)$ of **fig 1.2-3.** It will be shown in Chapter 4 that these two functions can be written as:

$$f_1(t) = \frac{1}{\pi}\left\{\mathrm{Sin}(\omega_0 t) + \frac{1}{3}\mathrm{Sin}(3\omega_0 t) + \frac{1}{5}\mathrm{Sin}(5\omega_0 t) + \frac{1}{7}\mathrm{Sin}(7\omega_0 t) + \ldots\right\} \qquad (1.2\text{-}4)$$

and

$$f_2(t) = \frac{4}{\pi^2}\left\{\mathrm{Sin}(\omega_0 t) - \frac{1}{9}\mathrm{Sin}(3\omega_0 t) + \frac{1}{25}\mathrm{Sin}(5\omega_0 t) - \frac{1}{49}\mathrm{Sin}(7\omega_0 t) + \ldots\right\} \quad (1.2\text{-}5)$$

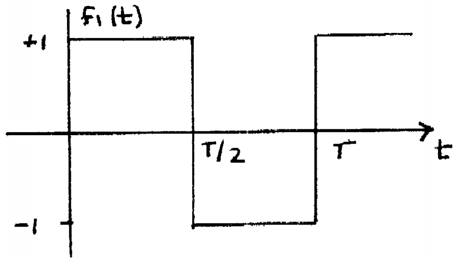


fig 1.2-2 Square Wave

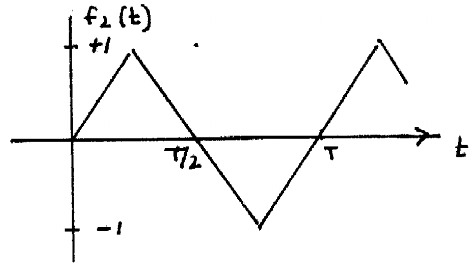


fig 1.2-3 Triangular Wave

In Chapter 4, we will see why these particular series expansions contain only odd harmonics.

Using MATLAB, we plot the partial sums of eq.1.2-4 in **fig 1.2-4** in order to show how this series converges to a square wave.

```
x = 0:0.1:2*pi;
y1 = (1/pi)*(sin(x));
subplot(221)
plot(x,y1)
y2 = (1/pi)*(sin(x)+(1/3)*sin(3*x));                                   (1.2-6)
subplot(222)
plot(x,y2)
y3 = (1/pi)*(sin(x)+(1/3)*sin(3*x)+(1/5)*sin(5*x));
subplot(223)
plot(x,y3)
y4 = (1/pi)*(sin(x)+(1/3)*sin(3*x)+(1/5)*sin(5*x)+(1/7)*sin(7*x));
subplot(224)
plot(x,y4)
```

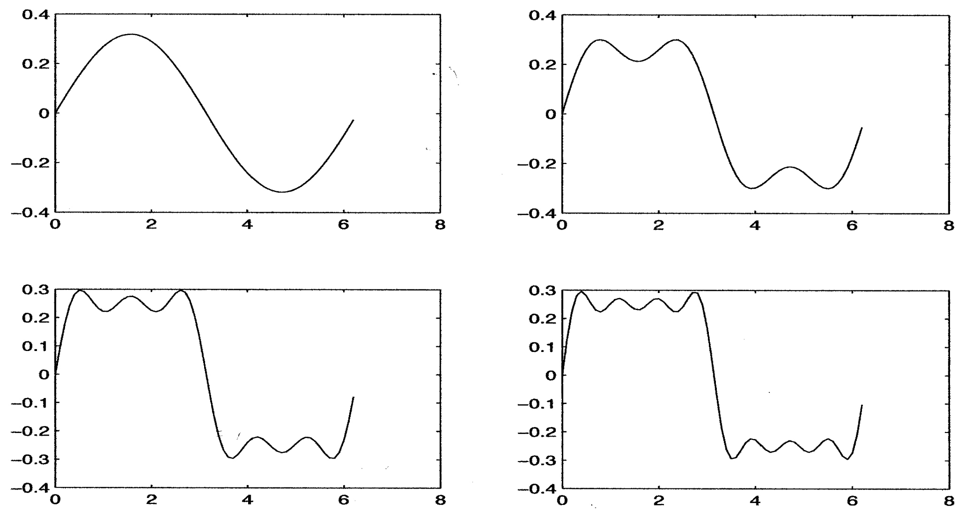


fig 1.2-4 Convergence to a Square Wave

The graph on the upper left is simply a sinusoid. To its right is the sum of the components having frequencies ω_0 and $3\omega_0$. On the lower left is the sum of ω_0, $3\omega_0$, and $5\omega_0$ and finally, on the lower right, we have $7\omega_0$. Although the series is clearly approaching a square wave, it does so in an oscillatory manner, referred to as "ringing" or the "Gibbs Phenomenon". We will see in Chapter 4 that this "ringing" corresponds to the existence of a discontinuity in the original waveform.

A similar convergence phenomenon is illustrated with the series representation of the triangular waveform of eq. 1.2-5 in **fig 1.2-5.** The MATLAB script for this demonstration is:

```
x = 0:0.1:2*pi;
y1 = (4/(pi*pi))*(sin(x));
subplot(221)
plot(x,y1)
y2 = (4/(pi*pi))*(sin(x)-(1/9)*sin(3*x));
subplot(222)
plot(x,y2)                                                        (1.2-7)
y3 = (4/(pi*pi))*(sin(x)-(1/9)*sin(3*x)+(1/25)*sin(5*x));
subplot(223)
plot(x,y3)
y4 = (4/(pi*pi))*(sin(x)-(1/9)*sin(3*x)+(1/25)*sin(5*x)-(1/49)*sin(7*x));
subplot(224)
plot(x,y4)
```

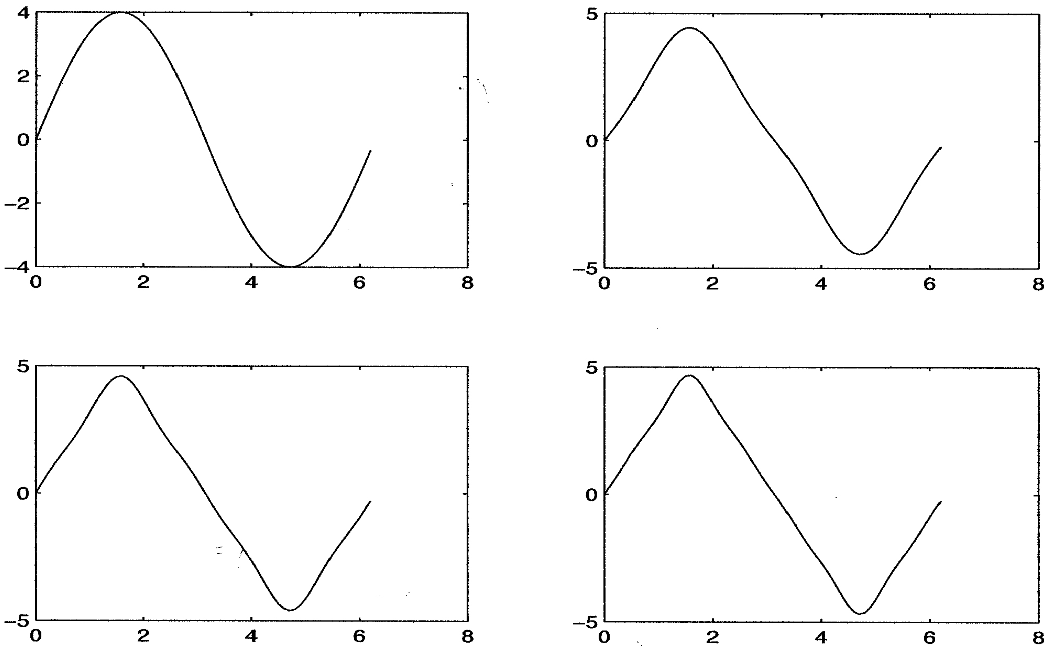


fig 1.2-5 Convergence to a Triangular Wave

Note that there are no discontinuities in the triangular function (there are discontinuities in the derivative, but that doesn't matter here) so that there is no ringing.

Turning to some of the subtleties and ambiguities of spectral analysis, let's jump ahead to Chapter 4 in order to take a quick look at the formula for generating the Fourier Series $\mathbf{f_n}$ coefficients in eq. 1.2-3b:

$$\mathbf{f_n = \frac{1}{T}\int_{t_1}^{t_1+T} f(t)e^{jn\omega_0 t}\,dt} \qquad (1.2\text{-}7)$$

We will not be doing anything with this formula just yet, just looking and having a bit of a conversation with it. All matters of computation will be reserved to Chapter 4. Now eq. 1.2-7 tells us that we can take any time interval of duration T of an arbitrary signal, **fig. 1.2-6a,** and find the $\mathbf{f_n}$ coefficients that will allow the series of eq. 1.2-3b to converge to the original function **in the interval** $\mathbf{[t_1,t_1+T]}$. Outside of that interval, the series approximation, $\mathbf{\hat{f}(t)}$, is periodic with period T; $\mathbf{\hat{f}(t)}$ is called the **periodic extension** of that portion of $\mathbf{f(t)}$ that is contained in the interval $\mathbf{[t_1,t_1+T]}$, **fig. 1.2-6b.**

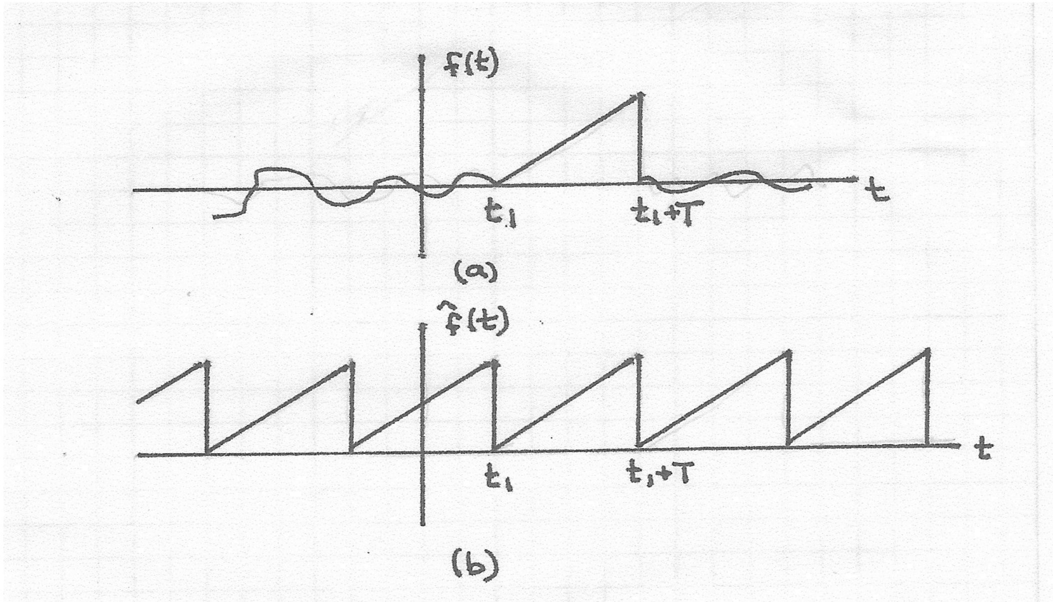


fig. 1.2-6

This means that in processing the arbitrary signal $\mathbf{f}(\mathbf{t})$, we get to choose the processing interval T and, therefore, the fundamental frequency $\boldsymbol{\omega_0 = 2\pi/T}$ and the harmonics $\mathbf{n\omega_0}$. Further, we can find the spectra (i.e.-find the Fourier Series coefficients) associated with successive intervals having duration T. In short, we can find successive approximations to the evolving spectrum of a changing signal.

This leads to the interesting question of what the spectrum looks like if the signal contains a frequency component that is not among the harmonics. The detailed answer to this question, reserved for Chapter 4, will provide some considerable insight into the character of the results of spectral analysis. It turns out that the analysis will indicate energy at the harmonics closest to that frequency. This means that the exact results of the spectral analysis will be dependent upon the observation interval T. The inverse relationship between $\boldsymbol{\omega_0}$ and T means that a long observation interval is required to observe lower frequency components and a shorter interval for higher frequency components. A long observation interval T makes the harmonics very dense and a short one makes them sparse.

This creates a substantial ambiguity in the process of spectral analysis. We shall subsequently learn that the result of spectral analysis yields information about where regions of energy occur in the spectrum. There is no fixed observation interval that is appropriate for all waveforms. The value of T should, in some sense, correspond to the rate of change of the waveform that is being analyzed. But, in general, this rate of change is unknown in advance. Consequently, the appropriate value of T for an arbitrary waveform is subject to experimentation

Some signals, however, can be made to contain a specifically highest frequency. These signals are “band limited”. Correspondingly, there is a smallest value of T that is required to fully encompass the frequency content of the signal. As we shall see in Chapter 5, Fourier Transforms, digital signal processing requires that signals be bandlimited.

1.3 A First Look at Linear Systems and Filters

Returning to the sound engineers equalizer board, the separation of the signal into adjacent frequency bands is accomplished by filters. An initial understanding of how filters work involves taking an introductory look at Continuous Linear Systems.

Continuous Linear Systems are a class of systems that are described by linear differential equations that have constant coefficients. The abbreviation LDE is used for such equations. We shall use a simple electric circuit as an example for an introductory look at some of the issues of these systems. There is a great diversity of additional classes of examples and physical phenomena to which ideas of linear systems can be applied, but in this text these ideas will be developed in relation to electric circuits.

An example of a linear differential equation having constant coefficients (LDE) is:

$$\mathbf{a_2\frac{d^2x(t)}{dt}+a_1\frac{dx(t)}{dt}+a_0x(t)=f(t)} \qquad (1.3\text{-}1)$$

where f(t) is called the input or forcing function and x(t) is the output. This basic input – output relationship is shown in the block diagram of **fig. 1.3-1.** Both f(t) and x(t) are continuous-time functions, i.e. – the variable ‘t’ is continuous.

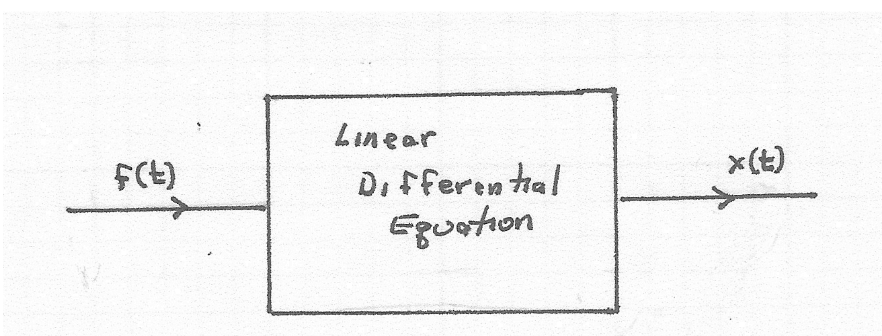


fig. 1.3-1

This equation is linear because neither x(t) nor its derivatives appear as the argument of any function; e.g. – nothing like $\sqrt{\mathbf{x(t)}}$ or $\mathbf{sin}\left[\frac{\mathbf{dx(t)}}{\mathbf{dt}}\right]$ appears in the equation. Similarly, the coefficients of the equation are constants, not functions of time. It is possible for the coefficients to be functions of time and still have the equation be linear, but that kind of equation will not be examined in this text.

Later in this text, in Chapter 6, we will look at the discrete – time version of the LDE, the linear difference equation:

$$\mathbf{a_2x[n-2]+a_1x[n-1]+a_0x[n]=f[n]} \tag{1.3-2}$$

where both the input **f[n]** and the output **x[n]** are functions of the discrete integer variable 'n' or 'nT', discrete intervals of time.

The characteristics of continuous – time and discrete – time signals, and their relations to each other form an important part of this book. Systems involving both kinds of signals exist in the engineering world and many systems that are continuous in nature are made into discrete – time systems in order to be solved digitally.

We will start with a simple example of a continuous time circuit. Let's take a look at the simple RC circuit shown in **fig 1.3-2.** Typically, the switch closes at t = 0, allowing the current to begin to flow.

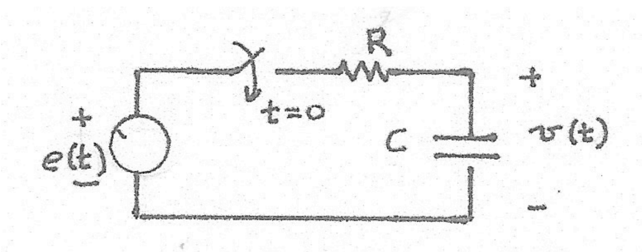


fig 1.3-2

Using the capacitor voltage **v(t)** as the variable and Kirchoff's Current Law, the equation describing this circuit is:

$$\mathbf{C\frac{dv(t)}{dt}+\frac{v(t)-e(t)}{R}=0} \tag{1.3-3a}$$

or:

$$\mathbf{RC\frac{dv(t)}{dt}+v(t)=e(t)} \tag{1.3-3b}$$

Eq. 1.3-3b is a first order linear differential equation having constant coefficients. This equation has a response for $\mathbf{t \geq 0}$ that consists of two components: the response to the initial capacitor voltage, $\mathbf{v(0^-) = V_C}$, and the response to the forcing function e(t). As a consequence of the linearity of the equation, these responses can be computed separately and then added together, resulting in the complete response of the circuit.

Setting e(t) = 0, the response to the initial capacitor voltage is:

$$\mathbf{v_T(t) = V_C e^{-t/RC}; \quad t \geq 0} \tag{1.3-4}$$

and is shown in **fig. 1.3-3.** The subscript T refers to this portion of the response as being "transient".

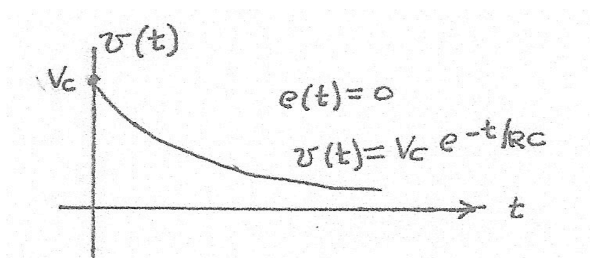
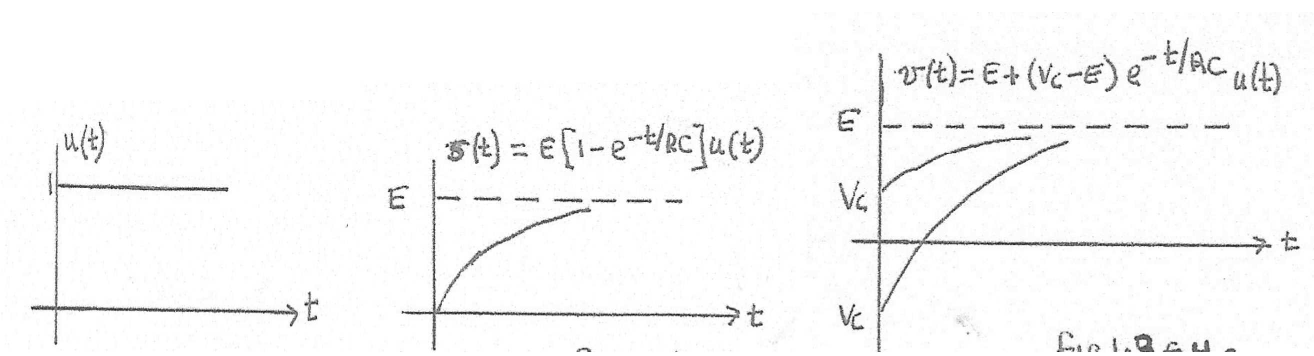


fig. 1.3-3

Turning now to the response to e(t), set $\mathbf{v(0^-) = 0}$ and consider what happens when $\mathbf{e(t) = E}$, a constant voltage. The combination of this constant voltage and the switch leads to the definition of the **unit step function**:

$$\mathbf{u(t) = \begin{cases} 1; \ t \geq 0 \\ 0; \ t < 0 \end{cases}} \tag{1.3-5}$$

as shown in **fig. 1.3-4a**.



(a) **(b)** **(c)**

fig. 1.3-4

With $\mathbf{e(t) = Eu(t)}$, the response voltage s(t) is:

$$s(t) = E\left[1 - e^{-t/RC}\right]u(t) \tag{1.3-6}$$

as shown in **fig. 1.3-4b.** This response, corresponding to $v(0^-) = 0$, is called the **step response** of the circuit. As we shall subsequently see, the step response of a circuit is a benchmark of the behavior of the circuit – and, by extension, the behavior of a linear system. This response consists of two components: the voltage E, which is reached after a theoretically infinite time, and an exponential voltage that decays to zero eventually. The latter portion is called the natural response. A distinction has been made between the decaying exponentials in eq. 1.3-4 and eq. 1.3-6. The former is called a **transient response** (arising from a non-zero initial condition) and the latter is called a **natural response** (corresponding to a zero initial condition). This distinction will be crucial to a formal definition of linearity that will be made in the next chapter.

More generally, the response of this RC circuit with a different initial condition $v(0^-) = V_C$ is shown in **fig. 1.3-4c** and is written as

$$v(t) = \left[E + (V_C - E)e^{-t/RC}\right]u(t) \tag{1.3-7a}$$

which shows the final value of E and the initial value (when t = 0) of $v(0^-) = V_C$. The response is shown with two different values of V_C.

Finally, eq. 1.6b can be rewritten as:

$$\begin{aligned} v(t) &= E\left[1 - e^{-t/RC}\right]u(t) + V_C e^{-t/RC}u(t) \\ &= s(t) + v_T(t) \end{aligned} \tag{1.3-7b}$$

separating the response to the unit step input function from the response to the non-zero initial condition. Having defined the **step response** of the RC circuit in **fig. 1.3-4**, we now shall define a more fundamental characterization of the circuit, the **impulse response.**

Let's start by reconsidering the definition of the unit step function. As $\varepsilon \to 0$ in **fig. 1.3-5a,** the function $u_\varepsilon(t) \to u(t)$, the unit step function. The time derivative of $u_\varepsilon(t)$, shown in **fig. 1.3-5b,** is called $\delta_\varepsilon(t) = {du_\varepsilon(t)}/{dt}$. It has width ε and amplitude 1/ε yielding an area equal to unity, independent of the value of ε. As $\varepsilon \to 0$, $\delta_\varepsilon(t)$ becomes narrower and higher, maintaining its area of unity. In the limit it has zero width and infinite amplitude and unit area. The limiting function $\delta(t)$ is called the

unit impulse function. It is formally defined as:

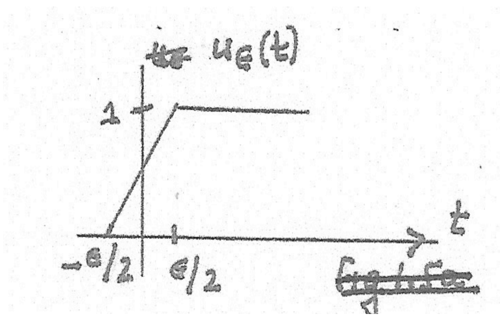
$$\int_{-\infty}^{\infty} \delta(t)dt = \int_{0^-}^{0^+} \delta(t)dt = 1 \tag{1.3-8a}$$

and

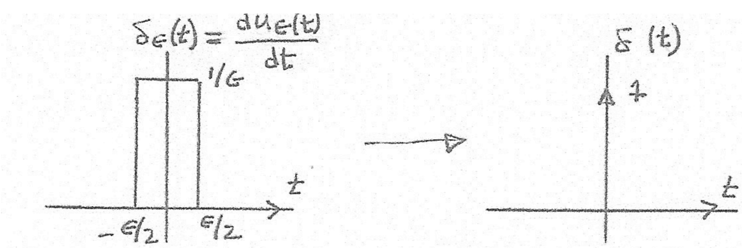
$$\delta(t) = \begin{cases} 0; \ t \neq 0 \\ \infty; \ t = 0 \end{cases} \tag{1.3-8b}$$

where

$$\delta(t) = \frac{du(t)}{dt} \tag{1.3--8c}$$



(a)



(b)

fig. 1.3-5

Using $\delta_\varepsilon(t)$ as the input to the RC circuit (again with zero initial conditions), we can see from eq. 1.3-7b that the response, shown in **fig. 1.3-6a,** will reach a peak value of:

$$v\left(\varepsilon/2\right) = \frac{1}{\varepsilon}\left[1 - e^{-\varepsilon/RC}\right] \tag{1.3-9a}$$

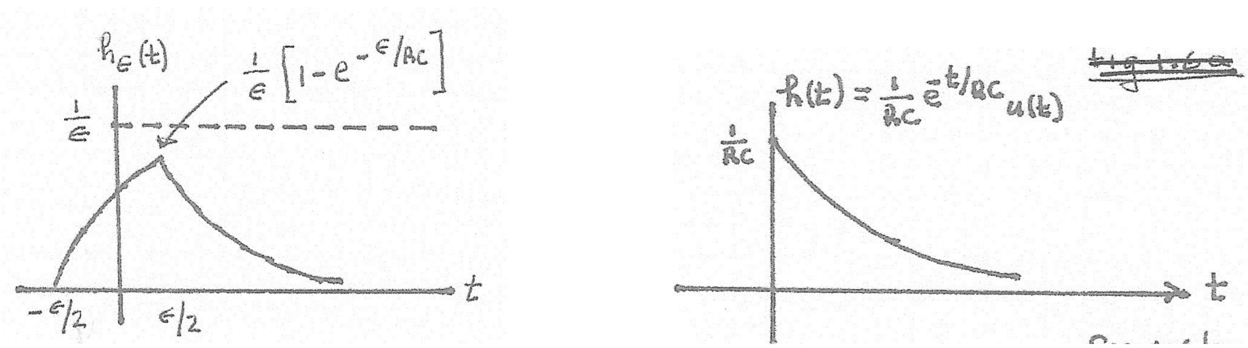
before decaying to zero. As $\varepsilon \to 0$, we shall use the approximation

$$e^{-\varepsilon/RC} \approx 1 - \varepsilon/RC \tag{1.3-9b}$$

so that

$$v(t) = h(t) = \frac{1}{RC}e^{-t/RC}u(t) \tag{1.3-9c}$$

where h(t) is defined as the **unit impulse response**, as shown in **fig. 1.3-6b.**



(a) **(b)**

fig. 1.3-6

The impulse response can also be obtained by differentiating the unit step response, eq. 1.6a, as follows:

$$\begin{aligned}\frac{ds(t)}{dt} &= \frac{d}{dt}\left\{\left[1-e^{-t/RC}\right]u(t)\right\} = \left\{\frac{du(t)}{dt} - \frac{d}{dt}\left[e^{-t/RC}u(t)\right]\right\} \\ &= \left\{\delta(t) - e^{-t/RC}\delta(t) + \frac{1}{RC}e^{-t/RC}u(t)\right\} = \frac{1}{RC}e^{-t/RC}u(t) = h(t)\end{aligned} \tag{1.3-10}$$

since $\left. e^{-t/RC}\right|_{t=0} = 1$.

It is worth noting that the impulse response h(t) does not have the dimensions of voltage; rather, it has the dimensions of voltage divided by time. This makes sense because it is the time derivative of the step response, which is a voltage.

The **unit impulse response h(t)** of a linear circuit (or system) is a unique defining characterization of that circuit. It is understood as the response of a circuit **having zero initial conditions** to an input impulse function having unit area.

The impulse responses of circuits more general than that of fig. 1.3-6 are not discussed in this Introduction. In order to have that discussion in a reasonable manner, it is necessary to use the analytic tool of the Laplace Transform. The Laplace Transform will be introduced in Chapter 2. Understanding that the impulse response is a **time domain** characterization of a circuit, based upon its **natural response**, we now shall turn to an alternative way of characterizing the circuit, its **frequency response.**

The **frequency response** of a circuit is defined as the forced response of the circuit when the input $\mathbf{e(t) = ECos(\omega t)}$ as $\mathbf{0 \le \omega < \infty}$.

We can easily find the frequency response by considering a somewhat more general issue.

If the input f(t) to any LDE having the form:

$$a_2\frac{d^2x(t)}{dt^2}+a_1\frac{dx(t)}{dt}+a_0x(t)=b_1\frac{df(t)}{dt}+b_0f(t) \qquad (1.3\text{-}11a)$$

is an exponential function $f(t) = Ae^{pt}$, then the forced output or response x(t) will have the form $x(t) = Be^{pt}$ where:

$$\left[a_2p^2+a_1p+a_0\right]Be^{pt} = A\left[b_1p+b_0\right]e^{pt} \qquad (1.3\text{-}11b)$$

or:

$$B = \frac{A\left[b_1p+b_0\right]}{\left[a_2p^2+a_1p+a_0\right]} = AH(p) \qquad (1.3\text{-}11c)$$

where H(p) is called the **transfer function** of the LDE. The conclusion is that if the input is an exponential, the forced response will also be an exponential **having the same exponent,** but differing in amplitude.

<u>Example 1.3-1:</u>

Consider the LDE

$$\frac{d^2x(t)}{dt^2}+3\frac{dx(t)}{dt}+2x(t) = Ae^{pt} \qquad (1.3\text{-}12a)$$

then

$$x(t) = \frac{A}{\left[p^2+3p+2\right]}e^{pt} = H(p)Ae^{pt} \qquad (1.3\text{-}12b)$$

If p = -4, then

$$x_1(t) = \frac{A}{\left[(-4)^2+3(-4)+2\right]}e^{-4t} = \frac{A}{6}e^{-4t} \qquad (1.3\text{-}12c)$$

where $H(p)\big|_{p=-4} = \left.\frac{1}{\left[p^2+3p+2\right]}\right|_{p=-4} = \frac{1}{6}$.

If p = 4j, then

$$x_2(t) = \frac{A}{\left[(4j)^2+3(4j)+2\right]}e^{j4t} = \frac{A}{\left[-14+j12\right]}e^{j4t} =$$
$$= \frac{A}{18.44}e^{-j40.6^0}e^{j4t} = \frac{A}{18.44}e^{j\left(4t-40.6^0\right)} \qquad (1.3\text{-}12d)$$

where

$$H(p)\Big|_{p=j4} = \frac{1}{\left[p^2+3p+2\right]}\Bigg|_{p=j4} = \frac{1}{18.44}e^{-j40.6^0} = |H(j4)|e^{j\theta(j4)}. \qquad (1.3\text{-}12e)$$

Continuing, let p = -4j, the conjugate of the preceding value. Under those circumstances, it is straightforward to demonstrate that:

$$H(p)\Big|_{p=-j4} = \frac{1}{\left[p^2+3p+2\right]}\Bigg|_{p=-j4} = \frac{1}{18.44}e^{j40.6^0} = |H(-j4)|e^{-j\theta(j4)} \qquad (1.3\text{-}12f)$$

and that

$$x_3(t) = \frac{A}{18.44}e^{j\left(4t+40.6^0\right)} \qquad (1.3\text{-}12g)$$

so that $\mathbf{x_2(t)}$ **and** $\mathbf{x_3(t)}$ are complex conjugates of each other.

End of example

Using the Euler equations, eq. 1.2-1, and the results for $\mathbf{x_2(t)}$ **and** $\mathbf{x_3(t)}$ in the example given above, it follows that if the input to the LDE is:

$$f(t) = E\mathrm{Cos}(\omega t) = \frac{E}{2}\left[e^{j\omega t} + e^{-j\omega t}\right] \qquad (1.3\text{-}13a)$$

then:

$$x(t) = \frac{E}{2}|H(j\omega)|\left[e^{j\left[\omega t+\theta(\omega)\right]} + e^{-j\left[\omega t+\theta(\omega)\right]}\right] \qquad (1.3\text{-}13b)$$

or:

$$x(t) = E|H(j\omega)|\mathrm{Cos}\left[\left[\omega t+\theta(\omega)\right]\right] \qquad (1.3\text{-}13c)$$

The result of eq. 1.3-13c is quite fundamental. It says that if the input function to an LDE is a cosine, then the corresponding output (forced response) is also a cosine having the **same frequency, different amplitude, and a different phase.**

The development from eq.1.3-11 through eq. 1.3-14 also yields the property of superposition in linear differential equations. If the input to the LDE of eq. 1.3-11a is:

$$f(t)=\sum_{n=1}^{N}E_n \mathrm{Cos}(\omega_n t) \tag{1.3-14a}$$

then the output is:

$$x(t)=\sum_{n=1}^{N}E_n\left|H(j\omega_n)\right|\mathrm{Cos}(\omega_n t+\theta_n). \tag{1.3-14b}$$

Clearly, the LDE treats each input cosine independently of the others; changing the amplitude and phase of the individual cosines while maintaining each of the frequencies. Referring to Section 1.2, this also means that if the input to the system is a Fourier Series, the output is a Fourier Series as well, but with modified amplitudes and phases.

The key expression determing these differences is the transfer function associated with the LDE. We return to the RC circuit of fig. 1.3-2 for a specific and illustrative example. The transfer function associated with the LDE of eq. 1.3-3b is easily found to be:

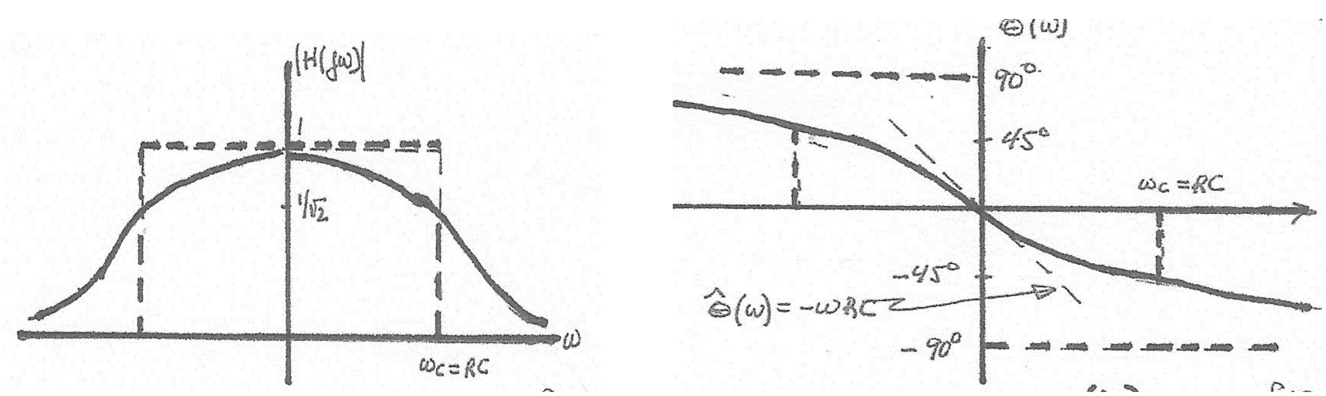
$$H(p)=\frac{1}{RCp+1} \tag{1.3-15a}$$

or:

$$H(j\omega)=\frac{1}{1+j\omega RC}=\sqrt{\frac{1}{1+(\omega RC)^2}}e^{-j\tan^{-1}(\omega RC)} \tag{1.3-15b}$$

The graph of both the magnitude and the phase of $H(j\omega)$ are illustrative of a most important concept in the understanding of LDE's, namely that of **filtering**.

The plot of $\left|H(j\omega)\right|$ **vs.** ω in **fig. 1.3-7a** shows that as the frequency increases, the magnitude of the transfer function decreases, with a critical frequency at $\omega_C = 1/RC$ where $\left|H(j\omega_C)\right| = 1/\sqrt{2} = .707$



(a) **(b)**

fig. 1.3-7

This decrease in amplitude with increasing frequency characterizes the RC circuit of fig. 1.1 as a **lowpass filter.** The phase characteristic of this circuit is shown in **fig. 1.3-7b.**

As we shall see in Chapter 3, the transfer function of a second order differential equation can be written as:

$$\mathbf{H(j\omega) = \frac{1}{1 + jQ_S\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}} \tag{1.3-16a}$$

where:

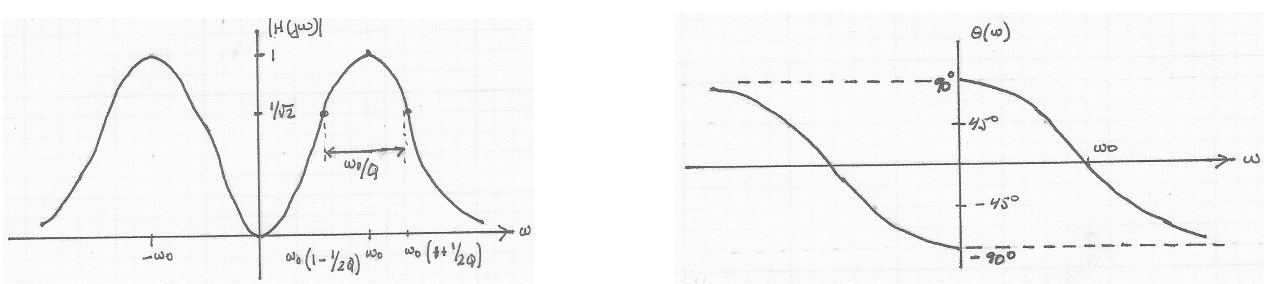
$$\mathbf{|H(j\omega)| = \frac{1}{\sqrt{1 + Q_S^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}}} \tag{1.3-16b}$$

and:

$$\mathbf{\theta(j\omega) = -\tan^{-1}\left[Q_S\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]} \tag{1.3-16c}$$

are the amplitude and phase characteristics.

Just as **fig. 1.3-7** characterized eq. 1.3-16b as a lowpass filter, **fig. 1.3-8** characterizes eq. 1.3-17a as a **bandpass filter.** Bandpass filters having differing center frequencies are used in audio equalizers to separate the frequency bands so that they can be selectively amplified or attenuated.



(a) **(b)**

fig. 1.3-8

The unifying issue of this section is that the simple RC circuit of fig. 1.3-2 has been characterized in two seemingly different ways, first by its impulse response:

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t) \qquad (1.3\text{-}9c)$$

which defines the time domain behavior of the circuit, and then by its transfer function:

$$H(j\omega) = \frac{1}{1 + j\omega RC} \qquad (1.3\text{-}15b)$$

which defines the forced response when the input is a cosine. These two characterizations were obtained through two very different lines of thought. Yet it turns out that these two quantities are mathematically closely related.

The relationship between the two, the impulse response and the transfer function, is at the very heart of linear system theory. It represents the essential duality of the time domain and the frequency domain. In fact, the transfer function is the Laplace Transform of the impulse response. In order to develop and understand this relationship, we will introduce a new tool, the Laplace Transform in Chapter 2.

1.4 Summary

This Chapter began by posing the problem of how an audio equalizer panel works. Explaining this simple device has led us into consideration of the complementary issues of spectral analysis and linear systems. In order to proceed, we now have to get a bit more technical. Chapters 2 and 3 will concentrate on linear systems and we shall return to spectral analysis in Chapter 4.

Chapter 1: Problems

1.1-1 A function f(t) is considered to be periodic with period T if f(t+T) = f(t). Find the period of each of the following functions:

a. $f(t) = \mathrm{Cos}(6\pi t)$

b. $f(t) = \mathrm{Cos}(8\pi t)$

c. $f(t) = \mathrm{Cos}(6\pi t) + \mathrm{Cos}(8\pi t)$

d. $f(t) = \mathrm{Cos}(6\pi t) + \frac{1}{2}\mathrm{Cos}(8\pi t)$

e. $f(t) = \mathrm{Cos}(6\pi t) + \mathrm{Cos}(8t)$ (this is a somewhat tricky question; think carefully)

1.1-2 **a.** Find the period of

$$f(t) = \mathrm{Cos}(2\pi t) + \frac{1}{2}\mathrm{Cos}(4\pi t) + \frac{1}{4}\mathrm{Cos}(6\pi t) + \frac{1}{8}\mathrm{Cos}(8\pi t)$$

b. Plot the amplitude spectrum of f(t)

1.1-3 **a.** Find the period of

$$f(t) = \mathrm{Cos}(\omega_0 t) + \frac{1}{2}\mathrm{Cos}(2\omega_0 t) + \frac{1}{4}\mathrm{Cos}(3\omega_0 t) + \frac{1}{8}\mathrm{Cos}(4\omega_0 t) \quad \text{where} \quad \omega_0 = {}^{2\pi}\!/_{T} .$$

b. Plot the amplitude spectrum of f(t)

Chapter 2

Transfer Functions and Impulse Responses in Linear Systems

Contents

Sections marked with an asterisk (*) can be skipped with no loss of continuity.

Chapter 2

Linear Systems and Laplace Transforms

This book is written from the point of view that the concepts of the transfer function and the impulse response are basic to the understanding and analysis of both linear systems and spectral analysis.

A linear system is mathematically defined as being represented by a linear differential equation (or, as we shall see in Chapter 7, by a linear difference equation). The transfer function **H(s)** represents that system as a function of frequency; the impulse response **h(t)** represents the system as a function of time. These two characterizations of the system are mathematically related; they are **dual** descriptions of the system.

The most direct way to begin to explore that relationship is through the use of the Laplace Transform. The advantage of beginning in this way is that the basic ideas involved in a discussion of linear systems: transfer functions, impulse responses, frequency response, convolution, duality, are put together in a single coherent package. As a coherent whole, they are readily accessible in discussions of continuous and discrete systems as well as in signal analysis.

2.1 Linear Systems and Laplace Transforms

Continuous Linear Systems were defined in Chapter 1 to be those systems that are described by linear differential equations having constant coefficients (LDE). The simple RC circuit of **fig. 1.3-1,** described by a first order linear differential equation was used as an example in order to develop a few key ideas. These ideas were the system **impulse response h(t)** and the system **transfer function H(p) or H(jω).** It was stated that these two different characterizations of the circuit, although arrived at in separate ways, were connected through the Laplace Transform as a single unified way of looking at the circuit, one through the “time domain” and the other through the “complex frequency domain”.

From a mathematical point point of view, the transform requires both background and proof. These will not be provided at this point; rather, they are provided in Chapter 10. Within this and subsequent chapters, the Laplace Transform will be used as a tool.

Let’s start with the LDE

$$a_2\frac{d^2x(t)}{dt^2}+a_1\frac{dx(t)}{dt}+a_0x(t)=b_1\frac{df(t)}{dt}+b_0f(t) \qquad (2.1\text{-}1)$$

where f(t) is the forcing function (or known input) to the system and we are trying to find the response (or output) x(t). A very elementary block digram of this process is shown in **fig. 2.1-1.**

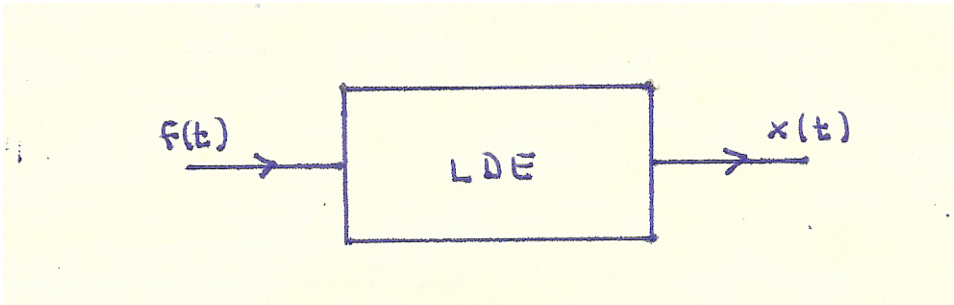


fig. 2.1-1

The one – sided Laplace Transform of a function f(t) is defined as:

$$F(s)=L[f(t)]=\int_{0^-}^{\infty} f(t)e^{-st}dt \qquad (2.1\text{-}2)$$

where f(t) = 0 for t < 0 (hence the one – sided transform), and where the lower limit of $t=0^-$ allows for consideration of the initial conditions in an LDE. The Laplace Transform **F(s)** is a function of the complex variable $\mathbf{s=\sigma+j\omega}$. An understanding of the significance of this variable will be developed as we proceed.

Note that the transform involves the multiplication of f(t) by the exponential function $\mathbf{e^{-st}}$. Recalling from Chapter 1 that the exponential function $\mathbf{e^{pt}}$ has a special relationship to the LDE, we can intuit that the effect of the Laplace Transform is to take advantage of that relationship.

Solving the LDE in eq. 2.1-1 using Laplace Transforms of involves three different tasks:

1. Finding the LT of the known input function $\mathbf{f(t)\Leftrightarrow F(s)}$;

2. Finding the LT of the LDE which includes the unknown function x(t), to which we assign the generic function X(s);

3. Finding an expression for X(s) and then finding the corresponding x(t), the inverse LT of X(s).

These three tasks will be undertaken in the subsequent three Sections.

Meanwhile, jumping ahead, in order to have the reader comfortable with the form of the LT of a function, we present the following listing of useful time functions and their transforms. Derivations and proofs of each of these items will be developed in Section 2.3

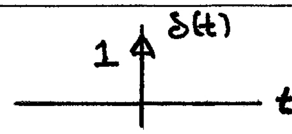
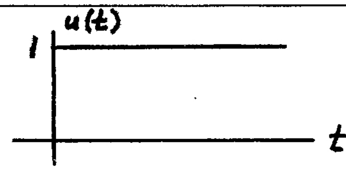
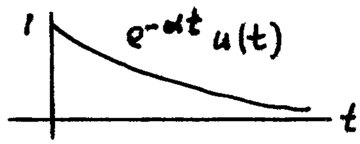
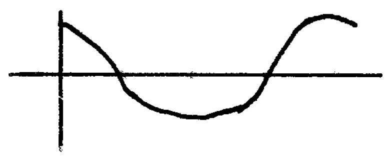
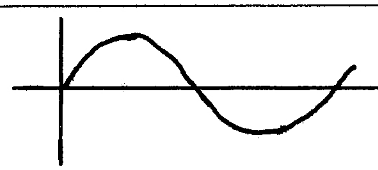
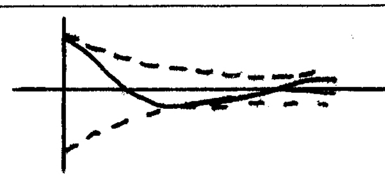
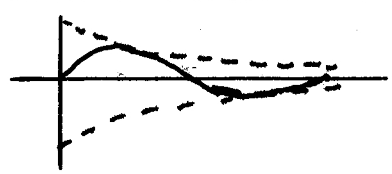
f(t)		**F(s)**
$\delta(t)$		1
$u(t)$		$\frac{1}{s}$
$e^{-\alpha t}u(t)$		$\frac{1}{s+\alpha}$
$\mathrm{Cos}(\beta t)u(t)$		$\frac{s}{s^2+\beta^2}$
$\mathrm{Sin}(\beta t)u(t)$		$\frac{\beta}{s^2+\beta^2}$
$e^{-\alpha t}\mathrm{Cos}(\beta t)u(t)$		$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$
$e^{-\alpha t}\mathrm{Sin}(\beta t)u(t)$		$\frac{\beta}{(s+\alpha)^2+\beta^2}$

Table 2.1-1

2.2 The Laplace Transform Solution of Linear Differential Equations

We want to use Laplace Transforms to find the solution of the differential equation

$$a_2 \frac{d^2x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = b_1 \frac{df(t)}{dt} + b_0 f(t) \qquad (2.2\text{-}1a)$$

where the initial conditions in the equation are:

$$x_0 = x(0^-) \quad \text{and} \quad x_1 = \left.\frac{dx(t)}{dt}\right|_{t=0^-} \qquad (2.2\text{-}1b)$$

The input, or forcing function, f(t) is a known function where f(t) = 0, t < 0. We shall discuss finding the LT of this function in the next section. The time functions and their Laplace Transforms listed in Table 2.1-1 are typical of the important functions that we will be considering. Since these functions are multiplied by u(t), they are all equal to zero for t < 0. Consequently there is no issue of initial conditions associated with the forcing function. But there are initial conditions associated with the internal variables of the system and the Laplace Transform of the derivatives of x(t) must take these into account.

This issue is addressed by the following theorem.

Theorem 2.1 Time Differentiation

If $L[x(t)] = X(s)$ then:

$$L\left[\frac{dx(t)}{dt}\right] = sX(s) - x(0^-) = sX(s) - x_0 \qquad (2.2\text{-}2a)$$

$$L\left[\frac{d^2x(t)}{dt^2}\right] = s^2X(s) - sx(0^-) - x'(0^-) = s^2X(s) - sx_0 - x_1 \qquad (2.2\text{-}2b)$$

and, in general:

$$L\left[\frac{d^nx(t)}{dt^n}\right] = s^nX(s) - s^{n-1}x(0^-) - s^{n-2}x'(0^-) - \cdots - x^{n-1}(0^-) \qquad (2.2\text{-}2c)$$

where $x(0^-), x'(0^-), x^{n-1}(0^-)$ are the values of x(t) and its derivatives at t = 0^-.

Proof:

$$L\left[\frac{dx(t)}{dt}\right] = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$$

Integrating by parts, we get:

$$\mathbf{L}\left[\frac{\mathbf{dx(t)}}{\mathbf{dt}}\right] = \mathbf{x(t)e^{-st}}\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty}\mathbf{(-s)x(t)e^{-st}dt = sX(s) - x\left(0^-\right)}$$

The proof of the general statement proceeds by recursion.

end of proof

Using Theorem 2.1, Time Differentiation, we can take the Laplace Transform of the entire differential equation eq 2.2-1a, yielding:

$$\mathbf{a_2\left[s^2X(s) - sx_0 - x_1\right] + a_1\left[sX(s) - x_0\right] + a_0X(s) = (b_1s + b_0)F(s)} \qquad (2.2\text{-}3a)$$

where we note that there is no initial condition associated with the derivative of f(t). Gathering terms,

$$\mathbf{\left[a_2s^2 + a_1s + a_0\right]X(s) = (b_1s + b_0)F(s) + a_2x_0s + (a_2x_1 + a_1x_0)} \qquad (2.2\text{-}3b)$$

and

$$\mathbf{X(s) = \frac{(b_1s + b_0)F(s) + a_2x_0s + (a_2x_1 + a_1x_0)}{\left[a_2s^2 + a_1s + a_0\right]}} \qquad (2.2\text{-}3c)$$

and, finally

$$\mathbf{X(s) = \frac{(b_1s + b_0)}{\left[a_2s^2 + a_1s + a_0\right]}F(s) + \frac{a_2x_0s + (a_2x_1 + a_1x_0)}{\left[a_2s^2 + a_1s + a_0\right]}} \qquad (2.2\text{-}3d)$$

In this format it is easy to distinguish between the part of the solution that is a response to non-zero initial conditions and the part of the solution that is a response simply to the forcing function and to zero initial conditions.

The value of the response x(t) and its derivatives are called the **state** of the system.

When the forcing function f(t) = 0 (which also means that F(s) = 0), the response depends entirely on the initial conditions and is called the **zero – input response**. The circumstance in which x(t) and its derivatives are all equal to zero at $t = 0^-$ is called the **zero – state** and and the corresponding response is called the **zero – state response**. In this instance, the response is due entirely to the forcing function F(s).

Linear system theory typically sets the initial conditions equal to zero so that there is a unique relationship between the input f(t) and the output x(t). Note that non – zero initial conditions result in an output term that does not depend on the input.

With zero initial conditions, eq. 2.2-3d undergoes a dramatic simplification and becomes:

$$\mathbf{X}(\mathbf{s}) = \frac{(\mathbf{b}_1\mathbf{s}+\mathbf{b}_0)}{\left[\mathbf{a}_2\mathbf{s}^2+\mathbf{a}_1\mathbf{s}+\mathbf{a}_0\right]}\mathbf{F}(\mathbf{s}) = \mathbf{H}(\mathbf{s})\mathbf{F}(\mathbf{s}) \tag{2.2-4}$$

where H(s) is called the **transfer function** of the system.

The reader may confirm that except for the variable, the transfer function of eq. 2.2-4 is exactly the same as **H(p)** developed in Chapter 1.3.

An important variant of the linear differential equation, one that arises in writing electric circuit equations, is the **integro – differential equation**

$$\mathbf{a}_2\frac{\mathbf{dx(t)}}{\mathbf{dt}} + \mathbf{a}_1\mathbf{x(t)} + \mathbf{a}_0\int_{-\infty}^{\mathbf{t}}\mathbf{x}(\lambda)\mathbf{d}\lambda = \mathbf{b}_1\frac{\mathbf{df(t)}}{\mathbf{dt}} + \mathbf{b}_0\mathbf{f(t)} \tag{2.2-5}$$

The keen-eyed reader will note the apparent contradiction between the lower limit of the integral in this equation and the assertion that the system is assumed to begin its operation at t = 0. This issue is addressed in the following theorem.

Theorem 2.2 Time Integration

$$\text{If } \mathbf{L}[\mathbf{x(t)}] = \mathbf{X(s)} \text{ then } \mathbf{L}\left[\int_{-\infty}^{\mathbf{t}}\mathbf{x(u)du}\right] = \frac{\mathbf{X(s)}}{\mathbf{s}} + \frac{\mathbf{1}}{\mathbf{s}}\int_{-\infty}^{\mathbf{0^-}}\mathbf{x(u)du} \tag{2.2-6}$$

Proof: Integrating by parts

$$\mathbf{L}\left[\int_{-\infty}^{\mathbf{t}}\mathbf{x(u)du}\right] = \int_{\mathbf{0^-}}^{\infty}\left[\int_{-\infty}^{\mathbf{t}}\mathbf{x(u)du}\right]\mathbf{e^{-st}dt} = -\frac{\mathbf{e^{-st}}}{\mathbf{s}}\int_{-\infty}^{\mathbf{t}}\mathbf{x(u)du}\Bigg|_{\mathbf{0^-}}^{\infty} + \frac{\mathbf{1}}{\mathbf{s}}\int_{\mathbf{0^-}}^{\infty}\mathbf{x(t)e^{-st}dt}$$

$$= \frac{\mathbf{1}}{\mathbf{s}}\int_{-\infty}^{\mathbf{0^-}}\mathbf{x(u)du} + \frac{\mathbf{X(s)}}{\mathbf{s}} = \frac{\mathbf{a}}{\mathbf{s}} + \frac{\mathbf{X(s)}}{\mathbf{s}}$$

where we assume that as $\mathbf{t} \to \infty$, $\mathbf{e^{-st}} \to 0$. Note that if

$$\int_{-\infty}^{\mathbf{0^-}}\mathbf{x(u)du} = \mathbf{0} \quad \textbf{then} \quad \mathbf{L}\left[\int_{-\infty}^{\mathbf{t}}\mathbf{x(u)du}\right] = \frac{\mathbf{X(s)}}{\mathbf{s}}$$

end of proof

The value $\mathbf{a} = \int_{-\infty}^{0^-} \mathbf{f(u)du}$ is an initial condition representing the accumulated past history of the variable x(t) up until the time t = 0^-, when the system is presumed to begin operating. It can represent the charge on a capacitor or the current in an inductance at t = 0.

The Laplace Transform of eq. 2.2-5 can now be written as:

$$\mathbf{a_2[sX(s) - x_0] + a_1X(s) + a_0\left[\frac{a}{s} + \frac{X(s)}{s}\right] = [b_1s + b_0]F(s)} \qquad (2.2\text{-}7a)$$

and collecting terms

$$\left[\mathbf{a_2s + a_1 + \frac{a_0}{s}}\right]\mathbf{X(s) = [b_1s + b_0]F(s) + \left[a_2x_0 - a_0\frac{a_1}{s}\right]} \qquad (2.2\text{-}7b)$$

and finally

$$\mathbf{X(s) = \frac{[b_1s + b_0]s}{[a_2s^2 + a_1s + a_0]}F(s) + \frac{[a_2x_0s - a_0a_1]}{[a_2s^2 + a_1s + a_0]}} \qquad (2.2\text{-}7c)$$

which is of the same form as eq. 2.2-3d.

When the initial conditions are all zero, eq. 2.2-7c becomes

$$\mathbf{X(s) = \frac{[b_1s + b_0]s}{[a_2s^2 + a_1s + a_0]}F(s) = H(s)F(s)} \qquad (2.2\text{-}8)$$

which has the same form as eq. 2.2-4.

This fundamental input – output relationship $\mathbf{X(s) = H(s)F(s)}$, where H(s) is called the **transfer function** and where F(s) and X(s) are the Laplace Transforms of f(t) and x(t), is shown in **fig 2.2-1.**

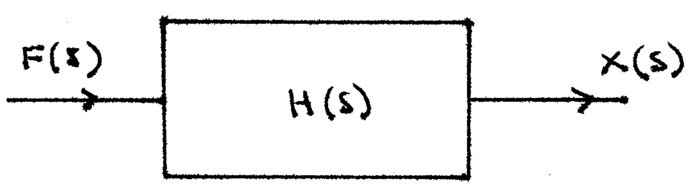


fig 2.2-1

This chapter began with the assertion that continuous linear systems are described by linear differential equations. This is the point at which the consequences and characteristics of linearity can begin to be explored. Closely associated with linearity is the idea of superposition. The following theorem begins the examination of these ideas.

Theorem 2.3 Linearity and Superposition

$$\text{If } \mathbf{L}\left[\mathbf{f}_1(\mathbf{t})\right]=\mathbf{F}_1(\mathbf{s}) \text{ and } \mathbf{L}\left[\mathbf{f}_2(\mathbf{t})\right]=\mathbf{F}_2(\mathbf{s}) \text{ then } \mathbf{L}\left[\alpha\mathbf{f}_1(\mathbf{t})+\beta\mathbf{f}_2(\mathbf{t})\right]=\alpha\mathbf{F}_1(\mathbf{s})+\beta\mathbf{F}_2(\mathbf{s}) \tag{2.2-9}$$

Proof:

$$\mathbf{L}\left[\alpha x_1(t)+\beta x_2(t)\right]=\int_{0^-}^{\infty}\left[\alpha x_1(t)+\beta x_2(t)\right]e^{-st}dt$$

$$=\alpha\int_{0^-}^{\infty}x_1(t)e^{-st}dt+\beta\int_{0^-}^{\infty}x_2(t)e^{-st}dt = \alpha X_1(s)+\beta X_2(s)$$

end of proof

The consequence of this theorem can be seen by examining its impact on eq. 2.2-3d or eq. 2.2-7c. If the forcing function **f(t)** is replaced with $\mathbf{f}(\mathbf{t})=\alpha\mathbf{f}_1(\mathbf{t})+\beta\mathbf{f}_2(\mathbf{t})$ then its Laplace Transform will become $\mathbf{F}(\mathbf{s})=\alpha\mathbf{F}_1(\mathbf{s})+\beta\mathbf{F}_2(\mathbf{s})$ and the response will be:

$$\mathbf{X}(\mathbf{s})=\mathbf{H}(\mathbf{s})\left[\alpha\mathbf{F}_1(\mathbf{s})+\beta\mathbf{F}_2(\mathbf{s})\right] + \mathbf{initial\_conditions}(\mathbf{s}) \tag{2.2-10}$$

A formal definition of linearity and superposition in systems is:

Definition 2.2-1 Linearity and Superposition in Systems

A linear system may be characterised by an operation that it performs on an input. Designating that operation as OP, f(t) as the input, and x(t) as the output, then linearity is defined as:

if: $\mathbf{f}_1(\mathbf{t})\rightarrow\mathbf{OP}\left[\mathbf{f}_1(\mathbf{t})\right]=\mathbf{x}_1(\mathbf{t})$

and: $\mathbf{f}_2(\mathbf{t})\rightarrow\mathbf{OP}\left[\mathbf{f}_2(\mathbf{t})\right]=\mathbf{x}_2(\mathbf{t})$

then:

$$\begin{aligned}&\alpha\mathbf{f}_1(\mathbf{t})+\beta\mathbf{f}_2(\mathbf{t})\rightarrow\mathbf{OP}\left[\alpha\mathbf{f}_1(\mathbf{t})+\beta\mathbf{f}_2(\mathbf{t})\right]=\\&\qquad=\alpha\mathbf{OP}\left[\mathbf{f}_1(\mathbf{t})\right]+\beta\mathbf{OP}\left[\mathbf{f}_2(\mathbf{t})\right]=\alpha\mathbf{x}_2(\mathbf{t})+\beta\mathbf{x}_2(\mathbf{t})\end{aligned} \tag{2.2-11}$$

Considering eq. 2.2-10 in relation to the definition of eq.1.2-11, we can see that if the term involving the initial conditions is equal to zero, then the definition of linearity and superposition is satisfied. Drawing on the definition of zero initial conditions as being the **zero – state**, we can characterize LDE's as systems having **zero – state linearity**.

Returning to fig 2.2-1, we see that **X(s) = H(s)F(s)** is a compact input – output relationship for the system in the Laplace Transform domain or "s – domain" if the initial conditions are zero. We shall now establish that there is a corresponding compact input – output relationship in the time domain that involves the impulse response h(t).

In order to show this, let's first find the Laplace Transform of the unit impulse function $\mathbf{f(t) = \delta(t)}$:

$$\begin{aligned} \mathbf{F(s) = L[f(t)]} &= \int_{0^-}^{\infty} \mathbf{f(t)e^{-st}dt} \\ &= \int_{0^-}^{\infty} \mathbf{\delta(t)e^{-st}dt} = \int_{0^-}^{0^+} \mathbf{\delta(t)dt = 1} \end{aligned} \qquad (2.2\text{-}12)$$

If the system input is $\mathbf{f(t) = \delta(t)}$, the unit impulse function, then the output will be $\mathbf{x(t) = h(t)}$, the impulse response of the system. In the Laplace Transform domain, this corresponds to an input $\mathbf{F(s) = 1}$, so that the system output is $\mathbf{X(s) = H(s)}$. We may therefore reasonably say that **H(s), the transfer function**, is the Laplace Transform of **h(t),** the **impulse response** of the system. Another way of saying this is that h(t) is the inverse Laplace Transform of H(s).

We are making a reasonable and basic assumption:

If a function x(t) has a Laplace Transform X(s), X(s) = L[x(t)], then the function X(s) has an inverse Laplace Transform x(t), x(t) = $\boldsymbol{L^{-1}}$***[X(s)].***

This reciprocal relationship is expressed mathematically as:

$$\begin{aligned} \mathbf{H(s)} &= \mathbf{L[h(t)]} \\ \mathbf{h(t)} &= \mathbf{L^{-1}[H(s)]} \end{aligned} \qquad (2.2\text{-}13)$$

We have a collection of Laplace Transform pairs: the input $\mathbf{f(t) \leftrightarrow F(s)}$; the output $\mathbf{x(t) \leftrightarrow X(s)}$; and the impulse response / transfer function $\mathbf{h(t) \leftrightarrow H(s)}$. It stands to reason that there will be an input – output relationship in the time domain that corresponds to $\mathbf{X(s) = H(s)F(s)}$, the input – output relationship in the Laplace Transform domain. This time domain relationship is called **<u>convolution</u>** . These two relationships are

$$\mathbf{X(s) = H(s)F(s)} \tag{2.2-14a}$$

$$\mathbf{x(t) = \int_{-\infty}^{\infty} h(\lambda)f(t-\lambda)d\tau = h(t) * f(t)} \tag{2.2-14b}$$

The proof that the two expressions of eq. 1.2-14 are a Laplace Transform pair is given in the following theorem.

<u>Theorem 2.4 Convolution</u>

Let L[x(t)] = X(s), L[F(t)] = F(s), and L[h(t)] = H(s). Then

$$\left\{\mathbf{x(t) = \int_{0^-}^{\infty} h(\lambda)f(t-\lambda)d\lambda}\right\} \quad \Leftrightarrow \quad \left\{\mathbf{X(s) = H(s)F(s)}\right\} \tag{2.2-15}$$

are a Laplace Transform pair.

<u>Proof:</u>

$$\mathbf{X(s) = L\left[x(t)\right] = \int_{0^-}^{\infty} x(t)e^{-st}dt = \int_{0^-}^{\infty}\left[\int_{0^-}^{\infty} h(\lambda)f(t-\lambda)d\lambda\right]e^{-st}dt =}$$

$$\mathbf{\int_{0^-}^{\infty}\int_{0^-}^{\infty} h(\lambda)e^{-s\lambda}f(t-\lambda)e^{-s(t-\lambda)}d\lambda dt = \int_{0^-}^{\infty} h(\lambda)e^{-s\lambda}d\lambda \int_{0^-}^{\infty} f(t-\lambda)e^{-s(t-\lambda)}dt}$$

Letting u = t - λ,

$$\mathbf{X(s) = H(s)\int_{-\lambda}^{\infty} f(u)e^{-su}\,du = H(s)\int_{0^-}^{\infty} f(u)e^{-su}\,du = H(s)F(s)}$$

where the change in the lower limit is a consequence of the one sided character of the Laplace Transform.

The proof of this Theorem also demonstrates that

$$\mathbf{x(t) = \int_{-\infty}^{\infty} h(\lambda)f(t-\lambda)d\tau = \int_{-\infty}^{\infty} f(\lambda)h(t-\lambda)d\tau} \tag{2.2-16a}$$

or:

$$x(t) = f(t) * h(t) = h(t) * f(t) \tag{2.2-16b}$$

or, in other words, convolution is a commutative operation.

end of proof

The mathematical relationships of eq. 2.2-15 are illustrated in **fig. 2.2-2**.

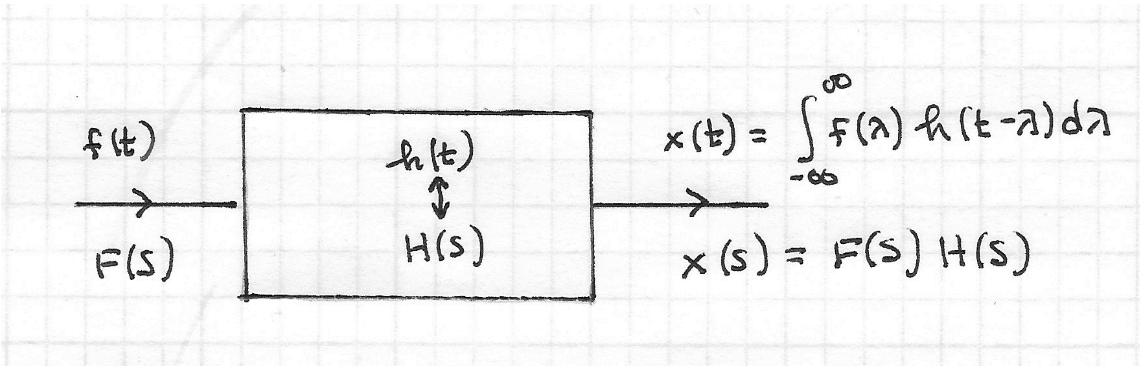


fig. 2.2-2

This figure sums up not only the basic input - output relationships of a linear system, but also the relationship between the time domain and the s – domain. As such, it represents a central idea in the theory of linear systems. We will no longer think about these systems as differential equations. Rather, we shall think of them either as transfer functions or as impulse responses.

Computing the system output using the convolution integral will be the subject of Section 2.5 of this Chapter. At this point, it is sufficient to recognize the conceptual importance of fig. 2.2-3.

This section has set up the framework for the understanding of the behavior of linear systems. In order to develop this understanding, we will have to:

1. Examine more of the theorems of Laplace Transforms and learn how to find the transforms of specific time functions. This will be accomplished in Section 2.3.

2. Learn how to find the inverse Laplace Transform of a function X(s) that is the ratio of two polynomials $X(s) = N(s)/D(s)$. This will be accomplished in Section 2.4.

The combination of Sections 2.3 and 2.4 will allow the computation of the transfer function from the impulse response and the impulse response from the transfer function.

3. Learn how to directly evaluate the convolution integral, eq. 2.2.16. This will be accomplished in Section 2.5.

* 2.2-1 Simultaneous Linear Differential Equations

Laplace Transforms also make the solution of simultaneous LDE's into an algebraic problem. Consider the following example:

Example 2.2-1 Two simultaneous LDE's are:

$$3\frac{\mathbf{d}^2\mathbf{x}_1(\mathbf{t})}{\mathbf{dt}^2} + 2\mathbf{x}_1(\mathbf{t}) + 5\frac{\mathbf{dx}_2(\mathbf{t})}{\mathbf{dt}} - 4\mathbf{x}_2(\mathbf{t}) = \mathbf{f}_1(\mathbf{t})$$

$$2\frac{\mathbf{dx}_1(\mathbf{t})}{\mathbf{dt}} - 4\mathbf{x}_1(\mathbf{t}) + 2\frac{\mathbf{d}^2\mathbf{x}_2(\mathbf{t})}{\mathbf{dt}^2} + \frac{\mathbf{dx}_2(\mathbf{t})}{\mathbf{dt}} + 3\mathbf{x}_2(\mathbf{t}) = \mathbf{f}_2(\mathbf{t}) \qquad (2.2\text{-}17a)$$

where the initial conditions are: $\mathbf{x_1(0^-) = a,\ x_1'(0^-) = b,\ x_2(0^-) = c,\ x_2'(0^-) = d}$. (Note that the initial conditions are being included in this process only for illustration. They will subsequently be set equal to zero, corresponding to the previous discussion.)

Again using the time differentiation theorem, the Laplace Transform of these two equations is:

$$\mathbf{3\left[s^2X_1(s) - as - b\right] + 2X_1(s) + 5\left[sX_2(s) - c\right] - 4X_2(s) = F_1(s)}$$

$$\mathbf{2\left[sX_1(s) - c\right] - 4X_1(s) + 2\left[s^2X_2(s) - cs - d\right] + \left[sX_2(s) - c\right] + 3X_2(s) = F_2(s)} \qquad (2.2\text{-}17b)$$

and gathering terms:

$$\mathbf{\left[3s^2 + 2\right]X_1(s) + \left[5s - 4\right]X_2(s) = F_1(s) + 3(as + b) + 5c}$$

$$\mathbf{\left[2s - 4\right]X_1(s) + \left[2s^2 + s + 3\right]X_2(s) = F_2(s) + 2c + 2(cs + d)} \qquad (2.2\text{-}17c)$$

Using standard algebraic techniques, these two equations may be solved simultaneously, yielding:

$$\mathbf{\left[\left(2s^2 + s + 3\right)\left(3s^2 + 2\right) - (5s - 4)(2s - 4)\right]X_1(s) =}$$
$$\mathbf{= \left(2s^2 + s + 3\right)\left[F_1(s) + 3(as + b) + 5c\right] - (5s - 4)\left[F_2(s) + 2c + 2(cs + d)\right]}$$

and:

$$\mathbf{-\left[\left(2s^2 + s + 3\right)\left(3s^2 + 2\right) - (5s - 4)(2s - 4)\right]X_2(s) =}$$
$$\mathbf{= (2s - 4)\left[F_1(s) + 3(as + b) + 5c\right] - \left(3s^2 + 2\right)\left[F_2(s) + 2c + 2(cs + d)\right]} \qquad (2.2\text{-}17d)$$

Now, solving explicitly for $\mathbf{X_1(s)}$ and $\mathbf{X_2(s)}$, we get:

$$\mathbf{X_1(s) = \frac{(2s^2+s+3)}{6s^4+3s^3+3s^2+30s-10}F_1(s) - \frac{(5s-4)}{6s^4+3s^3+3s^2+30s-10}F_2(s)}$$

$$\mathbf{+\frac{(2s^2+s+3)[3(as+b)+5c]-(5s-4)[2c+2(cs+d)]}{6s^4+3s^3+3s^2+30s-10}}$$

$$\mathbf{X_2(s) = \frac{(2s-4)}{6s^4+3s^3+3s^2+30s-10}F_1(s) - \frac{(3s^2+2)}{6s^4+3s^3+3s^2+30s-10}F_2(s)}$$

(2.2-17e)

$$\mathbf{+\frac{(2s-4)[3(as+b)+5c]-(3s^2+2)[2c+2(cs+d)]}{6s^4+3s^3+3s^2+30s-10}}$$

Finally, if we set the initial conditions equal to zero:

$$\begin{aligned}\mathbf{X_1(s)} &= \mathbf{H_{11}(s)F_1(s) + H_{12}(s)F_2(s)}\\ \mathbf{X_2(s)} &= \mathbf{H_{21}(s)F_1(s) + H_{22}(s)F_2(s)}\end{aligned} \qquad (2.2\text{-}17f)$$

The structure of this set of equations is shown in fig 2.2-3

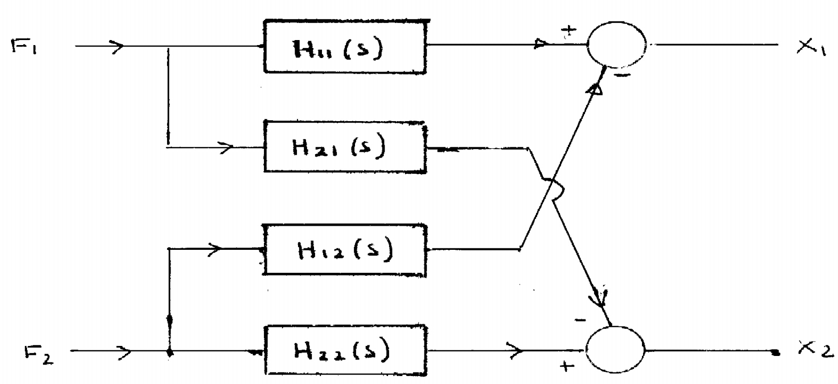


fig. 2.2-3

There are four separate transfer functions associated with this system and it is worth noting they all have the same denominator. The consequence of this observation will become clear when we discuss the inverse Laplace Transform.

<u>**end of example**</u>

***2.2-2 Electric Circuits and Transfer Functions**

As we have seen above, the Laplace Transform of the **integro – differential equation**

$$\mathbf{a}_2 \frac{\mathbf{dx(t)}}{\mathbf{dt}} + \mathbf{a}_1\mathbf{x(t)} + \mathbf{a}_0 \int_{-\infty}^{\mathbf{t}} \mathbf{x(\lambda)d\lambda} = \mathbf{b}_1 \frac{\mathbf{df(t)}}{\mathbf{dt}} + \mathbf{b}_0\mathbf{f(t)} \qquad (2.2\text{-}5)$$

is:

$$\mathbf{a}_2\left[\mathbf{sX(s)} - \mathbf{x}_0\right] + \mathbf{a}_1\mathbf{X(s)} + \mathbf{a}_0\left[\frac{\mathbf{a}}{\mathbf{s}} + \frac{\mathbf{X(s)}}{\mathbf{s}}\right] = \left[\mathbf{b}_1\mathbf{s} + \mathbf{b}_0\right]\mathbf{F(s)} \qquad (2.2\text{-}7a)$$

and, with initial conditions set to zero,

$$\mathbf{a_2sX(s) + a_1X(s) + a_0 \frac{X(s)}{s} = [b_1s + b_0]F(s)} \qquad (2.2\text{-}18)$$

In short, the Laplace Transform turns a differential equation into an algebraic equation. This has a profound impact on how electric circuits may be analyzed.

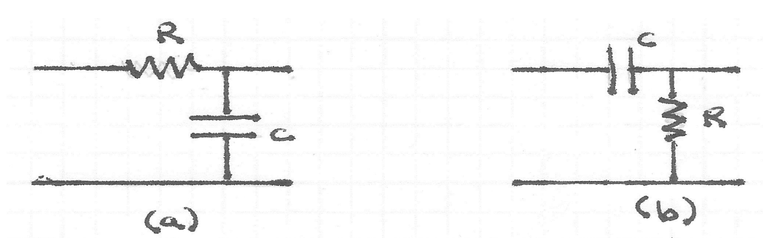
The relationships between voltage and current for various circuit elements in the time domain and in the Laplace Transform domains (with zero initial conditions) are:

Resistance: $\mathbf{v(t) = Ri(t) \quad \Leftrightarrow \quad V(s) = RI(s)}$ (2.2-19a)

Capacitance: $\mathbf{v(t) = \frac{1}{C}\int i(\lambda)d\lambda \quad \Leftrightarrow \quad V(s) = \frac{1}{sC}I(s)}$ (2.2-19b)

Inductance: $\mathbf{v(t) = L\frac{di(t)}{dt} \quad \Leftrightarrow \quad V(s) = sLI(s)}$ (2.2-19c)

As a consequence, in the Laplace Transform domain, each of the circuit elements looks like a resistance (R, sL, 1/sC) and circuit analysis may proceed according to the simple rules of resistance circuits including the voltage and current division rules. Accordingly, the transfer functions of the circuits in **fig. 2.2-4** are:



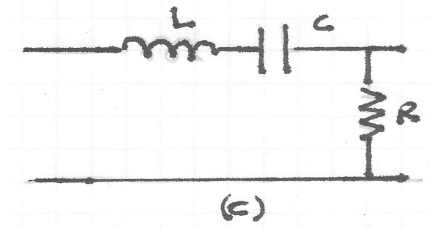


fig. 2.2-4

for fig. 2.2-4a

$$\mathbf{V(s) = \frac{1/sC}{R + 1/sC} E(s) \quad \Rightarrow \quad \frac{V(s)}{E(s)} = H(s) = \frac{1}{1+sRC}} \tag{2.2-20a}$$

for fig. 2.2-4b

$$\mathbf{V(s) = \frac{R}{R + 1/sC} E(s) \quad \Rightarrow \quad \frac{V(s)}{E(s)} = H(s) = \frac{sRC}{1+sRC}} \tag{2.2-20b}$$

and for fig. 2.2-4c

$$\mathbf{V(s) = \frac{R}{R + 1/sC + sL} E(s) \quad \Rightarrow \quad \frac{V(s)}{E(s)} = H(s) = \frac{sRC}{1+sRC+s^2LC}} \tag{2.2-20c}$$

For more complex circuits, we may simply assume zero initial conditions and take the Laplace Transform of the circuit as shown in fig 2.2-5.

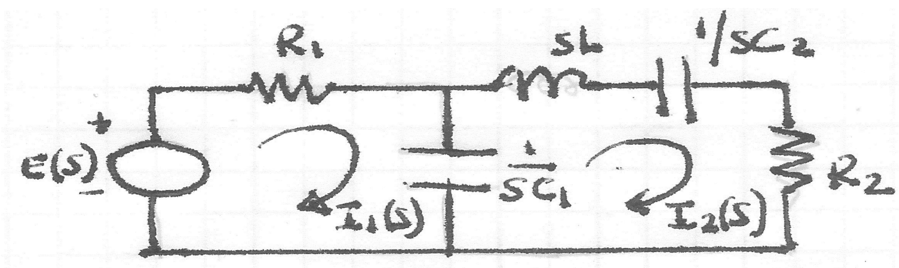


fig 2.2-5

We may then simply use Kirchoff's Laws for resistance circuits to write circuit equations directly from fig 2.2-5b. Using Kirchoff's Voltage Law to write loop equations:

$$\mathbf{E(s) = R_1 I_1(s) + \frac{1}{sC_1}\left[I_1(s) - I_2(s)\right]} \tag{2.2-21a}$$

$$\mathbf{0} = \frac{1}{sC_1}\left[I_2(s) - I_1(s)\right] + \left[sL + R_2 + \frac{1}{sC_2}\right] I_2(s) \qquad (2.2\text{-}21b)$$

Eliminating $\mathbf{I_1(s)}$:

$$E(s) = \left[\left(R_1 + \frac{1}{sC_1}\right)\left(1 + \frac{C_1}{C_2} + sC_1R_2 + s^2LC_1\right) - \frac{1}{sC_1}\right] I_2(s) \qquad (2.2\text{-}21c)$$

and, since $\mathbf{V(s)} = \frac{1}{sC_2} \mathbf{I_2(s)}$:

$$\frac{V(s)}{E(s)} = H(s) = \frac{1}{sC_2\left[\left(R_1 + \frac{1}{sC_1}\right)\left(1 + \frac{C_1}{C_2} + sC_1R_2 + s^2LC_1\right) - \frac{1}{sC_1}\right]} \qquad (2.2\text{-}21d)$$

or:

$$\frac{V(s)}{E(s)} = H(s) = \frac{1}{R_1LC_1C_2} \frac{1}{s^3 + \left(\frac{R_2}{L} + \frac{1}{R_1C_1}\right)s^2 + \left(\frac{1}{LC_1} + \frac{1}{LC_2} + \frac{R_2}{R_1LC_1}\right)s + \frac{1}{R_1LC_1C_2}} \qquad (2.2\text{-}21e)$$

yielding the transfer function of the circuit.

These examples serve to illustrate the point that Laplace Transforms make it straightforward to find a transfer function from a circuit. It is important to make this point because we shall subsequently discussing the transfer functions and frequency responses of filters and we want to be able to associate some of these responses with the transfer functions of particular circuits.

We shall conclude with a discussion of the Initial Value and Final Value Theorems. The Initial Value is the value of a variable in a differential equation or a circuit at $\mathbf{t = 0^+}$, immediately after a switch is closed or an input – perhaps an impulse function – is applied. The Final Value is the value of a variable after a long time – perhaps the final value of a voltage on a capacitor or current in ans inductor.

Theorem 2.5 The Initial Value Theorem

If $\mathbf{LT[f(t)] = F(s)}$ then $\mathbf{f}(0^+) = \lim\limits_{s \to \infty}[sF(s)]$ (2.2-22)

Proof: It follows from the time differentiation theorem that

$$\mathbf{L}\left[\frac{\mathbf{df(t)}}{\mathbf{dt}}\right]=\int_{0^-}^{\infty}\frac{\mathbf{df(t)}}{\mathbf{dt}}\mathbf{e^{-st}dt}=\int_{0^-}^{0^+}\frac{\mathbf{df(t)}}{\mathbf{dt}}\mathbf{e^{-st}dt}+\int_{0^+}^{\infty}\frac{\mathbf{df(t)}}{\mathbf{dt}}\mathbf{e^{-st}dt}=$$
$$=\mathbf{sF(s)}-\mathbf{f}\left(0^-\right)$$

where:

$$\int_{0^-}^{0^+}\frac{\mathbf{df(t)}}{\mathbf{dt}}\mathbf{e^{-st}dt}=\int_{0^-}^{0^+}\frac{\mathbf{df(t)}}{\mathbf{dt}}\mathbf{dt}=\int_{0^-}^{0^+}\mathbf{df(t)}=\mathbf{f}\left(0^+\right)-\mathbf{f}\left(0^-\right)$$

so that

$$\mathbf{f}\left(0^+\right)-\mathbf{f}\left(0^-\right)+\int_{0^+}^{\infty}\frac{\mathbf{df(t)}}{\mathbf{dt}}\mathbf{e^{-st}dt}=\mathbf{sF(s)}-\mathbf{f}\left(0^-\right)$$

or

$$\mathbf{f}\left(0^+\right)+\int_{0^+}^{\infty}\frac{\mathbf{df(t)}}{\mathbf{dt}}\mathbf{e^{-st}dt}=\mathbf{sF(s)}$$

As $s \to \infty$ the integral on the left hand side goes to zero yielding the result.

end of proof

The Initial Value Theorem can help us to understand what happens when a unit impulse function is applied to a system that has zero initial conditions at $t = 0^-$. Recall that in the Laplace Transform domain the output will be X(s) = H(s) and, correspondingly, in the time domain the output will be h(t). Applying an impulse function to the input of an LDE (or circuit) at t = 0 results in a change in the initial conditions from $t = 0^-$ to $t = 0^+$. The Initial Value Theorem is useful in finding these changes. In circuits, these changes appear as changes in capacitor voltages and inductor currents. If we want to find the new initial condition on a capacitor C, we write the transfer function $V_C(s) = H(s)E(s)$, allow E(s) = 1, corresponding to e(t) = δ(t), and then apply the Theorem.

Example 2.2-2

In the circuit of **fig 2.2-6**, the initial conditions $\mathbf{v}_1\left(0^-\right)=\mathbf{v}_2\left(0^-\right)=0$. The object is

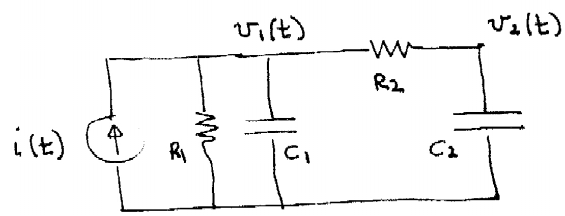


fig. 2.2-6

to find the values of $\mathbf{v}_1(0^+)$ **and** $\mathbf{v}_2(0^+)$ with the input current $i(t) = A\delta(t)$. After a bit of manipulation it can be shown that

$$\mathbf{V}_1(\mathbf{s}) = \frac{\mathbf{R}_1(1+\mathbf{s}\mathbf{R}_2\mathbf{C}_2)}{\mathbf{s}^2\mathbf{R}_1\mathbf{C}_1\mathbf{R}_2\mathbf{C}_2 + \mathbf{s}(\mathbf{R}_1\mathbf{C}_1 + \mathbf{R}_2\mathbf{C}_2 + \mathbf{R}_1\mathbf{C}_2) + 1}\mathbf{I}(\mathbf{s}) \tag{2.2-23a}$$

and

$$\mathbf{V}_2(\mathbf{s}) = \frac{\mathbf{R}_1}{\mathbf{s}^2\mathbf{R}_1\mathbf{C}_1\mathbf{R}_2\mathbf{C}_2 + \mathbf{s}(\mathbf{R}_1\mathbf{C}_1 + \mathbf{R}_2\mathbf{C}_2 + \mathbf{R}_1\mathbf{C}_2) + 1}\mathbf{I}(\mathbf{s}) \tag{2.2-23b}$$

Invoking the initial value theorem with I(s) = A

$$\mathbf{v}_1(0^+) = \lim_{\mathbf{s}\to\infty} \frac{\mathbf{s}\mathbf{R}_1(1+\mathbf{s}\mathbf{R}_2\mathbf{C}_2)}{\mathbf{s}^2\mathbf{R}_1\mathbf{C}_1\mathbf{R}_2\mathbf{C}_2 + \mathbf{s}(\mathbf{R}_1\mathbf{C}_1 + \mathbf{R}_2\mathbf{C}_2 + \mathbf{R}_1\mathbf{C}_2) + 1}\mathbf{A} = \frac{\mathbf{A}}{\mathbf{C}_1} \tag{2.2-23c}$$

and

$$\mathbf{v}_2(0^+) = \lim_{\mathbf{s}\to\infty} \frac{\mathbf{s}\mathbf{R}_1}{\mathbf{s}^2\mathbf{R}_1\mathbf{C}_1\mathbf{R}_2\mathbf{C}_2 + \mathbf{s}(\mathbf{R}_1\mathbf{C}_1 + \mathbf{R}_2\mathbf{C}_2 + \mathbf{R}_1\mathbf{C}_2) + 1}\mathbf{A} = 0 \tag{2.2-23d}$$

This demonstrates that an impulse function at t = 0 can change the initial conditions in a circuit and shows how to find these new initial conditions easily.

end of example

Just as the Initial Value Theorem allows us to find the value of a circuit variable at $t = 0^+$, the Final Value Theorem allows us to find the value of that variable at t = ∞. A small bit of thought reveals that if the input to a circuit is a unit step, effectively a dc, then the final value of an inductor current or a capacitor voltage may be nonzero. The use of this Theorem will make finding these final values very straightforward.

Theorem 2.6 The Final Value Theorem

If $\mathbf{LT}[\mathbf{f}(\mathbf{t})] = \mathbf{F}(\mathbf{s})$ then $\lim_{\mathbf{t}\to\infty} \mathbf{f}(\mathbf{t}) = \lim_{\mathbf{s}\to 0}[\mathbf{s}\mathbf{F}(\mathbf{s})]$ (2.2-24)

Proof: From the proof of the initial value theorem

$$\mathbf{f}(0^+) + \int_{0^+}^{\infty} \frac{\mathbf{df}(\mathbf{t})}{\mathbf{dt}} \mathbf{e}^{-\mathbf{st}}\mathbf{dt} = \mathbf{s}\mathbf{F}(\mathbf{s})$$

As $s \to 0$

$$\mathbf{f}(0^+) + \int_{0^+}^{\infty} \frac{\mathbf{df(t)}}{\mathbf{dt}} \mathbf{dt} = \lim_{\mathbf{s} \to 0} \mathbf{sF(s)}$$

or

$$\mathbf{f}(0^+) + \mathbf{f}(\infty) - \mathbf{f}(0^+) = \lim_{\mathbf{s} \to 0} \mathbf{sF(s)} = \mathbf{f}(\infty)$$

end of proof

Example 2.2-3

Let the input current in the circuit of **fig. 2.2-7** be $i(t) = Au(t)$ so that I(s) = A/s. We have seen that:

$$\mathbf{V}_1(\mathbf{s}) = \frac{\mathbf{R}_1(1 + \mathbf{sR}_2\mathbf{C}_2)\mathbf{A}}{\mathbf{s}\left[\mathbf{s}^2\mathbf{R}_1\mathbf{C}_1\mathbf{R}_2\mathbf{C}_2 + \mathbf{s}(\mathbf{R}_1\mathbf{C}_1 + \mathbf{R}_2\mathbf{C}_2 + \mathbf{R}_1\mathbf{C}_2) + 1\right]} \qquad (2.2\text{-}25a)$$

and

$$\mathbf{V}_2(\mathbf{s}) = \frac{\mathbf{R}_1\mathbf{A}}{\mathbf{s}\left[\mathbf{s}^2\mathbf{R}_1\mathbf{C}_1\mathbf{R}_2\mathbf{C}_2 + \mathbf{s}(\mathbf{R}_1\mathbf{C}_1 + \mathbf{R}_2\mathbf{C}_2 + \mathbf{R}_1\mathbf{C}_2) + 1\right]} \qquad (2.2\text{-}25b)$$

According to the final value theorem,

$$\mathbf{v}_1(\infty) = \lim_{\mathbf{s} \to 0} \frac{\mathbf{sR}_1(1 + \mathbf{sR}_2\mathbf{C}_2)\mathbf{A}}{\mathbf{s}\left[\mathbf{s}^2\mathbf{R}_1\mathbf{C}_1\mathbf{R}_2\mathbf{C}_2 + \mathbf{s}(\mathbf{R}_1\mathbf{C}_1 + \mathbf{R}_2\mathbf{C}_2 + \mathbf{R}_1\mathbf{C}_2) + 1\right]} = \mathbf{AR}_1 \qquad (2.2\text{-}25c)$$

$$\mathbf{v}_2(\infty) = \lim_{\mathbf{s} \to 0} \frac{\mathbf{sR}_1\mathbf{A}}{\mathbf{s}\left[\mathbf{s}^2\mathbf{R}_1\mathbf{C}_1\mathbf{R}_2\mathbf{C}_2 + \mathbf{s}(\mathbf{R}_1\mathbf{C}_1 + \mathbf{R}_2\mathbf{C}_2 + \mathbf{R}_1\mathbf{C}_2) + 1\right]} = \mathbf{AR}_1 \qquad (2.2\text{-}25d)$$

which is the voltage across the resistor R_1 in the steady state.

end of example

2.3 Laplace Transforms and Their Properties

The purpose of this section is to develop a library of Laplace Transforms of some common functions that arise in the analysis of linear differential equations and, simultaneously, through a collection of Theorems, uncover some of the properties of Laplace Transforms. We will first find the Laplace Transform of the unit step function. As we shall see, this Laplace Transform of this function is the foundation for finding the transforms of the entire class of functions associated with LDE's.

Example 2.3-1 The Laplace Transform of the Unit Step u(t)

The **unit step function u(t)** is defined as

$$u(t) = \begin{cases} 1; & t \geq 0 \\ 0; & t < 0 \end{cases} \tag{2.3-1a}$$

and is shown in **fig 2.3-1a.**

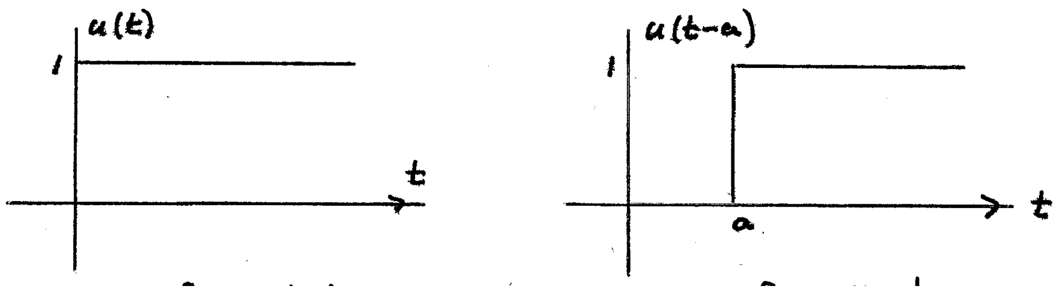


fig. 2.3-1.

The same step function shifted to the right by **t = a** is shown in **fig. 2.3-1b.** It is defined as

$$\mathbf{u(t-a)} = \begin{cases} \mathbf{1;} & \mathbf{t \geq a} \\ \mathbf{0;} & \mathbf{t < a} \end{cases} \tag{2.3-1b}$$

Using the defining integral of the Laplace Transform, eq. 2.1-1,

$$\mathbf{L[u(t)]} = \int_{0^-}^{\infty} \mathbf{u(t)e^{-st}dt} = \int_{0^+}^{\infty} \mathbf{u(t)e^{-st}dt} = \left.\frac{\mathbf{e^{-st}}}{\mathbf{-s}}\right|_{0^+}^{\infty} = \frac{1}{\mathbf{s}} \tag{2.3-1c}$$

In evaluating this integral, we have taken into account that u(t) = 0 for t < 0 by changing the lower limit to 0^+. Evaluating the integral at the upper limit yields zero and, at the lower limit, yields 1/s. The Laplace Transform of the time shifted step function will be easily derived a bit later.

The next theorem is the basis for extending the LT of the unit step to finding the transforms of a larger collection of functions.

Theorem 2.5 Multiplication by an Exponential

If $\mathbf{L}\left[\mathbf{f}(\mathbf{t})\right]=\mathbf{F}(\mathbf{s})$ then $\mathbf{L}\left[\mathbf{e}^{-\alpha\mathbf{t}}\mathbf{f}(\mathbf{t})\right]=\mathbf{F}(\mathbf{s}+\alpha)$ (2.3-2)

Proof:

$$\mathbf{L}\left[\mathbf{e}^{-\alpha\mathbf{t}}\mathbf{f}(\mathbf{t})\right]=\int_{0^-}^{\infty}\mathbf{e}^{-\alpha\mathbf{t}}\mathbf{f}(\mathbf{t})\mathbf{e}^{-\mathbf{st}}\mathbf{dt}=\int_{0^-}^{\infty}\mathbf{f}(\mathbf{t})\mathbf{e}^{-(\mathbf{s}+\alpha)\mathbf{t}}\mathbf{dt}=\mathbf{F}(\mathbf{s}+\alpha)$$

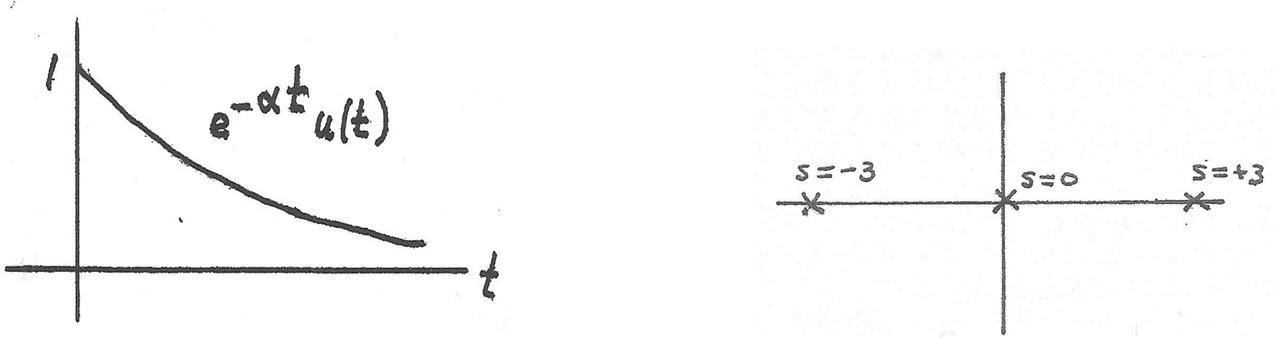
end of proof

Example 2.3-2 The Laplace Transform of a Decaying Exponential

A one sided decaying exponential is defined as

$$\mathbf{f}(\mathbf{t})=\mathbf{e}^{-\alpha\mathbf{t}}\mathbf{u}(\mathbf{t}) \tag{2.3-3a}$$

and is shown in **fig. 2.3-2.**



(a) **(b)**

fig. 2.3-2

Since $\mathbf{L}\left[\mathbf{u}(\mathbf{t})\right]=\frac{1}{\mathbf{s}}$, Theorem 2.3 implies

$$\mathbf{L}\left[\mathbf{e}^{-\alpha\mathbf{t}}\mathbf{u}(\mathbf{t})\right]=\frac{\mathbf{1}}{\mathbf{s}+\alpha} \tag{2.3-3b}$$

According to the Theorem, if $\mathbf{f}(\mathbf{t})=\mathbf{e}^{-3\mathbf{t}}\mathbf{u}(\mathbf{t})$ then $\mathbf{F}(\mathbf{s})=\frac{\mathbf{1}}{\mathbf{s}+\mathbf{3}}$.

end of example

Theorem 2.5 also raises the interesting question of the Laplace Transform of an increasing exponential: for example $\mathbf{g(t) = e^{3t}u(t)}$. According to the formalism of the Theorem, its Laplace Transform would be $\mathbf{G(s) = \frac{1}{s-3}}$.

In the first instance, s = -3 is characterized as a pole of F(s), the value of s that causes F(s) to become infinite. In the second instance, the pole of G(s) is s = +3.

Plotting these poles in the complex plane is a useful visual guide to understanding the LT of various functions. As shown in **fig. 2.3-2b,** the pole s = -3, associated with the decaying exponential, is on the negative real axis of the complex plane and the pole s = +3 is on the positive real axis of that plane. We may generalize to characterize all poles on the negative real axis to represent decaying exponentials and all poles on the positive real axis to represent increasing exponentials. In addition, from eq.2.3-1c, the pole at s = 0 associated with the unit step function is at the origin.

In Chapter 1 we saw that the impulse response of an RC circuit is a decaying exponential. This should make intuitive sense because that circuit has no internal energy sources. Furthermore, an output that is an increasing exponential would soon have the circuit start to smoke, burst into flames, and disintegrate. This condition is called "instability", something to be avoided.

Example 2.3-3 The Laplace Transform of Sinusoids

According to the Euler equations, sinusoidal functions can be represented by a sum of exponentials:

$$\mathbf{Cos(x)} = \frac{\mathbf{e^{jx} + e^{-jx}}}{2} \tag{2.3-4a}$$

$$\mathbf{Sin(x)} = \frac{\mathbf{e^{jx} - e^{-jx}}}{2\mathbf{j}} \tag{2.3-4b}$$

According to Theorem 2.5, $\mathbf{L}\left[\mathbf{e^{-j\beta t}u(t)}\right] = \frac{1}{\mathbf{s + j\beta}}$ and $\mathbf{L}\left[\mathbf{e^{j\beta t}u(t)}\right] = \frac{1}{\mathbf{s - j\beta}}$. So, using Theorem 2.3, Linearity and Superposition

1.

$$\mathbf{L}\left[\mathbf{Cos}(\beta \mathbf{t})\mathbf{u}(\mathbf{t})\right] = \mathbf{L}\left[\left(\frac{\mathbf{e}^{\mathbf{j}\beta\mathbf{t}} + \mathbf{e}^{-\mathbf{j}\beta\mathbf{t}}}{2}\right)\mathbf{u}(\mathbf{t})\right]$$

$$= \frac{1}{2}\left[\frac{1}{\mathbf{s}-\mathbf{j}\beta} + \frac{1}{\mathbf{s}+\mathbf{j}\beta}\right] = \frac{\mathbf{s}}{\mathbf{s}^2+\beta^2} \quad (2.3\text{-}5a)$$

2.

$$\mathbf{L}\left[\mathbf{Sin}(\beta \mathbf{t})\mathbf{u}(\mathbf{t})\right] = \mathbf{L}\left[\left(\frac{\mathbf{e}^{\mathbf{j}\beta\mathbf{t}} - \mathbf{e}^{-\mathbf{j}\beta\mathbf{t}}}{2\mathbf{j}}\right)\mathbf{u}(\mathbf{t})\right]$$

$$= \frac{1}{2\mathbf{j}}\left[\frac{1}{\mathbf{s}-\mathbf{j}\beta} - \frac{1}{\mathbf{s}+\mathbf{j}\beta}\right] = \frac{\beta}{\mathbf{s}^2+\beta^2} \quad (2.3\text{-}5b)$$

This gives us the Laplace Transforms of basic Sine and Cosine. Note that since these functions are multiplied by u(t), they are zero for t < 0.

End of example

The function $\mathbf{f_1(t) = Sin(\beta t)u(t)}$ and its poles are shown in **fig 2.3-3a.** Note that these poles $\mathbf{s = \pm j\beta}$ are complex conjugates of each other and lie on the imaginary axis. The sinusoid goes on forever without either decay or increase, corresponding to the poles lying on the boundary between stability and instability.

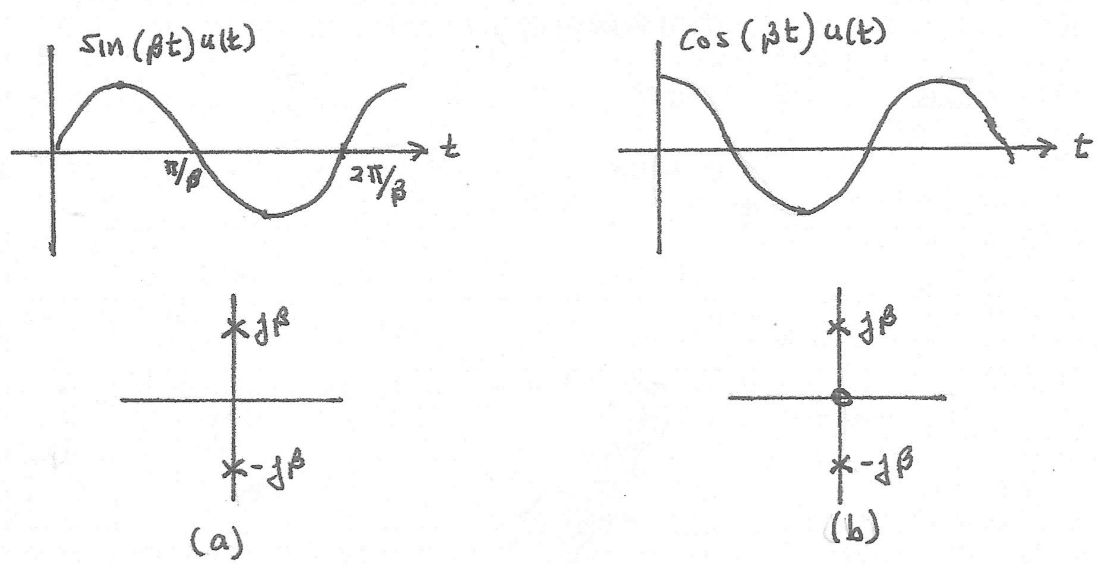


fig. 2.3-3

The function $\mathbf{f_2(t) = Cos(\beta t)u(t)}$ and its poles are shown in **fig 2.3-3b.** Note that the poles of $\mathbf{f_1(t)}$ and $\mathbf{f_2(t)}$ are the same but, corresponding to the **s** in the numerator of eq. 2.3-5b, there is also a zero at the origin.

Example 2.3-4 The Laplace Transform of Decaying Sinusoids We may easily extend Theorem 2.5 and the previous examples in order to find the transforms of the damped sinusoids:

$$1.\ \mathbf{L\left[e^{-\alpha t}Cos(\beta t)u(t)\right]} = \frac{\mathbf{s+\alpha}}{(\mathbf{s+\alpha})^2+\beta^2} \qquad (2.3\text{-}6a)$$

$$2.\ \mathbf{L\left[e^{-\alpha t}Sin(\beta t)u(t)\right]} = \frac{\beta}{(\mathbf{s+\alpha})^2+\beta^2} \qquad (2.3\text{-}6b)$$

end of example

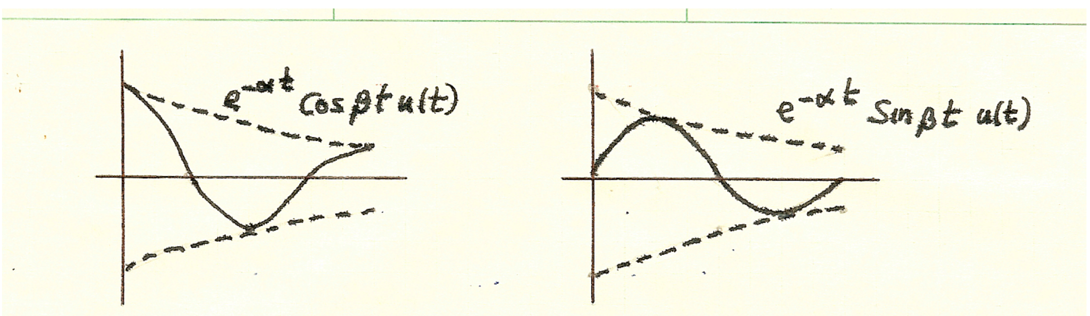


fig 2.3-4a

fig 2.3-4b

Note that the denominator of these two Laplace Transforms may be expanded and factored as

$$(s+\alpha)^2+\beta^2 = \left[s^2+2\alpha s+(\alpha+\beta)^2\right] = (s+\alpha-j\beta)(s+\alpha+j\beta) \qquad (2.3\text{-}6c)$$

As with Example 2.3-2, if $\alpha > \mathbf{0}$, the oscillation of the sinusoid will be multiplied by a decaying exponential, but if $\alpha < \mathbf{0}$ then the oscillation will continue to increase in magnitude, again resulting in a breakdown of a system.

These waveforms, together with the s – plane diagram of their poles and zeros are shown in **fig 2.3-4a,b.** In both instances, the poles are located at $\mathbf{s = -\alpha \pm j\beta}$, complex conjugate poles located in the left half of the complex plane. The decaying cosine also has a zero at $\mathbf{s = -\alpha}$.

Example 2.3-5

The purpose of this example to show what the Laplace Transform of a somewhat more complicated time function looks like. Let:

$$\mathbf{f_1(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)} \tag{2.3-7a}$$

Using linearity and superposition together with eq. 2.3-3b:

$$\mathbf{F_1(s) = \frac{3}{s+2} - \frac{2}{s+1} = \frac{(s-1)}{(s+2)(s+1)}} \tag{2.3-7b}$$

Next let:

$$\mathbf{f_2(t) = 3e^{-2t}u(t) + 2e^{-t}Cos(2t)u(t)} \tag{2.3-7c}$$

so that

$$\begin{aligned}\mathbf{F_2(s)} &= \mathbf{\frac{3}{(s+2)} + \frac{2(s+1)}{(s+1)^2+4}} = \\ &= \mathbf{\frac{3\left[(s+1)^2+4\right]+2(s+1)(s+2)}{(s+2)(s+1-j2)(s+1+j2)}} \\ &= \mathbf{\frac{5s^2+12s+19}{(s+2)(s+1-j2)(s+1+j2)}}\end{aligned} \tag{2.3-7d}$$

Note that F(s) is the ratio of two polynomials F(s) = N(s)/D(s). Its poles are at $\mathbf{s = -2}$ and $\mathbf{s = -1 \pm j2}$. This is typical of the Laplace Transforms that will be encountered in solving differential equations. Exercises that illustrate this are given in the Problems.

end of example

The next step is to find the Laplace Transforms of a class of functions having the form $\mathbf{t^n f(t)}$ where the LT of f(t) is known to be F(s). This requires a new theorem.

Theorem 2.6 Multiplication by $\mathbf{t^n}$.

$$\text{If } \mathbf{L[f(t)] = F(s)} \text{ then } \mathbf{L\left[t^n f(t)\right] = (-1)^n \frac{d^n F(s)}{ds^n}} \tag{2.3-8}$$

This Theorem is also called the **Frequency Differentiation Theorem**.

Proof:

1. let n =1, and integrating by parts,

$$L\left[tf(t)\right] = \int_{0^-}^{\infty} tf(t)e^{-st}dt = \int_{0^-}^{\infty} f(t)\left[-\frac{d}{ds}e^{-st}\right]dt =$$

$$= -\frac{d}{ds}\int_{0^-}^{\infty} f(t)e^{-st}dt = -\frac{dF(s)}{ds}$$

2. in general

$$L\left[t^n f(t)\right] = \int_{0^-}^{\infty} t^n f(t)e^{-st}dt = \int_{0^-}^{\infty} f(t)\left[(-1)^n \frac{d^n}{ds^n}e^{-st}\right]dt$$

$$= (-1)^n \frac{d^n F(s)}{ds^n}$$

end of proof

Using Theorem 2.6, the Laplace Transforms of the family of functions $f_n(t) = t^n u(t)$ can easily be found. Note that $f_0(t) = u(t)$, **the unit step function**. The **unit ramp function** $r(t) = f_1(t) = tu(t)$, shown in fig 2.3-5a, is also defined as

$$r(t) = \begin{cases} t; \ t \geq 0 \\ 0; \ t < 0 \end{cases} \tag{2.3-9a}$$

and its time shifted version $r(t-t_0)$:

$$r(t-t_0) = \begin{cases} (t-t_0); \ t \geq t_0 \\ 0; \qquad t < t_0 \end{cases} \tag{2.3-9b}$$

is shown in fig 2.3-5b.

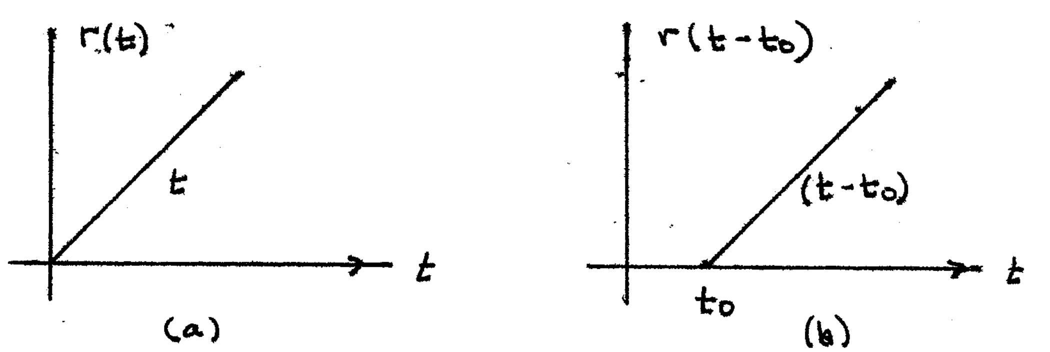


fig 2.3-5

The relationship between the unit step u(t) and the unit ramp r(t) is

$$\mathbf{r(t)} = \int_{-\infty}^{t} \mathbf{u(\lambda)d\lambda} \qquad \textbf{and} \qquad \mathbf{u(t)} = \frac{\mathbf{dr(t)}}{\mathbf{dt}} \tag{2.3-9c}$$

Note that the value of the ramp function is the accumulated area under the unit step as "t" increases.

Example 2.3-5

a. Since $\mathbf{L}\left[\mathbf{u(t)}\right] = \frac{1}{\mathbf{s}}$, it follows from Theorem 2.6 that the Laplace Transform of a unit ramp function is

$$\mathbf{L}\left[\mathbf{r(t)}\right] = \mathbf{L}\left[\mathbf{tu(t)}\right] = (-1)^1 \frac{\mathbf{d}}{\mathbf{ds}}\left[\frac{1}{\mathbf{s}}\right] = \frac{1}{\mathbf{s}^2} \tag{2.3-10a}$$

b. Similarly,

$$\mathbf{L}\left[\mathbf{t}^2\mathbf{u(t)}\right] = (-1)^2 \frac{\mathbf{d}^2}{\mathbf{ds}^2}\left[\frac{1}{\mathbf{s}^2}\right] = \frac{2}{\mathbf{s}^3} \tag{2.3-10b}$$

c. And, in general

$$\mathbf{L}\left[\mathbf{t^n u(t)}\right] = \frac{\mathbf{n!}}{\mathbf{s^{n+1}}} \tag{2.3-10c}$$

where n! = n(n –1)(n – 2)…..(2)(1) and 0! = 1.

d. Extending eq. 2.3-8b, it follows that

$$\mathbf{L}\left[\mathbf{t^n e^{-\alpha t} u(t)}\right] = \frac{\mathbf{n!}}{(\mathbf{s}+\alpha)^{\mathbf{n+1}}} \tag{2.3-11}$$

end of example

Recall that Theorem 2.1 was called the Time – Differentiation Theorem. It and Theorem 2.6, Frequency Differentiation, constitute a pair of dual theorems:

$$\left\{ \mathbf{L}\left[\frac{\mathbf{d^n f(t)}}{\mathbf{dt^n}}\right] = \mathbf{s^n F(s)} \right\} \xleftrightarrow{\textbf{dual}} \left\{ \mathbf{L}\left[\mathbf{t^n f(t)}\right] = (-1)^{\mathbf{n}} \frac{\mathbf{d^n F(s)}}{\mathbf{ds^n}} \right\} \tag{2.3-12}$$

where we assume zero initial conditions.

Both of these theorems show how a specific operation on f(t) in the time domain has a corresponding operation in the s – domain. Observe that time differentiation leads to multiplication by 's'. The dual operation is that multiplication by 't' leads to differentiation with respect to 's'.

We shall see that there are many dualities in Laplace Transforms and, subsequently, in Fourier Transforms. These dualities are at the basis of many powerful operations in signal processing. Part of the journey through this text will be the demonstration and application of dual properties of transforms.

The next theorem that we will consider, Theorem 2.7, Time Shifting, is the dual to Theorem 2.5, Multiplication by an Exponential.

Theorem 2.7 Time Shifting

$$\text{If } \mathbf{L}[\mathbf{f}(\mathbf{t})] = \mathbf{F}(\mathbf{s}) \text{ then } \mathbf{L}[\mathbf{f}(\mathbf{t}-\mathbf{t}_0)] = \mathbf{F}(\mathbf{s})\mathbf{e}^{-\mathbf{st}_0} \qquad (2.3\text{-}13)$$

Proof:

$$\mathbf{L}[\mathbf{f}(\mathbf{t}-\mathbf{t}_0)] = \int_{0^-}^{\infty} \mathbf{f}(\mathbf{t}-\mathbf{t}_0)\mathbf{e}^{-\mathbf{st}}\mathbf{dt} = \int_{-\mathbf{t}_0}^{\infty} \mathbf{f}(\mathbf{u})\mathbf{e}^{-\mathbf{s}(\mathbf{t}-\mathbf{t}_0)}\mathbf{du}$$

$$= \mathbf{e}^{-\mathbf{st}_0} \int_{-\mathbf{t}_0}^{\infty} \mathbf{f}(\mathbf{u})\mathbf{e}^{-\mathbf{su}}\mathbf{du}$$

Since f(t) = 0 for t < 0, the lower limit on the integral can be changed to u = 0^- so that

$$\mathbf{L}[\mathbf{f}(\mathbf{t}-\mathbf{t}_0)] = \mathbf{e}^{-\mathbf{st}_0} \int_{0^-}^{\infty} \mathbf{f}(\mathbf{u})\mathbf{e}^{-\mathbf{su}}\mathbf{du} = \mathbf{e}^{-\mathbf{st}_0}\mathbf{F}(\mathbf{s})$$

end of proof

The duality of Theorems 2.5 and 2.7 is easily seen below:

$$\left\{ \mathbf{L}\left[\mathbf{e}^{-\alpha \mathbf{t}}\mathbf{f}(\mathbf{t})\right] = \mathbf{F}(\mathbf{s}+\alpha) \right\} \xleftrightarrow{\mathbf{dual}} \left\{ \mathbf{L}[\mathbf{f}(\mathbf{t}-\mathbf{t}_0)] = \mathbf{F}(\mathbf{s})\mathbf{e}^{-\mathbf{st}_0} \right\} \qquad (2.3\text{-}14)$$

Example 2.3-6 Using the Time Shifting Theorem

The definitions of unit step function u(t), the unit ramp function r(t), and the unit impulse function δ(t) included definitions of their time shifted versions. The Laplace transforms of these time shifted versions are very simply determined through the use of the time shifting theorem.

$$\begin{aligned}
&\mathbf{u}(\mathbf{t}) \leftrightarrow \frac{1}{\mathbf{s}} && \mathbf{u}(\mathbf{t}-\mathbf{t}_0) \leftrightarrow \frac{1}{\mathbf{s}}\,\mathbf{e}^{-\mathbf{s}\mathbf{t}_0} \\
&\mathbf{r}(\mathbf{t}) \leftrightarrow \frac{1}{\mathbf{s}^2} && \mathbf{r}(\mathbf{t}-\mathbf{t}_0) \leftrightarrow \frac{1}{\mathbf{s}^2}\mathbf{e}^{-\mathbf{s}\mathbf{t}_0} \\
&\delta(\mathbf{t}) \leftrightarrow 1 && \delta(\mathbf{t}-\mathbf{t}_0) \leftrightarrow 1\,\mathbf{e}^{-\mathbf{s}\mathbf{t}_0} \\
&\mathbf{e}^{-\alpha\mathbf{t}}\mathbf{u}(\mathbf{t}) \leftrightarrow \frac{1}{\mathbf{s}+\alpha} && \mathbf{e}^{-\alpha(\mathbf{t}-\mathbf{t}_0)}\mathbf{u}(\mathbf{t}-\mathbf{t}_0) \leftrightarrow \frac{1}{\mathbf{s}+\alpha}\mathbf{e}^{-\mathbf{s}\mathbf{t}_0}
\end{aligned} \tag{2.3-14}$$

There is a caution in the use of this theorem. For example, the function $\mathbf{f}_1(\mathbf{t}) = \mathbf{tu}(\mathbf{t}-3)$ shown in **fig 2.3-6a**

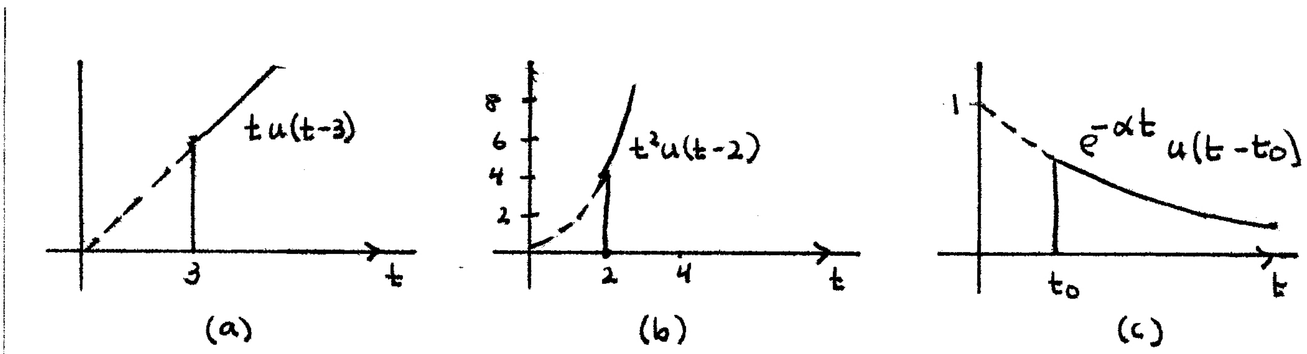


fig 2.3-6

must be reworked to be

$$\mathbf{f}_1(\mathbf{t}) = \mathbf{tu}(\mathbf{t}-3) = (\mathbf{t}-3)\mathbf{u}(\mathbf{t}-3) + 3\mathbf{u}(\mathbf{t}-3) \tag{2.3-15a}$$

in order to be consistent with the statement of the theorem. With this reworking, the Laplace Transform easily becomes

$$\mathbf{F}_1(\mathbf{s}) = \left[\frac{1}{\mathbf{s}^2} + 3\frac{1}{\mathbf{s}}\right]\mathbf{e}^{-3\mathbf{s}} \tag{2.3-15b}$$

Similarly, we can see that the function $\mathbf{f}(\mathbf{t}) = \mathbf{t}^2\mathbf{u}(\mathbf{t}-2)$ in **fig 2.3-6b** can be rewritten as:

$$\begin{aligned}
\mathbf{f}_2(\mathbf{t}) = \mathbf{t}^2\mathbf{u}(\mathbf{t}-2) &= (\mathbf{t}-2)^2\mathbf{u}(\mathbf{t}-2) + 4\mathbf{tu}(\mathbf{t}-2) - 4\mathbf{u}(\mathbf{t}-2) \\
&= (\mathbf{t}-2)^2\mathbf{u}(\mathbf{t}-2) + 4(\mathbf{t}-2)\mathbf{u}(\mathbf{t}-2) + 4\mathbf{u}(\mathbf{t}-2)
\end{aligned} \tag{2.3-16a}$$

can be transformed directly using the theorem:

$$\mathbf{F}_2(\mathbf{s}) = \left[\frac{2}{\mathbf{s}^3} + \frac{4}{\mathbf{s}^2} - \frac{4}{\mathbf{s}}\right]\mathbf{e}^{-2\mathbf{s}} \tag{2.3-16b}$$

Similarly, as shown in **fig. 2.3-6c**

$$\mathbf{LT}\left[e^{-\alpha t}u(t-t_0)\right] = e^{-\alpha t_0}\mathbf{LT}\left[e^{-\alpha(t-t_0)}u(t-t_0)\right] = e^{-\alpha t_0}\frac{1}{s+\alpha}e^{-st_0}. \quad (2.3\text{-}17)$$

In short, it's important to carefully examine the time functions that are to be transformed in order to properly apply the appropriate theorems.

Example 2.3-7 The Laplace Transform of Composite Waveforms

We want to find the Laplace Transforms of waveforms that can be decomposed into step and ramp functions that are scaled and shifted in time. The underlying justification for this operation is linearity and superposition: the LT of the sum of functions is the sum of the LT of each of them.

Consider the functions shown in fig 2.3-7.

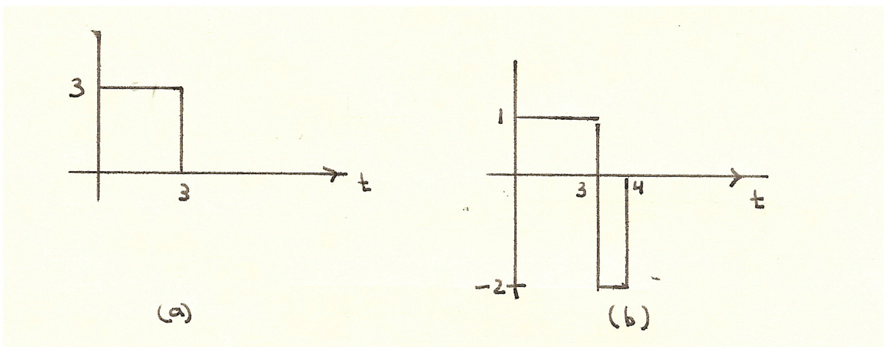


fig. 2.3-7

1. The pulse in fig 2.3-7a is $\mathbf{f_1(t) = u(t) - u(t-3)}$ so that $\mathbf{F}_1(\mathbf{s}) = \frac{1}{\mathbf{s}}\left(1 - \mathbf{e}^{-3\mathbf{s}}\right)$.
2. The pulse in fig 2.3-7b is $\mathbf{f}_2(\mathbf{t}) = \mathbf{u}(\mathbf{t}) - 3\mathbf{u}(\mathbf{t}-3) + 2\mathbf{u}(\mathbf{t}-4)$ so that
$$\mathbf{F}_2(\mathbf{s}) = \frac{1}{\mathbf{s}}\left(1 - 3\mathbf{e}^{-3\mathbf{s}} + 2\mathbf{e}^{-4\mathbf{s}}\right).$$
3. The waveform in fig 2.3-8a is:
$$f_3(t) = \begin{cases} \frac{3}{2}t; & 0 \le t \le 2 \\ 0; & t > 2 \end{cases}$$

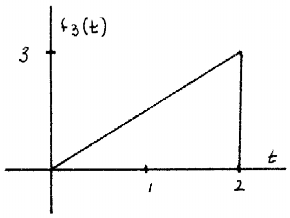


fig 2.3-8a

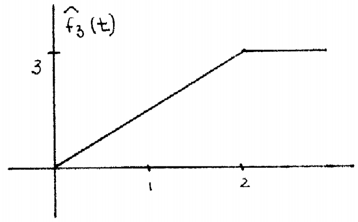


fig 2.3-8b

The object is to construct this function using ramps and steps. First consider the function

$$\hat{f}_3(t) = \frac{3}{2}r(t) - \frac{3}{2}r(t-2) \qquad (2.3\text{-}18a)$$

shown in fig 2.3-8b**.** It is clearly necessary to rid this function of the constant tail for t > 2. This can be accomplished by subtracting an appropriate step function. Therefore:

$$f_3(t) = \frac{3}{2}r(t) - \frac{3}{2}r(t-2) - 3u(t-2) \qquad (2.3\text{-}18b)$$

and the Laplace Transform is:

$$F_3(s) = \frac{3}{2}\frac{1}{s^2} - \frac{3}{2}\frac{1}{s^2}e^{-2s} - 3\frac{1}{s}e^{-2s} \qquad (2.3\text{-}18c)$$

4. Consider the function $f_4(t)$ shown in **fig. 2.3-9a**

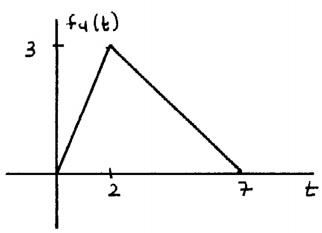


fig. 2.3-9a

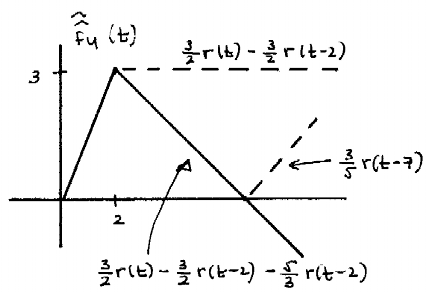


fig. 2.3-9b

We can see that

$$\hat{f}_4(t) = \frac{3}{2}r(t) - \frac{3}{2}r(t-2) \qquad (2.3\text{-}19a)$$

will result in the waveform of **fig. 2.3-9b** (horizontal dashed line) that is similar to the waveform of **fig 2.3-8b**. We should, additionally, subtract a ramp of $\frac{3}{5}r(t-2)$ in order to yield the required negative slope resulting in the solid waveform of **fig. 2.3-9b**. The equation then becomes

$$\hat{\hat{f}}_4(t) = \frac{3}{2}r(t) - \frac{3}{2}r(t-2) - \frac{3}{5}r(t-2) \qquad (2.3\text{-}19b)$$

In order to complete this construction it is necessary to add another ramp that makes

$\mathbf{f_4(t) = 0, \text{ for } t > 7}$ so the final expression is

$$\mathbf{f_4(t) = \frac{3}{2}r(t) - \frac{3}{2}r(t-2) - \frac{3}{5}r(t-2) + \frac{3}{5}r(t-7)} \qquad (2.3\text{-}19c)$$

The Laplace Transform then becomes

$$\mathbf{F_4(s) = \frac{1}{s^2}\left[\frac{3}{2} - \frac{3}{2}e^{-2s} - \frac{3}{5}e^{-2s} + \frac{3}{5}e^{-7s}\right]} \qquad (2.3\text{-}19d)$$

end of example

Unlike the previous examples of dual theorems, the **Time Scaling Theorem** is a single theorem that is embodiment of duality. When the theorem is subsequently applied to Fourier Transforms, it will be seen to be central to the relationship of time and frequency in signal analysis. In order to visualize the scaling process consider the rectangular pulse

$$\mathbf{f_1(x) = \begin{cases} 1; & -1 \le x \le 1 \\ 0; & \text{otherwise} \end{cases}} \qquad (2.3\text{-}20)$$

in **fig. 2.3-10.** This function has discontinuities at x = ±1. Strictly following the functional notation, it follows that the function $f_1(2t)$ has its discontinuities at t = ± ½ and $f_1(t/2)$ has its discontinuities at t = ± 2.

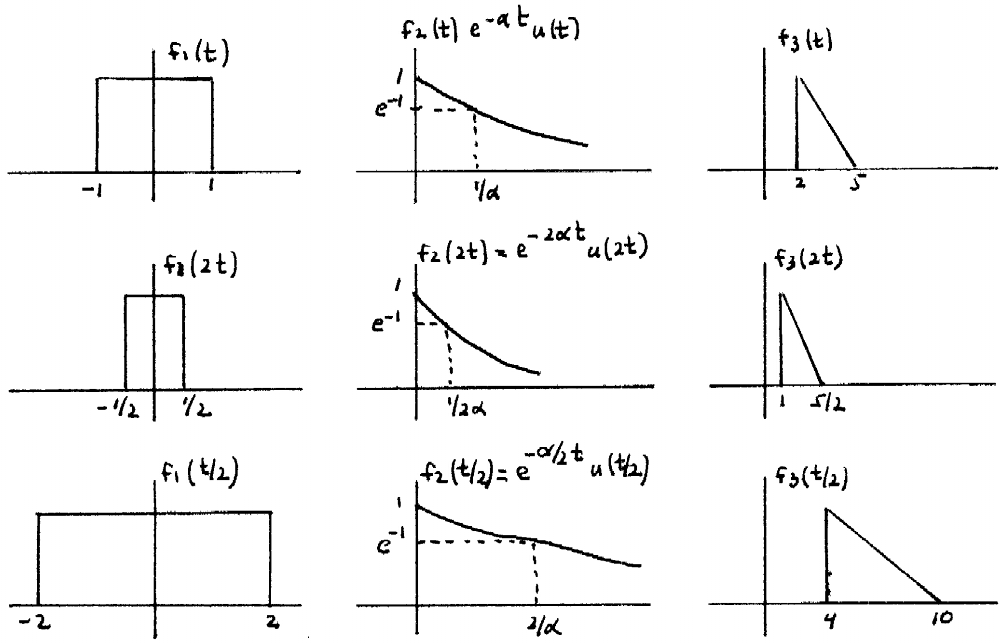


fig. 2.3-10 Time Scaling of Signals

Consequently, $f_1(2t)$ is a **contraction** of $f_1(t)$ and $f_1(t/2)$ is an **expansion** of $f_1(t)$. In subsequent discussions of Fourier Series and Fourier Transforms we will see that contraction in time leads to expansion of signal bandwidth and expansion in time leads to contraction in signal bandwidth.

This establishes the scaling theorem as a statement of duality between the time domain and the s – domain.

Theorem 2.8 The Scaling Theorem

$$\text{If } \mathbf{L}\left[\mathbf{f}(\mathbf{t})\right] = \mathbf{F}(\mathbf{s}) \text{ then, for } a > 0,\ \mathbf{L}\left[\mathbf{f}(\mathbf{at})\right] = \frac{1}{\mathbf{a}}\mathbf{F}\left(\frac{\mathbf{s}}{\mathbf{a}}\right) \tag{2.3-20}$$

Note that the restriction that a > 0 arises from the requirement that the one – sided Laplace Transform requires that the function being transformed be zero for t < 0.

Proof:

$$\mathbf{L}\left[\mathbf{f}(\mathbf{at})\right] = \int_{0^-}^{\infty} \mathbf{f}(\mathbf{at})\mathbf{e}^{-\mathbf{st}}\mathbf{dt} = \int_{0^-}^{\infty} \mathbf{f}(\mathbf{u})\mathbf{e}^{-\frac{\mathbf{s}}{\mathbf{a}}\mathbf{u}}\frac{\mathbf{du}}{\mathbf{a}} = \frac{1}{\mathbf{a}}\mathbf{F}\left(\frac{\mathbf{s}}{\mathbf{a}}\right)$$

end of proof

Example 2.3-8

As an example of the use of the Scaling Theorem, note that

$$\mathbf{L}\left[\ \mathbf{e}^{-2\mathbf{t}}\mathbf{u}(\mathbf{t})\ \right] = \frac{1}{(\mathbf{s}+2)}.$$

Scaling this function by the factor 3 yields

$$\mathbf{L}\left[\ \mathbf{e}^{-3\cdot 2\mathbf{t}}\mathbf{u}(3\mathbf{t})\ \right] = \frac{1}{3}\frac{1}{\left(\frac{\mathbf{s}}{3}+2\right)} = \frac{1}{\mathbf{s}+6}.$$

end of example

Example 2.3-9

This example illustrates the combined use of the scaling and time - shifting theorems. Let:

$$\mathbf{F}(\mathbf{s}) = \mathbf{L}\left[\mathbf{f}(\mathbf{t})\right] = \frac{(\mathbf{s}+\mathbf{2})}{(\mathbf{s}+\mathbf{1})(\mathbf{s}+\mathbf{3})}$$

Now we want to find $\mathbf{L}\left[\mathbf{f}(\mathbf{3t}-\mathbf{2})\right]$.

Step 1: Let $\mathbf{f(3t-2) = f\left[3\left(t-\frac{2}{3}\right)\right] = f(3x) = f_1(x)}$. Now using the scaling theorem

$$\mathbf{L[f_1(x)] = F_1(s) = \frac{1}{3}F\left(\frac{s}{3}\right) = \frac{1}{3}\frac{\left(\frac{s}{3}+2\right)}{\left(\frac{s}{3}+1\right)\left(\frac{s}{3}+3\right)} = \frac{(s+6)}{(s+3)(s+9)}}$$

Step 2: Let x = t – 2/3 so that

$$\mathbf{L[f(3t-2)] = L\left[\,f_1\left(t-\tfrac{2}{3}\right)\,\right] = F_1(s)e^{-\frac{2}{3}s}}$$

end of example

Finally, we shall return to look a bit more deeply into the unit impulse function $\delta(t)$. From both an analytic and a practical point of view, the most important application of impulse functions is in the sampling of continuous functions.

Recall that the unit impulse function was defined as:

$$\delta(t) = \begin{cases} 0; & t \neq 0 \\ \infty; & t = 0 \end{cases} \qquad \int_{-\infty}^{\infty}\delta(t)dt = \int_{0^-}^{0^+}\delta(t)dt = 1 \qquad (2.3\text{-}21)$$

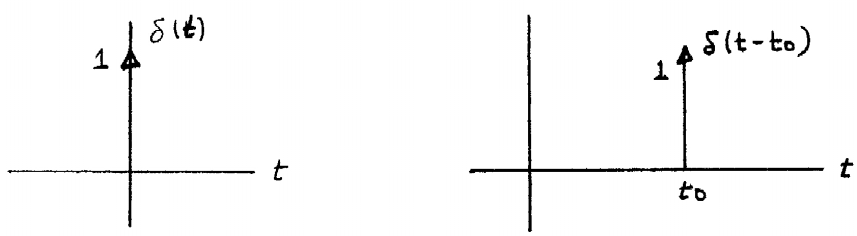


fig. 2.3-11

as shown in fig. 2.3-11, both centered at the origin and shifted to the right by $\mathbf{t_0}$.

The sampling property is first illustrated in the determination of the Laplace Transform of the impulse function:

$$\mathbf{LT[\delta(t)] = \int_{0^-}^{\infty}\delta(t)e^{-st}dt = \int_{0^-}^{0^+}\delta(t)e^{-st}dt = 1} \qquad (2.3\text{-}22)$$

The reasoning is that between the limits $\mathbf{[0^-, 0^+]}$, the function $\mathbf{e^{-st} = e^0 = 1}$ is a constant. The integral as a whole simply reduces to the integral of the impulse function.

More generally, if f(t) is continuous at $t = t_0$, then

$$\int_{-\infty}^{\infty} \mathbf{f(t)\delta(t - t_0)dt = f(t_0)} \qquad (2.3\text{-}23)$$

The function f(t) and the impulse function $\delta(t - t_0)$ are shown in **fig 2.3-11.** Note that f(t) is continuous at $t = t_0$, so there is unique value $f(t_0)$. Since the impulse function is equal to zero everywhere except where its argument is zero

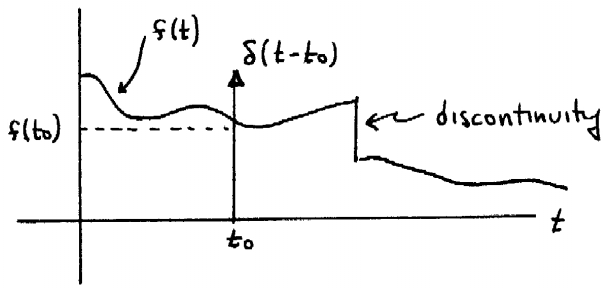


fig 2.3-12 Sampling Property of Impulse Functions

$$\int_{-\infty}^{\infty} \mathbf{f(t)\delta(t - t_0)dt} = \int_{t_0^-}^{t_0^+} \mathbf{f(t)\delta(t - t_0)dt} = \int_{t_0^-}^{t_0^+} \mathbf{f(t_0)\delta(t - t_0)dt = f(t_0)} \qquad (2.3\text{-}24)$$

At points of discontinuity, f(t) does not have a unique value. The integral of eq. 2.3-24 becomes the average value of f(t) around the discontinuity. The proof of this will be left to the Problems.

Example 2.3-10 Let $f(t) = 3t^2 + 2t + 1$ so that

$$\int_{-\infty}^{\infty} \mathbf{f(t)\delta(t + 2)dt = f(-2)} = 3(-2)^2 + 2(-2) + 1 = 9$$

end of example

The following simplification is often made as an extension of the sampling property:

$$\mathbf{f(t)\delta(t - t_0) = f(t_0)\delta(t - t_0)} \qquad (2.3\text{-}25)$$

since the integration of either of the two expressions yields the same result.

When we discuss the Sampling Theorem in Chapter 5, Fourier Transforms we will make use of an analytic expression for periodic samples of a time function f(t). This expression is:

$$\mathbf{f^*(t)} = \sum_{\mathbf{n=-\infty}}^{\infty} \mathbf{f(nT)\delta(t - nT)} \qquad (2.3\text{-}26)$$

The unit impulse function was shown in Chapter 1 to be the derivative of the unit step function u(t):

$$\delta(t) = \frac{du(t)}{dt} \tag{2.3-27}$$

The limiting process shown in **fig. 2.3-12** has the impulse being the limit as $\varepsilon \rightarrow 0$ of a rectangular pulse having width ε and height $1/\varepsilon$.

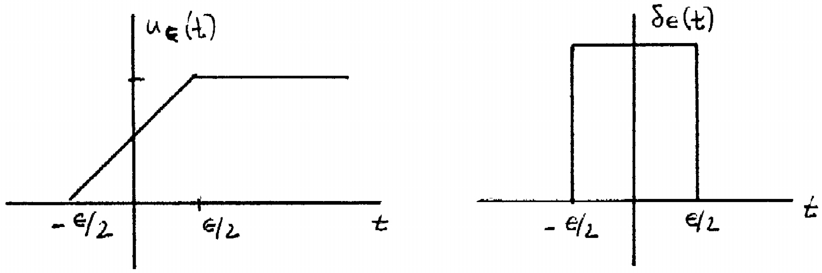


fig. 2.3-12

The derivation of δ(t) as a result of a limiting process is not unique to the rectangular pulse of **fig. 2.3-12**; it can be derived as the limit of a variety of functions. One of these, a triangle, is shown in **fig. 2.3-13**. Note that the area is

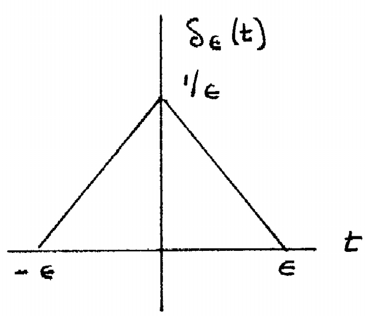


fig 2.3-13

equal to '1' independent of ε and the triangle gets very narrow and very high as $\varepsilon \rightarrow 0$. Additional examples of functions that become impulses as a result of a limiting process will be discussed in Chapters 4 and 5.

The derivative of the triangle in **fig. 2.3-13** is an approximation to the derivative of the unit impulse function, defined as the unit doublet.

$$\delta'(t) = \frac{d\delta(t)}{dt} \tag{2.3-28a}$$

An approximation to the unit doublet is obtained by differentiating the triangular approximation to the unit impulse shown in **fig 2.3-13.** The result of this differentiation is shown in **fig. 2.3-14.**

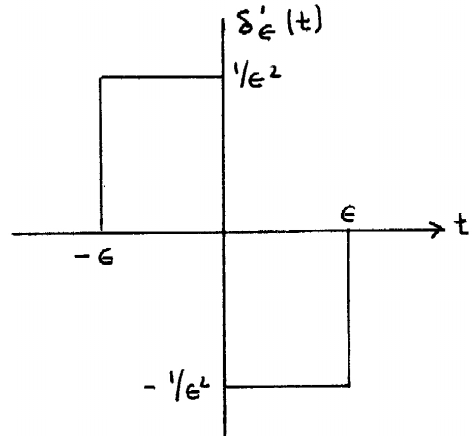


fig. 2.3-14 Approximation to the Doublet

As $\varepsilon \to 0$, the waveform of **fig. 2.3-14** will approach the unit doublet. By observation, we can see that:

a. $$\delta'(t - t_0) = 0; \qquad t \neq t_0 \qquad (2.3\text{-}28b)$$

b. $$\int_{-\infty}^{\infty} \delta'(t)dt = 0 \qquad (2.3\text{-}28c)$$

which seems to lead to the conclusion that the doublet doesn't do much of anything. The doublet does, however, have an interesting sampling property of sampling the **derivative** of a function:

c. $$\int_{-\infty}^{\infty} f(t)\delta'(t - t_0)dt = -\left.\frac{df(t)}{dt}\right|_{t=t_0} = -f'(t_0) \qquad (2.3\text{-}28d)$$

This can be shown by integrating by parts:

$$\int_{-\infty}^{\infty} f(t)\delta'(t - t_0)dt = f(t)\delta(t - t_0)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{df(t)}{dt}\delta(t - t_0)dt = -f'(t_0) \qquad (2.3\text{-}28e)$$

Finally, the Laplace Transform of the unit doublet can be found with a simple application of Theorem 2.1, Time Differentiation. Since $L[\delta(t)] = 1$, it follows that

$$L\left[\frac{d\delta(t)}{dt}\right] = s \qquad (2.3\text{-}29a)$$

and, continuing

$$\mathbf{L}\left[\frac{d^2\delta(t)}{dt^2}\right] = s^2 \tag{2.3-29b}$$

and, more generally,

$$\mathbf{L}\left[\frac{d^n\delta(t)}{dt^n}\right] = s^n \tag{2.3-29c}$$

This hierarchy of impulse functions and their derivatives is of theoretical use in system analysis and approximations to these functions are often useful in the sampling of signals.

Finally, we want to establish the dual theorem to Theorem 2.2, Time Integration.

Theorem 2.9 The Frequency Integration Theorem (Division by t)

$$\text{If } \mathbf{L}[f(t)] = F(s) \text{ then } \mathbf{L}\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(u)du \tag{2.3-30}$$

Proof:

$$\mathbf{L}\left[\frac{f(t)}{t}\right] = \int_{0^-}^{\infty} \frac{f(t)}{t} e^{-st} dt = \int_{0^-}^{\infty} f(t)\left[\int_s^{\infty} e^{-ut} du\right] dt$$

$$= \int_s^{\infty}\left[\int_{0^-}^{\infty} f(t) e^{-ut} dt\right] du = \int_s^{\infty} F(u) du$$

end of proof

Since this theorem often involves functions that are not bounded at t = 0, it does not find direct use at this point. It will, however, be very useful in subsequent discussions where it is applied to Fourier Transforms.

The formal duality, between Theorem 2.2, Time Integration and Theorem 2.9, Frequency Integration is:

$$\left\{\mathbf{L}\left[\int_0^t f(\lambda)d\lambda\right] = \frac{F(s)}{s}\right\} \xleftrightarrow{\text{dual}} \left\{\mathbf{L}\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(u)du\right\} \tag{2.3-31}$$

Having established these theorems of the Laplace Transform and demonstrated the validity of the list of transforms in Tables 2.2 and 2.3, we are now able to examine how to find the inverse Laplace Transform.

2.4 The Inverse Laplace Transform Using Partial Fractions

The Laplace Transform of the solution of a linear differential equation having constant coefficients and zero initial conditions was established in Section 2.2 to be

$$\mathbf{X}(\mathbf{s}) = \frac{(\mathbf{b}_1\mathbf{s}+\mathbf{b}_0)}{\left[\mathbf{a}_2\mathbf{s}^2+\mathbf{a}_1\mathbf{s}+\mathbf{a}_0\right]}\mathbf{F}(\mathbf{s}) = \mathbf{H}(\mathbf{s})\mathbf{F}(\mathbf{s}) \tag{2.4-1a}$$

where H(s) is the Transfer Function.

If, for example, the input to the system is $\mathbf{f}(\mathbf{t}) = \mathbf{Cos}(\beta\mathbf{t})\mathbf{u}(\mathbf{t})$, its Laplace Transform is $\mathbf{F}(\mathbf{s}) = \frac{\mathbf{s}}{\mathbf{s}^2+\beta^2}$ and

$$\mathbf{X}(\mathbf{s}) = \frac{(\mathbf{b}_1\mathbf{s}+\mathbf{b}_0)}{\left[\mathbf{a}_2\mathbf{s}^2+\mathbf{a}_1\mathbf{s}+\mathbf{a}_0\right]}\frac{\mathbf{s}}{\mathbf{s}^2+\beta^2} \tag{2.4-1b}$$

where **X(s)** is the ratio of two polynomials in s.

Having found X(s), the problem is now to find the corresponding time function x(t). This problem will be approached by seeking to divide the ratio of polynomials into small pieces, each of which is recognizable as the Laplace Transform of a well – known function.

In order to get an idea of this process, first consider the problem in reverse. Let:

$$\mathbf{f}(\mathbf{t}) = 2\mathbf{e}^{-\mathbf{t}}\mathbf{u}(\mathbf{t}) + 4\mathbf{e}^{-3\mathbf{t}}\mathbf{u}(\mathbf{t}) \tag{2.4-2a}$$

so that:

$$\mathbf{F}(\mathbf{s}) = \frac{2}{\mathbf{s}+1} + \frac{4}{\mathbf{s}+3} \tag{2.4-2b}$$

and

$$\mathbf{F}(\mathbf{s}) = \frac{6\mathbf{s}+10}{(\mathbf{s}+1)(\mathbf{s}+3)} = \frac{(6\mathbf{s}+10)}{(\mathbf{s}^2+4\mathbf{s}+3)} = \frac{\mathbf{N}(\mathbf{s})}{\mathbf{D}(\mathbf{s})} \tag{2.4-2c}$$

so the final result is the ratio of two polynomials in s. The method of partial fractions involves starting from eq. 2.4-2c, moving to 2.4-2b, and finally to 2.4-2a.

By examining a few small variations to f(t) in eq. 2.4-2a, we can deepen our understanding of the character of F(s). Let:

$$\mathbf{f}_1(\mathbf{t}) = \mathbf{2e}^{-\mathbf{t}}\mathbf{u}(\mathbf{t}) - \mathbf{4e}^{-\mathbf{3t}}\mathbf{u}(\mathbf{t}) \tag{2.4-2d}$$

so that:

$$\mathbf{F}_1(\mathbf{s}) = \frac{\mathbf{2}}{\mathbf{s}+\mathbf{1}} - \frac{\mathbf{4}}{\mathbf{s}+\mathbf{3}} \tag{2.4-2e}$$

or:

$$\mathbf{F_1(s) = -\frac{2(s-1)}{(s+1)(s+3)}} \qquad (2.4\text{-}2f)$$

from which we can conclude that the denominator is completely determined by the exponential time functions but that the numerator changes with the magnitudes and signs of those functions.

Now let:

$$\mathbf{f_2(t) = 2e^{-t}u(t) + 4e^{+3t}u(t)} \qquad (2.4\text{-}2g)$$

which is considered to be "unstable" because the increasing exponential causes the entire function to go to infinity. In this instance, the Laplace Transform is:

$$\mathbf{F_2(s) = \frac{2}{s+1} + \frac{4}{s-3} = \frac{(6s-2)}{(s+1)(s-3)}} \qquad (2.4\text{-}2h)$$

The values of **s** that cause the denominator of the Laplace Transform to go to zero are called poles. It is clear that those poles that are negative (or, as shall be seen in the Problems, have negative real parts) correspond to time functions that decay to zero. These functions are designated as "stable". Those having positive real parts correspond to functions having values that increase exponentially are called "unstable".

The issue of the significance of the roots of both the denominator and numerator polynomials will further be explored in Chapter 3.

This small example indicates that the roots of the denominator polynomial govern the character of the time function in the inverse transform. According to the algebra of such polynomials, there are a limited number of possibilities for the roots. The roots may be real, having the form $(\mathbf{s}+\mathbf{a})$, or complex, having the form $(\mathbf{s}+\alpha+\mathbf{j}\beta)(\mathbf{s}+\alpha-\mathbf{j}\beta)=\left[\mathbf{s}^2+2\alpha\mathbf{s}+\left(\alpha^2+\beta^2\right)\right]$. Both real and complex roots may be multiple, i.e. – they may be raised to the n^{th} power.

The partial fraction process may be defined by a few steps:

<u>**Step 1**</u> The degree of N(s) should be strictly less than the degree of D(s). If it is not, then divide N(s) by D(s), resulting in a polynomial in 's' plus a new ratio of two polynomials N'(s) and D(s) where degN'(s) < degD(s).

Example 2.4-1:

1. Let $\mathbf{F(s)} = \frac{\mathbf{s}^2+6\mathbf{s}+2}{\mathbf{s}^2+4\mathbf{s}+3}$. Since deg N(s) = deg D(s), it is necessary to divide

$$\begin{array}{r} 1 \\ \left(\mathbf{s}^2+4\mathbf{s}+3\right)\overline{\big)\mathbf{s}^2+6\mathbf{s}+2} \\ \underline{\mathbf{s}^2+4\mathbf{s}+3} \\ \mathbf{2s-1} \end{array}$$

so that $\mathbf{F}(\mathbf{s}) = 1 + \dfrac{2\mathbf{s}-1}{\mathbf{s}^2+4\mathbf{s}+3}$.

2. Let $\mathbf{F}(\mathbf{s}) = \dfrac{2\mathbf{s}^3+\mathbf{s}^2+6\mathbf{s}-2}{\mathbf{s}^2+4\mathbf{s}+3}$ so that after dividing

$$\mathbf{F}(\mathbf{s}) = 2\mathbf{s} - 7 + \frac{28\mathbf{s}+19}{\mathbf{s}^2+4\mathbf{s}+3}$$

<u>end of example</u>

When it comes to finding the corresponding time function, the polynomial in s resulting from the division (in the example given above, 2s - 7) will become the impulse function and its derivatives according to:

$$\mathbf{L[A\delta(t)] = A}$$

$$\mathbf{L[A\delta'(t)] = As}$$

and, in general: $\mathbf{L[A\delta^{(n)}(t)] = As^n}$

as shown in Section 2.3.

As we proceed in this chapter, we will come across some examples in which the numerator and denominator of a transfer function have the same degree. These are important examples but they are idealized because they do not take into account the unavoidable "stray" capacitance to ground (or its equivalent) that accompanies all real systems. This capacitance manifests itself as an additional (s+a) factor in the denominator, resulting in deg N(s) < deg D(s). As these examples arise, the reader will be reminded of their limitations.

<u>Step 2.</u> With deg N(s) < deg D(s), the next step is to factor the denominator polynomial by finding its roots. Since the coefficients of D(s) are real the roots may be real or complex but, if complex, must occur in complex conjugate pairs. These roots, real or complex, may be multiple. Factoring D(s) may easily be accomplished by using software tools such as Matlab.

The polynomial $\mathbf{D(s) = a_3s^3 + a_2s^2 + a_1s + a_0}$ is represented as:

$$\mathbf{> D = [\ a_3\ a_2\ \ a_1\ \ a_0\ ]}$$

and

> r = roots(D)

as shown in Example 2.4-2.

Example 2.4-2

The Laplace Transform of a time function x(t) is

$$\mathbf{X}(\mathbf{s}) = \frac{\mathbf{N}(\mathbf{s})}{\mathbf{D}(\mathbf{s})} = \frac{\mathbf{s}+3}{\left(\mathbf{s}^5 + 3\mathbf{s}^4 + 5\mathbf{s}^3 + 7\mathbf{s}^2 + 5\mathbf{s} + 2\right)}$$

In order to find the inverse LT x(t), we must first find the roots of D(s). According to MATLAB:

```
D = [1 3 5 7 5 2]
D =
    1    3    5    7    5    2
r = roots(D)
r =
  -1.7031
  -0.1079 + 1.3952i
  -0.1079 - 1.3952i
  -0.5405 + 0.5546i
  -0.5405 - 0.5546i
```

We will subsequently see that these roots determine the form of x(t):

$$\mathbf{x}(\mathbf{t}) = \mathbf{A}_1\mathbf{e}^{-1.7031\mathbf{t}}\mathbf{u}(\mathbf{t}) + \mathbf{A}_2\mathbf{e}^{-0.1079\mathbf{t}}\mathbf{Cos}(1.3952\mathbf{t} + \theta_2)\mathbf{u}(\mathbf{t}) + \mathbf{A}_3\mathbf{e}^{-0.5405\mathbf{t}}\mathbf{Cos}(0.5546\mathbf{t} + \theta_3)\mathbf{u}(\mathbf{t})$$

where the constants A_1, A_2, θ_2, A_3, θ_3 are to be determined through the partial fraction method.

End of example

Step 3. Having accomplished this factoring, we next proceed with the partial fraction expansion. In order to explain how this works, consider the case of D(s) having **three real roots**:

$$\mathbf{F}(\mathbf{s}) = \frac{\mathbf{N}(\mathbf{s})}{(\mathbf{s}+\mathbf{a})(\mathbf{s}+\mathbf{b})(\mathbf{s}+\mathbf{c})} \tag{2.4-3a}$$

where, according to eq. 2.4-2) the expansion should result in:

$$\mathbf{F}(\mathbf{s}) = \frac{\mathbf{A}}{(\mathbf{s}+\mathbf{a})} + \frac{\mathbf{B}}{(\mathbf{s}+\mathbf{b})} + \frac{\mathbf{C}}{(\mathbf{s}+\mathbf{c})} \tag{2.4-3b}$$

Note that:

$$(\mathbf{s}+\mathbf{a})\mathbf{F}(\mathbf{s}) = \mathbf{A} + \frac{\mathbf{B}(\mathbf{s}+\mathbf{a})}{(\mathbf{s}+\mathbf{b})} + \frac{\mathbf{C}(\mathbf{s}+\mathbf{a})}{(\mathbf{s}+\mathbf{c})} \qquad (2.4\text{-}3c)$$

so that:

$$(\mathbf{s}+\mathbf{a})\mathbf{F}(\mathbf{s})\Big|_{\mathbf{s}=-\mathbf{a}} = \mathbf{A} \qquad (2.4\text{-}4a)$$

and similarly

$$(\mathbf{s}+\mathbf{b})\mathbf{F}(\mathbf{s})\Big|_{\mathbf{s}=-\mathbf{b}} = \mathbf{B} \qquad (2.4\text{-}4b)$$

$$(\mathbf{s}+\mathbf{c})\mathbf{F}(\mathbf{s})\Big|_{\mathbf{s}=-\mathbf{c}} = \mathbf{C} \qquad (2.4\text{-}4c)$$

Example 2.4-3 Simple Real Roots

$$\mathbf{F}(\mathbf{s}) = \frac{(\mathbf{s}+2)^2}{\mathbf{s}(\mathbf{s}+1)(\mathbf{s}+3)} = \frac{\mathbf{A}}{\mathbf{s}} + \frac{\mathbf{B}}{(\mathbf{s}+1)} + \frac{\mathbf{C}}{(\mathbf{s}+3)}$$

Even without knowing the values for A, B, C we can find the inverse Laplace Transform of F(s) by recalling that $\mathbf{L}\left[\mathbf{e}^{-\mathbf{at}}\mathbf{u}(\mathbf{t})\right] = \dfrac{1}{\mathbf{s}+\mathbf{a}}$ so that

$$\mathbf{f}(\mathbf{t}) = \mathbf{Au}(\mathbf{t}) + \mathbf{Be}^{-\mathbf{t}}\mathbf{u}(\mathbf{t}) + \mathbf{Ce}^{-3\mathbf{t}}\mathbf{u}(\mathbf{t})$$

The values of A, B, C are found as:

$$\mathbf{A} = \mathbf{sF}(\mathbf{s})\Big|_{\mathbf{s}=0} = \frac{\mathbf{s}(\mathbf{s}+2)^2}{\mathbf{s}(\mathbf{s}+1)(\mathbf{s}+3)}\Bigg|_{\mathbf{s}=0} = \frac{(\mathbf{s}+2)^2}{(\mathbf{s}+1)(\mathbf{s}+3)}\Bigg|_{\mathbf{s}=0} = \frac{(2)^2}{(1)(3)} = \frac{4}{3}$$

$$\mathbf{B} = (\mathbf{s}+1)\mathbf{F}(\mathbf{s})\Big|_{\mathbf{s}=-1} = \frac{(\mathbf{s}+1)(\mathbf{s}+2)^2}{\mathbf{s}(\mathbf{s}+1)(\mathbf{s}+3)}\Bigg|_{\mathbf{s}=-1} = \frac{(\mathbf{s}+2)^2}{\mathbf{s}(\mathbf{s}+3)}\Bigg|_{\mathbf{s}=-1} = \frac{(-1+2)^2}{(-1)(-1+3)} = -\frac{1}{2}$$

$$\mathbf{C} = (\mathbf{s}+3)\mathbf{F}(\mathbf{s})\Big|_{\mathbf{s}=-3} = \frac{(\mathbf{s}+3)(\mathbf{s}+2)^2}{\mathbf{s}(\mathbf{s}+1)(\mathbf{s}+3)}\Bigg|_{\mathbf{s}=-3} = \frac{(\mathbf{s}+2)^2}{\mathbf{s}(\mathbf{s}+1)}\Bigg|_{\mathbf{s}=-3} = \frac{(-3+2)^2}{(-3)(-3+1)} = \frac{1}{6}$$

so that

$$\mathbf{f}(\mathbf{t}) = \frac{4}{3}\mathbf{u}(\mathbf{t}) + \frac{1}{2}\mathbf{e}^{-\mathbf{t}}\mathbf{u}(\mathbf{t}) + \frac{1}{6}\mathbf{e}^{-3\mathbf{t}}\mathbf{u}(\mathbf{t})$$

end of example

We now want to continue the discussion by considering the issue of **complex conjugate roots**. Let:

$$\mathbf{F(s)} = \frac{\mathbf{N(s)}}{\left[\mathbf{s}^2 + 2\alpha\mathbf{s} + \left(\alpha^2 + \beta^2\right)\right]} = \frac{\mathbf{N(s)}}{(\mathbf{s} + \alpha - \mathbf{j}\beta)(\mathbf{s} + \alpha + \mathbf{j}\beta)} = \qquad (2.4\text{-}5a)$$

$$= \frac{\mathbf{A}}{(\mathbf{s} + \alpha - \mathbf{j}\beta)} + \frac{\mathbf{A}^*}{(\mathbf{s} + \alpha + \mathbf{j}\beta)}$$

Since the coefficients of both N(s) and D(s) are assumed to be real numbers, it automatically follows that in the partial fraction expansion the complex conjugate factors in the denominator will have complex conjugate coefficients A and A^*. This makes it unnecessary to find both, one will do. Accordingly:

$$\mathbf{A} = (\mathbf{s} + \alpha - \mathbf{j}\beta)\mathbf{F(s)}\Big|_{\mathbf{s}=-\alpha+\mathbf{j}\beta} = \mathbf{A} + \left.\frac{(\mathbf{s} + \alpha - \mathbf{j}\beta)\mathbf{A}^*}{(\mathbf{s} + \alpha + \mathbf{j}\beta)}\right|_{\mathbf{s}=-\alpha+\mathbf{j}\beta} = |\mathbf{A}|\mathbf{e}^{\mathbf{j}\theta} \qquad (2.4\text{-}5b)$$

where $\mathbf{A} = |\mathbf{A}|\mathbf{e}^{\mathbf{j}\theta}$ is a complex number.

Using Theorem 1.3, Linearity and Superposition, and Theorem 1.7, Multiplication by an Exponential, the inverse Laplace Transform of eq. 2.4-4a is:

$$\begin{aligned}\mathbf{f(t)} &= |\mathbf{A}|\mathbf{e}^{\mathbf{j}\theta}\mathbf{e}^{-(\alpha - \mathbf{j}\beta)\mathbf{t}}\mathbf{u(t)} + |\mathbf{A}|\mathbf{e}^{-\mathbf{j}\theta}\mathbf{e}^{-(\alpha + \mathbf{j}\beta)\mathbf{t}}\mathbf{u(t)} \\ &= |\mathbf{A}|\mathbf{e}^{-\alpha\mathbf{t}}\left[\mathbf{e}^{\mathbf{j}(\beta\mathbf{t} + \theta)} + \mathbf{e}^{-\mathbf{j}(\beta\mathbf{t} + \theta)}\right]\mathbf{u(t)} \qquad (2.4\text{-}5c) \\ &= 2|\mathbf{A}|\mathbf{e}^{-\alpha\mathbf{t}}\mathbf{Cos}(\beta\mathbf{t} + \theta)\mathbf{u(t)}\end{aligned}$$

As a consequence, we can go from eq. 2.4-4b to eq. 2.4-4c directly, without any intermediate steps.

<u>**Example 2.4-4**</u> Simple Complex Conjugate Roots

$$\mathbf{F(s)} = \frac{(\mathbf{s} + 2)(\mathbf{s} + 3)}{(\mathbf{s} + 1)\left(\mathbf{s}^2 + 4\mathbf{s} + 8\right)} = \frac{(\mathbf{s} + 2)(\mathbf{s} + 3)}{(\mathbf{s} + 1)(\mathbf{s} + 2 - \mathbf{j}2)(\mathbf{s} + 2 + \mathbf{j}2)}$$

$$= \frac{\mathbf{A}}{(\mathbf{s} + 1)} + \frac{\mathbf{B}}{(\mathbf{s} + 2 - \mathbf{j}2)} + \frac{\mathbf{B}^*}{(\mathbf{s} + 2 + \mathbf{j}2)}$$

which immediately yields

$$\mathbf{f(t)} = \mathbf{A}\mathbf{e}^{-\mathbf{t}}\mathbf{u(t)} + 2|\mathbf{B}|\mathbf{e}^{-2\mathbf{t}}\mathbf{Cos}(2\mathbf{t} + \theta)\mathbf{u(t)}$$

$$\mathbf{A} = (\mathbf{s}+1)\mathbf{F}(\mathbf{s})\Big|_{\mathbf{s}=-1} = \left.\frac{(\mathbf{s}+2)(\mathbf{s}+3)}{(\mathbf{s}^2+4\mathbf{s}+8)}\right|_{\mathbf{s}=-1} = \frac{2}{5}$$

$$\mathbf{B} = (\mathbf{s}+2-\mathbf{j}2)\mathbf{F}(\mathbf{s})\Big|_{\mathbf{s}=-2+\mathbf{j}2} = \left.\frac{(\mathbf{s}+2)(\mathbf{s}+3)}{(\mathbf{s}+1)(\mathbf{s}+2+\mathbf{j}2)}\right|_{\mathbf{s}=-2+\mathbf{j}2} = \frac{(\mathbf{j}2)(1+\mathbf{j}2)}{(-1+\mathbf{j}2)(\mathbf{j}4)} = \frac{1}{2}\mathbf{e}^{-\mathbf{j}53^0}$$

so that, using eq. 2.4-4c

$$\mathbf{f}(\mathbf{t}) = \frac{2}{5}\mathbf{e}^{-\mathbf{t}}\mathbf{u}(\mathbf{t}) + \mathbf{e}^{-2\mathbf{t}}\mathbf{Cos}(2\mathbf{t}-53^0)\mathbf{u}(\mathbf{t})$$

end of example

There is a MATLAB command called **residue** that computes the partial fraction expansion, but only for systems having simple poles, real or complex. Regrettably, this function is not expandable to poles or roots of order greater than one. Using **residue** for the previous example yields:

```
EDU>> b =[1 5 6]
b =
    1     5     6
EDU>> a = [1 5 12 8]
a =
    1     5    12     8
EDU>> [r,p,k] = residue(b,a)
r =
   0.3000 - 0.4000i
   0.3000 + 0.4000i
   0.4000
p =
  -2.0000 + 2.0000i
  -2.0000 - 2.0000i
  -1.0000
k =    []
```

Before turning to partial fraction examples for multiple roots, we shall use the results for simple roots as a way of making some general observations about the inverse Laplace Transform.

The Laplace Transform solution X(s) to an LDE was shown to be

$$\mathbf{X}(\mathbf{s}) = \frac{(\mathbf{b}_1\mathbf{s} + \mathbf{b}_0)}{\left[\mathbf{a}_2\mathbf{s}^2 + \mathbf{a}_1\mathbf{s} + \mathbf{a}_0\right]}\mathbf{F}(\mathbf{s}) = \mathbf{H}(\mathbf{s})\mathbf{F}(\mathbf{s}) \qquad (2.4\text{-}1a)$$

where H(s) is the transfer function and F(s) is the LT of the input f(t). It is assumed that there are zero initial conditions. We have seen that if the input $\mathbf{f}(\mathbf{t}) = \boldsymbol{\delta}(\mathbf{t})$ then $\mathbf{F}(\mathbf{s}) = \mathbf{1}$ and the output **X(s) = H(s).** In turn**,** the inverse LT is the time function h(t), the impulse response of the system. This is an assertion that has been made previously, but with the ability to find the inverse LT using partial fractions, the relationship between transfer function and impulse response is made in a much more specific way.

Example 2.4-5

The transfer function of an LDE is

$$\mathbf{H}(\mathbf{s}) = \frac{(\mathbf{s}+\mathbf{2})}{(\mathbf{s}+\mathbf{1})(\mathbf{s}+\mathbf{4})} = \frac{\mathbf{A}}{(\mathbf{s}+\mathbf{1})} + \frac{\mathbf{B}}{(\mathbf{s}+\mathbf{4})}$$

so that:

$$\mathbf{A} = \left.\frac{(\mathbf{s}+\mathbf{2})}{(\mathbf{s}+\mathbf{4})}\right|_{\mathbf{s}=-\mathbf{1}} = \frac{\mathbf{1}}{\mathbf{3}} \qquad \mathbf{B} = \left.\frac{(\mathbf{s}+\mathbf{2})}{(\mathbf{s}+\mathbf{1})}\right|_{\mathbf{s}=-\mathbf{4}} = \frac{\mathbf{2}}{\mathbf{3}}$$

and the impulse response is:

$$\mathbf{h}(\mathbf{t}) = \frac{\mathbf{1}}{\mathbf{3}}\mathbf{e}^{-\mathbf{t}}\mathbf{u}(\mathbf{t}) + \frac{\mathbf{2}}{\mathbf{3}}\mathbf{e}^{-\mathbf{4t}}\mathbf{u}(\mathbf{t})$$

end of example

Now let's examine the complete response of the system (again with zero initial conditions) when the input is a cosine. We have seen that the output of the system under those circumstances is:

$$\mathbf{X}(\mathbf{s}) = \frac{(\mathbf{b}_1\mathbf{s} + \mathbf{b}_0)}{\left[\mathbf{a}_2\mathbf{s}^2 + \mathbf{a}_1\mathbf{s} + \mathbf{a}_0\right]}\frac{\mathbf{s}}{\mathbf{s}^2 + \boldsymbol{\beta}^2} = \mathbf{H}(\mathbf{s})\frac{\mathbf{s}}{(\mathbf{s}-\mathbf{j}\boldsymbol{\beta})(\mathbf{s}+\mathbf{j}\boldsymbol{\beta})} \qquad (2.4\text{-}1b)$$

In order to find the cosine component of the output, we must find the partial fraction coefficient corresponding to the term $(\mathbf{s}-\mathbf{j}\boldsymbol{\beta})$:

$$\begin{aligned} A &= (s - j\beta)H(s)\frac{s}{(s-j\beta)(s+j\beta)}\bigg|_{s=j\beta} = \\ &= H(j\beta)\frac{j\beta}{j2\beta} = \frac{1}{2}H(j\beta) = \frac{1}{2}|H(j\beta)|e^{j\theta} \end{aligned} \qquad (2.4\text{-}6)$$

This result exactly corresponds to the result of Chapter 1 in eq. 1.12c:

$$x(t) = E|H(j\omega)|\mathrm{Cos}\big[[\omega t + \theta(\omega)]\big]u(t) \qquad (1.12c)$$

We have now demonstrated the validity of the basic assertion that was made in Chapter 1, that the impulse reponse h(t) and the transfer function H(s) are essentially the same description of the system, one in the time domain and one in the s domain. This duality provides for the ability to analyze sytems in both of the domains and to directly relate the outcomes of these analyses to each other.

The combination of equations 2.4-6 and 1.12c provide the basis for discussing the **frequency response of a linear system** and the idea of filtering. This will be reserved for Chapter 3.

Example 2.4-6 The Complete Response of a System

With the system of Example 2.4-5

$$H(s) = \frac{(s+2)}{(s+1)(s+4)}$$

and the input

$$F(s) = \frac{3}{(s^2+9)}$$

the output is

$$\begin{aligned} X(s) = H(s)F(s) &= \frac{(s+2)}{(s+1)(s+4)}\frac{3}{(s^2+9)} \\ &= \frac{A}{(s+1)} + \frac{B}{(s+4)} + \frac{C}{(s-j3)} + \frac{C^*}{(s+j3)} \end{aligned}$$

where

$$C = (s-j3)X(s)\big|_{s=j3} = \frac{1}{2}\frac{(j3+2)}{(j3+1)(j3+4)} = \frac{1}{2}\frac{\sqrt{130}}{25}e^{-j52^0}$$

$$A = (s+1)X(s)\big|_{s=-1} = \frac{(s+2)}{(s+4)}\frac{3}{(s^2+9)}\bigg|_{s=-1} = \frac{1}{10}$$

$$\mathbf{B} = (s+4)X(s)\Big|_{s=-4} = \frac{(s+2)}{(s+1)}\frac{3}{(s^2+9)}\Bigg|_{s=-4} = \frac{2}{25}$$

Consequently the system output is:

$$x(t) = \frac{1}{10}e^{-t}u(t) + \frac{2}{25}e^{-4t}u(t) + \frac{\sqrt{130}}{25}\mathrm{Cos}\left(3t - 52^0\right)u(t) \qquad (2.4\text{-}7)$$

where the decaying exponential terms are the natural response (but NOT the impulse response) and the output cosine has the same frequency as that of the input, but a different amplitude and phase.

End of example

2.4-1 Partial Fractions in Systems Having Multiple Roots

Having established these basic features of the Inverse Laplace Transform, we now turn to dealing with **multiple roots**. Consider the following function where the denominator has a root of order 3.

$$F(s) = \frac{N(s)}{(s+a)^3(s+b)} \qquad (2.4\text{-}8a)$$

The partial fraction expansion must be assumed to be of the form

$$F(s) = \frac{A_3}{(s+a)^3} + \frac{A_2}{(s+a)^2} + \frac{A_1}{(s+a)} + \frac{B}{(s+b)} \qquad (2.4\text{-}8b)$$

because this is the most general form that will give the denominator $(s+a)^3(s+b)$. This can be directly generalized to a root of any order. Once A_3, A_2, A_1, and B are determined, the inverse Laplace Transform can easily be found. To find f(t), recall that

$$L\left[t^n e^{-at}u(t)\right] = \frac{n!}{(s+a)^{n+1}} \qquad (2.4\text{-}8c)$$

so that

$$f(t) = 2A_3t^2e^{-at}u(t) + A_2te^{-t}u(t) + A_1e^{-t}u(t) + Be^{-bt}u(t) \qquad (2.4\text{-}8d)$$

Finding B is a straightforward application of the technique for simple roots. From eq. 2.4-8a and eq. 2.4-8b:

$$\mathbf{B} = (\mathbf{s}+\mathbf{b})\mathbf{F}(\mathbf{s})\Big|_{\mathbf{s}=-\mathbf{b}} = \frac{\mathbf{N(s)}}{(\mathbf{s}+\mathbf{a})^3}\Bigg|_{\mathbf{s}=-\mathbf{b}} =$$

$$= \left\{\mathbf{B} + \frac{\mathbf{A}_3(\mathbf{s}+\mathbf{b})}{(\mathbf{s}+\mathbf{a})^3} + \frac{\mathbf{A}_2(\mathbf{s}+\mathbf{b})}{(\mathbf{s}+\mathbf{a})^2} + \frac{\mathbf{A}_1(\mathbf{s}+\mathbf{b})}{(\mathbf{s}+\mathbf{a})}\right\}_{\mathbf{s}=-\mathbf{b}} \tag{2.4-9a}$$

Using the same approach, we may find A_3 in a straightforward manner by multiplying eq. 2.4-8a or eq. 2.4-8b by $(\mathbf{s}+\mathbf{a})^3$:

$$\mathbf{A}_3 = (\mathbf{s}+\mathbf{a})^3\mathbf{F}(\mathbf{s})\Big|_{\mathbf{s}=-\mathbf{a}} = \left\{\mathbf{A}_3 + \mathbf{A}_2(\mathbf{s}+\mathbf{a}) + \mathbf{A}_1(\mathbf{s}+\mathbf{a})^2 + \frac{\mathbf{B}(\mathbf{s}+\mathbf{a})^3}{(\mathbf{s}+\mathbf{b})}\right\}_{\mathbf{s}=-\mathbf{a}} \tag{2.4-9b}$$

Note that when s = - a all of the terms except A_3 go to zero.

We now have the problem of finding A_2 and A_1. Considering eq. 2.4-9b closely, we can see that A_2 can be found through differentiation:

$$\mathbf{A}_2 = \frac{\mathbf{d}}{\mathbf{ds}}\left\{(\mathbf{s}+\mathbf{a})^3\mathbf{F}(\mathbf{s})\right\}\Big|_{\mathbf{s}=-\mathbf{a}} = \left\{\mathbf{A}_2 + 2\mathbf{A}_1(\mathbf{s}+\mathbf{a}) + \frac{\mathbf{d}}{\mathbf{ds}}\frac{\mathbf{B}(\mathbf{s}+\mathbf{a})^3}{(\mathbf{s}+\mathbf{b})}\right\}_{\mathbf{s}=-\mathbf{a}} \tag{2.4-9c}$$

and, differentiating again:

$$2\mathbf{A}_1 = \frac{\mathbf{d}^2}{\mathbf{ds}^2}\left\{(\mathbf{s}+\mathbf{a})^3\mathbf{F}(\mathbf{s})\right\}\Big|_{\mathbf{s}=-\mathbf{a}} = \left\{2\mathbf{A}_1 + \frac{\mathbf{d}^2}{\mathbf{ds}^2}\frac{\mathbf{B}(\mathbf{s}+\mathbf{a})^3}{(\mathbf{s}+\mathbf{b})}\right\}_{\mathbf{s}=-\mathbf{a}} \tag{2.4-9d}$$

In general, it can be extrapolated that if

$$\mathbf{F}(\mathbf{s}) = \frac{\mathbf{N(s)}}{(\mathbf{s}+\mathbf{a})^{\mathbf{n}}(\mathbf{s}+\mathbf{b})} \tag{2.4-10a}$$

so that

$$(\mathbf{s}+\mathbf{a})^{\mathbf{n}}\mathbf{F}(\mathbf{s}) = \mathbf{A}_{\mathbf{n}} + \mathbf{A}_{\mathbf{n}-1}(\mathbf{s}+\mathbf{a}) + \ldots + \mathbf{A}_1(\mathbf{s}+\mathbf{a})^{\mathbf{n}-1} + \frac{\mathbf{B}(\mathbf{s}+\mathbf{a})^{\mathbf{n}}}{(\mathbf{s}+\mathbf{b})} \tag{2.4-10b}$$

and

$$\mathbf{A}_{\mathbf{n}-\mathbf{k}} = \frac{1}{\mathbf{k}!}\frac{\mathbf{d}^{\mathbf{k}}}{\mathbf{ds}^{\mathbf{k}}}\left[(\mathbf{s}+\mathbf{a})^{\mathbf{n}}\mathbf{F}(\mathbf{s})\right]\Bigg|_{\mathbf{s}=-\mathbf{a}} \tag{2.4-10c}$$

A simple example of this process is given below.

Example 2.4-5 Multiple Real Roots

Let's start with the Laplace Transform:

$$\mathbf{F}(\mathbf{s})=\frac{(\mathbf{s}+2)}{(\mathbf{s}+1)^3(\mathbf{s}+3)}=\frac{\mathbf{A}_3}{(\mathbf{s}+1)^3}+\frac{\mathbf{A}_2}{(\mathbf{s}+1)^2}+\frac{\mathbf{A}_1}{(\mathbf{s}+1)}+\frac{\mathbf{B}}{(\mathbf{s}+3)}$$

$$\mathbf{B}=(\mathbf{s}+3)\mathbf{F}(\mathbf{s})\Big|_{\mathbf{s}=-3}=\left.\frac{(\mathbf{s}+2)}{(\mathbf{s}+1)^3}\right|_{\mathbf{s}=-3}=\frac{1}{8}$$

$$\mathbf{A}_3=(\mathbf{s}+1)^3\mathbf{F}(\mathbf{s})\Big|_{\mathbf{s}=-1}=\left.\frac{(\mathbf{s}+2)}{(\mathbf{s}+3)}\right|_{\mathbf{s}=-1}=\frac{1}{2}$$

$$\mathbf{A}_2=\frac{\mathbf{d}}{\mathbf{ds}}(\mathbf{s}+1)^3\mathbf{F}(\mathbf{s})\Big|_{\mathbf{s}=-1}=\left.\frac{\mathbf{d}}{\mathbf{ds}}\frac{(\mathbf{s}+2)}{(\mathbf{s}+3)}\right|_{\mathbf{s}=-1}=$$

$$=\left.\frac{(\mathbf{s}+3)-(\mathbf{s}+2)}{(\mathbf{s}+3)^2}\right|_{\mathbf{s}=-1}=\left.\frac{1}{(\mathbf{s}+3)^2}\right|_{\mathbf{s}=-1}=\frac{1}{4}$$

$$\mathbf{A}_1=\frac{1}{2}\frac{\mathbf{d}^2}{\mathbf{ds}^2}(\mathbf{s}+1)^3\mathbf{F}(\mathbf{s})\Big|_{\mathbf{s}=-1}=\left.\frac{1}{2}\frac{\mathbf{d}}{\mathbf{ds}}\frac{1}{(\mathbf{s}+3)^2}\right|_{\mathbf{s}=-1}=$$

$$=\left.\frac{1}{2}\frac{-2}{(\mathbf{s}+3)^3}\right|_{\mathbf{s}=-1}=-\frac{1}{8}$$

so that

$$\mathbf{F}(\mathbf{s})=\frac{(\mathbf{s}+2)}{(\mathbf{s}+1)^3(\mathbf{s}+3)}=\frac{1/2}{(\mathbf{s}+1)^3}+\frac{1/4}{(\mathbf{s}+1)^2}-\frac{1/8}{(\mathbf{s}+1)}+\frac{1/8}{(\mathbf{s}+3)}$$

and following eq.2.4-8d:

$$\mathbf{f}(\mathbf{t})=\frac{1}{4}\mathbf{t}^2\mathbf{e}^{-\mathbf{t}}\mathbf{u}(\mathbf{t})+\frac{1}{4}\mathbf{t}\mathbf{e}^{-\mathbf{t}}\mathbf{u}(\mathbf{t})-\frac{1}{8}\mathbf{e}^{-\mathbf{t}}\mathbf{u}(\mathbf{t})+\frac{1}{8}\mathbf{e}^{-3\mathbf{t}}\mathbf{u}(\mathbf{t})$$

end of example

As we can see, this is quite a tedious process. Even more tedious is separating into partial fractions a Laplace Transform having multiple complex poles:

Example 2.4-6 Multiple Complex Roots

Let's start with the Laplace Transform:

$$\mathbf{F}(\mathbf{s}) = \frac{(\mathbf{s}+3)}{(\mathbf{s}+1)(\mathbf{s}^2+4\mathbf{s}+8)^2} = \frac{(\mathbf{s}+3)}{(\mathbf{s}+1)(\mathbf{s}+2-2\mathbf{j})^2(\mathbf{s}+2+2\mathbf{j})^2}$$

$$= \frac{\mathbf{A}}{(\mathbf{s}+\mathbf{1})} + \frac{\mathbf{B}_2}{(\mathbf{s}+2-2\mathbf{j})^2} + \frac{\mathbf{B}_1}{(\mathbf{s}+2-2\mathbf{j})} + \frac{\mathbf{B}_2^*}{(\mathbf{s}+2+2\mathbf{j})^2} + \frac{\mathbf{B}_1^*}{(\mathbf{s}+2+2\mathbf{j})}$$

It follows that the inverse transform is

$$\mathbf{f}(\mathbf{t}) = \mathbf{A}\mathbf{e}^{-\mathbf{t}}\mathbf{u}(\mathbf{t}) + 2|\mathbf{B}_2|\mathbf{t}\mathbf{e}^{-2\mathbf{t}}\mathbf{Cos}(2\mathbf{t}+\theta_2)\mathbf{u}(\mathbf{t}) + 2|\mathbf{B}_1|\mathbf{e}^{-2\mathbf{t}}\mathbf{Cos}(2\mathbf{t}+\theta_1)\mathbf{u}(\mathbf{t})$$

where:

$$\mathbf{A} = (\mathbf{s}+1)\mathbf{F}(\mathbf{s})\Big|_{\mathbf{s}=-1} = \left.\frac{(\mathbf{s}+3)}{(\mathbf{s}^2+4\mathbf{s}+8)^2}\right|_{\mathbf{s}=-1} = \frac{2}{25}$$

$$\mathbf{B_2} = (\mathbf{s}+\mathbf{2}-\mathbf{2j})^2\mathbf{F}(\mathbf{s})\Big|_{\mathbf{s}=-2+2\mathbf{j}} = \left.\frac{(\mathbf{s}+\mathbf{3})}{(\mathbf{s}+\mathbf{1})(\mathbf{s}+\mathbf{2}+\mathbf{2j})^2}\right|_{\mathbf{s}=-2+2\mathbf{j}} = \frac{(\mathbf{1}+\mathbf{2j})}{(-\mathbf{1}+\mathbf{2j})(\mathbf{4j})^2} =$$

$$= \frac{\mathbf{1}}{\mathbf{16}}\frac{(\mathbf{1}+\mathbf{2j})}{(\mathbf{1}-\mathbf{2j})} = \frac{\mathbf{1}}{\mathbf{16}}\frac{(\mathbf{1}+\mathbf{2j})^2}{\mathbf{5}} = \frac{\mathbf{1}}{\mathbf{80}}(-\mathbf{3}+\mathbf{4j}) = \frac{\mathbf{1}}{\mathbf{16}}\mathbf{e}^{\mathbf{j127^0}}$$

$$\mathbf{B_1} = \frac{\mathbf{d}}{\mathbf{ds}}(\mathbf{s}+\mathbf{2}-\mathbf{2j})^2\mathbf{F}(\mathbf{s})\Big|_{\mathbf{s}=-2+2\mathbf{j}} = \frac{\mathbf{d}}{\mathbf{ds}}\left.\frac{(\mathbf{s}+\mathbf{3})}{(\mathbf{s}+\mathbf{1})(\mathbf{s}+\mathbf{2}+\mathbf{2j})^2}\right|_{\mathbf{s}=-2+2\mathbf{j}} =$$

$$= \left.\frac{(\mathbf{s}+\mathbf{1})(\mathbf{s}+\mathbf{2}+\mathbf{2j})^2 - (\mathbf{s}+\mathbf{3})\left[(\mathbf{s}+\mathbf{2}+\mathbf{2j})^2 + \mathbf{2}(\mathbf{s}+\mathbf{1})(\mathbf{s}+\mathbf{2}+\mathbf{2j})\right]}{(\mathbf{s}+\mathbf{1})^2(\mathbf{s}+\mathbf{2}+\mathbf{2j})^4}\right|_{\mathbf{s}=-2+2\mathbf{j}}$$

$$= \frac{(-\mathbf{1}+\mathbf{2j})(\mathbf{4j})^2 - (\mathbf{1}+\mathbf{2j})\left[(\mathbf{4j})^2 + \mathbf{2}(-\mathbf{1}+\mathbf{2j})(\mathbf{4j})\right]}{(-\mathbf{1}+\mathbf{2j})^2(-\mathbf{4j})^4}$$

$$= \frac{\mathbf{4j}\left[(\mathbf{-8}-\mathbf{4j}) - (\mathbf{1}+\mathbf{2j})(-\mathbf{2}+\mathbf{8j})\right]}{(-\mathbf{1}+\mathbf{2j})^2(-\mathbf{4j})^4} = \frac{[\mathbf{10}-\mathbf{8j}]}{(\mathbf{4j})^3(-\mathbf{3}-\mathbf{4j})} = \frac{\mathbf{2}(\mathbf{5}-\mathbf{4j})}{\mathbf{64j}(\mathbf{3}+\mathbf{4j})}$$

$$= \frac{\mathbf{6.4e^{-j39^0}}}{\mathbf{80e^{j127^0}}} = \mathbf{.08e^{-j166^0}}$$

It follows that

$$\mathbf{f}(\mathbf{t}) = \frac{\mathbf{2}}{\mathbf{25}}\mathbf{e}^{-\mathbf{t}}\mathbf{u}(\mathbf{t}) + \frac{\mathbf{1}}{\mathbf{8}}\mathbf{t}\mathbf{e}^{-\mathbf{2t}}\mathbf{Cos}\left(\mathbf{2t}+\mathbf{127^0}\right)\mathbf{u}(\mathbf{t}) + \mathbf{.08e}^{-\mathbf{2t}}\mathbf{Cos}\left(\mathbf{2t}-\mathbf{166^0}\right)\mathbf{u}(\mathbf{t}) \qquad (2.4\text{-}9)$$

<u>**end of example**</u>

Clearly, finding the coefficients for partial fraction expansions for systems having multiple poles is not a simple matter. Fortunately, multiple poles are typically theoretical rather than realistic constructs. Even the slightest variation of parameters in a real system will result in a double pole becoming two closely separated poles. That sensitivity to numerical variation is the basic reason that MATLAB residue algorithms exclude multiple poles.

This exhausts the possibilities of partial fraction expansions for the sort of systems that we are discussing. The method, as outlined, is an absolutely general and straightforward approach to finding the inverse Laplace Transforms that arise from linear, time-invariant differential equations. Further insights and applications are found in Chapter 3.

2.5 Convolution

It has so far been shown that the Laplace Transform of an LDE (having zero intial conditions) has the form:

$$\mathbf{X(s) = H(s)F(s)} \tag{2.5-1}$$

where F(s) is the Laplace Transform of the input f(t), X(s) is the LT of the output x(t), and H(s) is called the transfer function of the system.

Since the Laplace Transform of the unit impulse function is $\mathbf{LT[\delta(t)] = 1}$, it follows that if the input function $\mathbf{f(t) = \delta(t)}$, then we may reasonably call $\mathbf{h(t) = LT^{-1}[H(s)]}$ the impulse response of the system.

The impulse response of a linear system has the property that h(t) = 0 for t < 0. This property, called causality, says that the response of the system to an impulse function occurring at t = 0 cannot begin before the impulse is applied. The system cannot laugh before it is tickled.

According to Theorem 2.4, Convolution, the inverse Laplace Transform of the s – domain multiplication in eq. 2.5-1 is the time domain operation:

$$\mathbf{LT^{-1}\{X(s) = H(s)F(s)\} = \left\{x(t) = \int_{0^-}^{\infty} h(\lambda)f(t-\lambda)d\lambda\right\}} \tag{2.5-2}$$

as shown in fig. 2.5-1

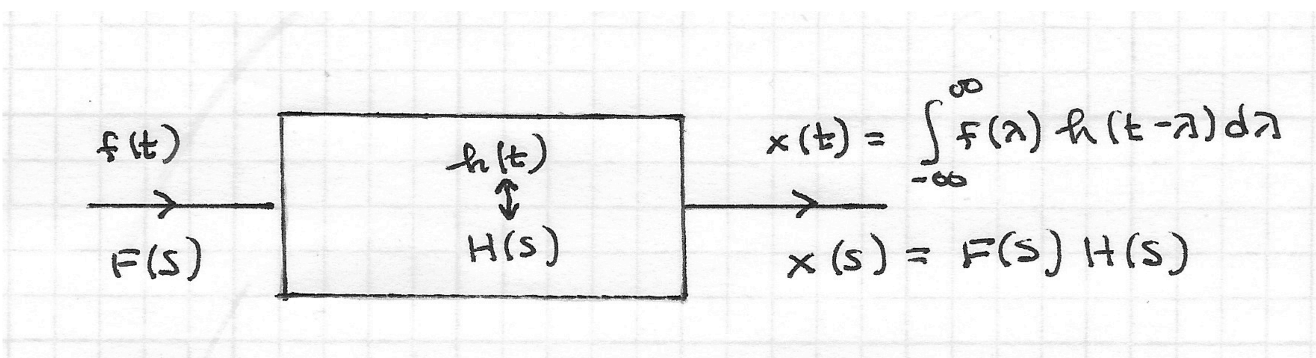


fig. 2.5-1

The purposes of this Section are to study the Convolution Integral of eq. 2.5-2 with the aims of:

- Learning how to perform the indicated integration;
- Understanding the reasons behind the particular form of the integral.

2.5-1 Evaluating the Convolution Integral

The first important observation to be made in examining the convolution integral of eq 2.5-2 is that the variable of integration is λ. Since the result of the integration, x(t), is a function of 't', the integral is saying that at any particular value of t, say t_1, that

$$x(t_1) = \int_{0^-}^{\infty} h(\lambda) f(t_1 - \lambda) d\lambda \qquad (2.5\text{-}3a)$$

The integral is also giving some instructions about how to visualize the two functions in its integrand. The term

$$f(t - \lambda) = f[-(\lambda - t)] \qquad (2.5\text{-}3b)$$

is to be understood as $f(\lambda)$ being subjected to reversal and then shifted to the right by the amount **'t'** **.** This operation can best be illustrated by example.

Example 2.5-1

Consider the input f(t) and the impulse response h(t) shown in **fig 2.5-2a**. Finding the convolution can best be accomplished by thinking of the integral as a set of instructions and sequentially following those instructions.

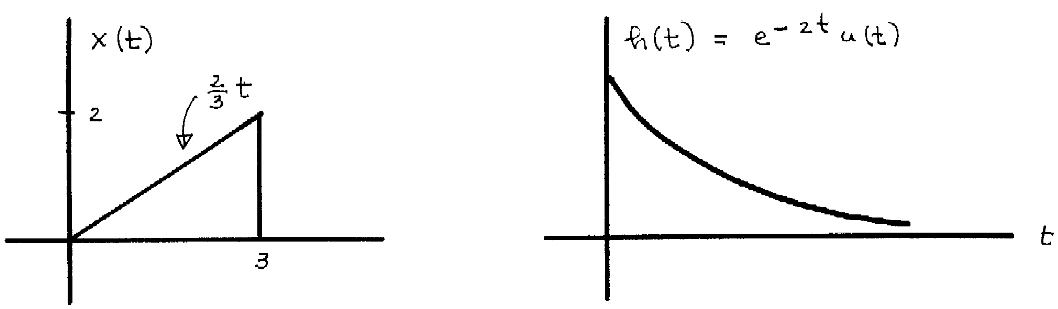


Fig. 2.5-2a

First place h(λ) down on the graph, taking care to recognize that λ is the variable of integration. Next, reverse f(λ) and shift the reversed signal to the right by 't', as shown in **fig 2.5-2b.** Note that three copies of the resulting f(t - λ) are shown: one copy for $\mathbf{t < 0}$ (negative 't' results in a left shift); one copy for $\mathbf{0 \le t \le 3}$; and one copy for $\mathbf{t > 3}$. This constitutes the entire range of 't' in relation to h(λ). The integral now says to multiply h(λ) and x(t - λ) and find the area under the resulting curve.

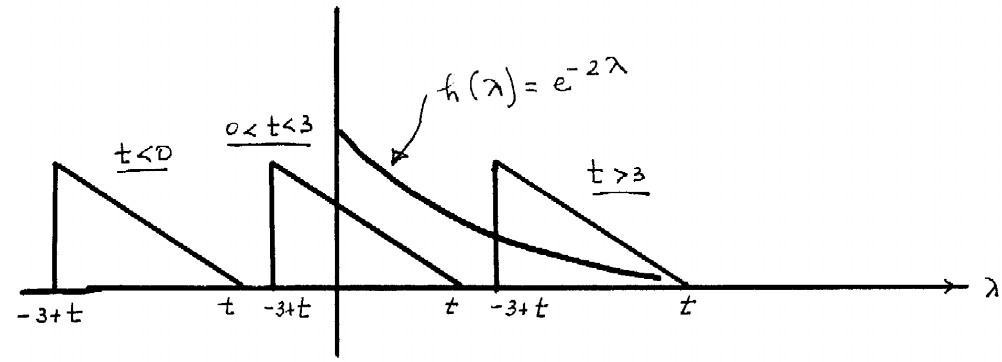


fig. 2.5-2b

For:

<u>t < 0</u> $\quad x(t) = 0 \quad$ since the product $h(\lambda)f(t - \lambda) = 0$ (2.5-4a)

<u>0 < t < 3</u>

$$\begin{aligned} \mathbf{x(t)} &= \int_0^{\mathbf{t}} \left[\frac{2}{3}(\mathbf{t}-\boldsymbol{\lambda})\right]\mathbf{e}^{-2\boldsymbol{\lambda}}\mathbf{d\lambda} \\ &= \frac{2}{3}\mathbf{t}\int_0^{\mathbf{t}} \mathbf{e}^{-2\boldsymbol{\lambda}}\mathbf{d\lambda} - \frac{2}{3}\int_0^{\mathbf{t}} \boldsymbol{\lambda}\mathbf{e}^{-2\boldsymbol{\lambda}}\mathbf{d\lambda} \\ &= \frac{2}{3}\mathbf{t}\frac{\mathbf{e}^{-2\boldsymbol{\lambda}}}{-2}\Bigg|_0^{\mathbf{t}} - \frac{2}{3}\boldsymbol{\lambda}\frac{\mathbf{e}^{-2\boldsymbol{\lambda}}}{-2}\Bigg|_0^{\mathbf{t}} + \frac{2}{3}\int_0^{\mathbf{t}}\frac{\mathbf{e}^{-2\boldsymbol{\lambda}}}{-2}\mathbf{d\lambda} \\ &= \frac{\mathbf{1}}{\mathbf{3}}\mathbf{t} - \frac{1}{6}\left(1-\mathbf{e}^{-2\mathbf{t}}\right) \end{aligned} \qquad (2.5\text{-}4b)$$

<u>3 < t</u>

$$\mathbf{x(t)} = \int_{-3+\mathbf{t}}^{\mathbf{t}} \left[\frac{2}{3}(\mathbf{t}-\boldsymbol{\lambda})\right]\mathbf{e}^{-2\boldsymbol{\lambda}}\mathbf{d\lambda} \qquad (2.5\text{-}4c)$$

The computation of this integral and the sketching of x(t) is left to the Problems.

<u>end of example</u>

Example 2.5.2

Consider the input f(t) and the impulse response h(t) shown in **fig 2.5-3a**. In this instance we will use the form of the convolution integral

$$\mathbf{x(t) = \int_{0}^{\infty} f(\lambda)h(t-\lambda)d\lambda} \tag{2.5-5}$$

as shown in **fig 2.5-3b**. Computing the integral:

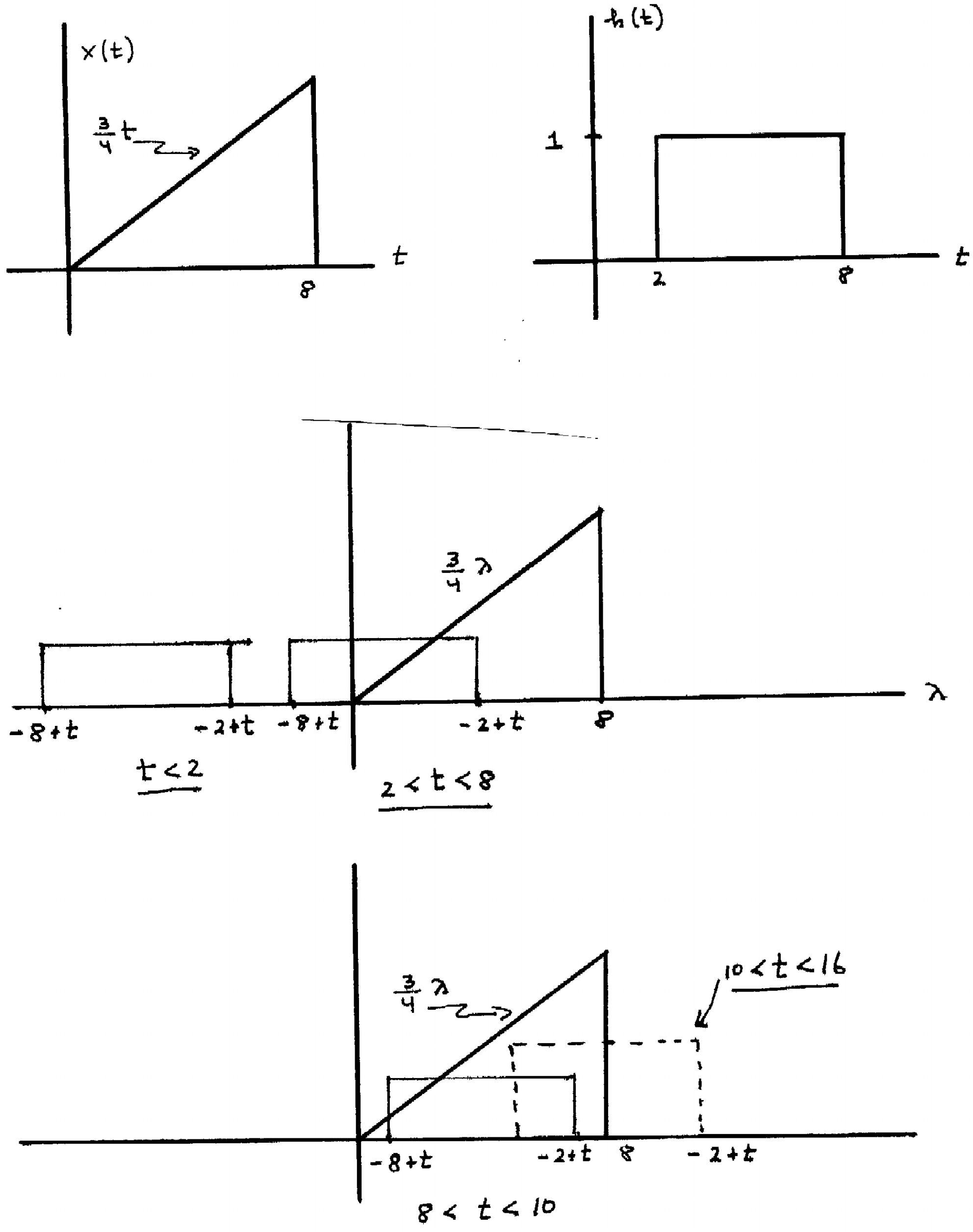


fig. 2.5-3

t < 2 $\qquad x(t) = 0$ since the product $h(\lambda)x(t - \lambda) = 0$ (2.5-6a)

2 < t < 8 $$\mathbf{x}(\mathbf{t}) = \int_0^{-2+\mathbf{t}} \frac{3}{4}\boldsymbol{\lambda} \mathbf{d}\boldsymbol{\lambda} = \frac{3}{4}\frac{\boldsymbol{\lambda}^2}{2}\Bigg|_0^{-2+\mathbf{t}} = \frac{3}{8}(\mathbf{t}-2)^2 \qquad (2.5\text{-}6b)$$

8 < t < 10 $$\mathbf{x}(\mathbf{t}) = \int_{-8+\mathbf{t}}^{-2+\mathbf{t}} \frac{3}{4}\boldsymbol{\lambda} \mathbf{d}\boldsymbol{\lambda} = \frac{3}{4}\frac{\boldsymbol{\lambda}^2}{2}\Bigg|_{-8+\mathbf{t}}^{-2+\mathbf{t}} = \frac{3}{8}\left[(\mathbf{t}-2)^2 - (\mathbf{t}-8)^2\right] \qquad (2.5\text{-}6c)$$

10 < t < 16 $$\mathbf{x}(\mathbf{t}) = \int_{-8+\mathbf{t}}^{8} \frac{3}{4}\boldsymbol{\lambda} \mathbf{d}\boldsymbol{\lambda} = \frac{3}{4}\frac{\boldsymbol{\lambda}^2}{2}\Bigg|_{-8+\mathbf{t}}^{8} = \frac{3}{8}\left[8^2 - (\mathbf{t}-8)^2\right] \qquad (2.5\text{-}6d)$$

Note that at t = 16, x (t) = 0 and, of course, remains equal to zero for t > 16.

Again, sketching the output x(t) will be left to the Problems.

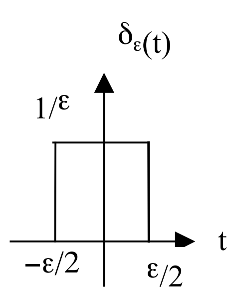
end of example

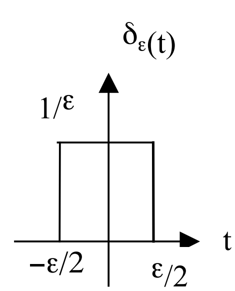
Example 2.5-3

This example illustrates the important result that the convolution of two impulse functions is also an impulse function:

$$\int_{-\infty}^{\infty} \delta(\lambda)\delta(t-\lambda)d\lambda = \delta(t) \qquad (2.5\text{-}7)$$

As shown in **fig. 2.5-4**, we begin with the convolution of two rectangular pulses, each of which becomes an impulse function in the limit as $\varepsilon \to 0$.





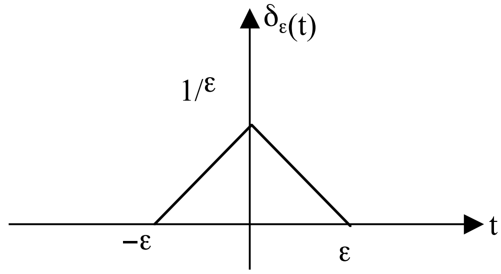


fig 2.5-4 Convolution of Two Impulses

The result of this convolution is the triangular pulse that also becomes an impulse function in the limit as $\varepsilon \to 0$. Time shifting also yields the more general result:

$$\delta(t-a) * \delta(t-b) = \int_{-\infty}^{\infty} \delta(\lambda - a)\delta((t-b)-\lambda)d\lambda = \delta(t-(a+b)) \qquad (2.5\text{-}8)$$

end of example

Example 2.5-4

This very important example is entitled **"Convolution with an Impulse is Easy".** It derives from the sampling property of impulse functions.

$$\delta(t-a) * f(t-b) = \int_{-\infty}^{\infty} \delta(\lambda - a)f((t-b)-\lambda)d\lambda = f(t-(a+b))$$

The value of the integral is simply a shifted version of the function f(t). This example will be used in a very crucial way in Chapter 5, Fourier Transforms.

Many more examples of convolution are given in the Problems.

2.5.2 Time Domain Derivation of the Convolution Integral

Having shown how to compute the convolution integral, the next issue is how to understand its format, particularly the reversal and shifting of one of its waveforms. In order to accomplish this, we shall construct a simple example.

As shown in **fig. 2.5-5,** the input signal to a linear system **f(t)** can approximated by a series of rectangular pulses $f_k(t)$:

$$f_k(t) = \sum_{k=-\infty}^{\infty} f(kT)p_T(t-kT) \cdot T \qquad (2.5\text{-}9a)$$

where $p_T(t)$ is a rectangular pulse having width T and height 1/T. Note that $\mathbf{p_T(t)\cdot T}$ is a pulse having unit amplitude and width T.

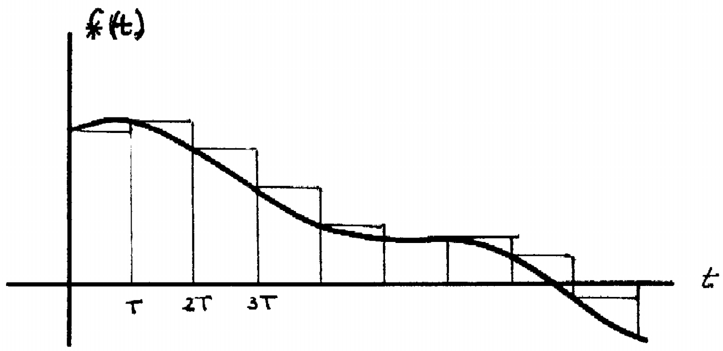


fig. 2.5-5 Rectangular Approximation to f(t)

As T gets very small, $p_T(t)$ becomes $\delta(t)$ and we may replace the discrete variable kT by the continuous variable λ and T by the differential $d\lambda$ so that the summation of eq. 3.9-1a becomes the integral:

$$\mathbf{f(t) = \int_{-\infty}^{\infty} f(\lambda)\delta(t-\lambda)d\lambda} \qquad (2.5\text{-}9b)$$

which can simply be understood as a slight variation of the sampling property of impulse functions where the impulse function $\delta(t - \lambda)$ samples $f(\lambda)$ at $\lambda = t$. Now we shall apply $\mathbf{f_k(t)}$ to a circuit having response $\mathbf{h_T(t)}$ to the pulse $\mathbf{p_T(t)}$. Since linearity and superposition hold in this circuit, it follows that:

$$\mathbf{x_k(t) = \sum_{k=-\infty}^{\infty} f(kT)h_T(t-kT)\cdot T} \qquad (2.5\text{-}10a)$$

Using the same limiting arguments that led from eq. 2.5-9a to 2.5-9b, it follows that

$$\mathbf{x(t) = \int_{-\infty}^{\infty} f(\lambda)h(t-\lambda)d\lambda} \qquad (2.5\text{-}10b)$$

which is the convolution integral.

We shall revisit the operation of convolution in Chapter 6, Discrete Time Systems, where the issue of "reversal and shifting" will become transparent.

2.5-3 Convolution Algebra

There are a variety of interesting and useful properties of the convolution integral that will be outlined below. Let:

$$\mathbf{z(t) = x(t) * y(t)} \tag{2.5-11a}$$

It follows that:

1. $$\mathbf{z(t) * \delta(t) = \int_{-\infty}^{\infty} z(\lambda)\delta(t-\lambda)d\lambda = z(t)} \tag{2.5-11b}$$

2. $$\mathbf{z(t) * u(t) = \int_{-\infty}^{\infty} z(\lambda)u(t-\lambda)d\lambda = \int_{-\infty}^{t} z(\lambda)d\lambda} \tag{2.5-11c}$$

3. Recalling that the doublet, the derivative of the impulse, is an odd function

$$\mathbf{\delta'(t-\lambda) = -\delta'(\lambda-t)} \tag{2.5-11d}$$

integrating by parts yields

$$\mathbf{z(t) * \delta'(t) = \int_{-\infty}^{\infty} z(\lambda)\delta'(t-\lambda)d\lambda = \frac{dz(t)}{dt}} \tag{2.5-11e}$$

4. As a special case of eq. 3.9-6e,

$$\mathbf{u(t) * \delta'(t) = \delta(t)} \tag{2.5-11f}$$

5. $$\mathbf{z(t) * \delta'(t) = x(t) * y(t) * \delta'(t) = x(t) * \frac{dy(t)}{dt} = \frac{dx(t)}{dt} * y(t)} \tag{2.5-11g}$$

6.

$$\begin{aligned}\mathbf{z(t) * \delta(t)} &= \mathbf{x(t) * y(t) * u(t) * \delta'(t)} \\ &= \mathbf{[x(t) * \delta'(t)] * [y(t) * u(t)]} \\ &= \mathbf{\frac{dx(t)}{dt} * \int_{-\infty}^{t} y(\lambda)d\lambda = \frac{dy(t)}{dt} * \int_{-\infty}^{t} x(\lambda)d\lambda}\end{aligned} \tag{2.5-11h}$$

Just as h(t) was defined as the impulse response of a system, we may also define the step response s(t) of the same system. Maintaining the criterion of zero initial conditions, the step response is the convolution of the impulse response with a unit step function:

$$\mathbf{s(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\lambda)u(t-\lambda)d\lambda = \int_{0}^{t} h(\lambda)d\lambda} \tag{2.5-12a}$$

The lower limit becomes zero as a consequence of the causality of the impulse response h(t). The result is very satisfying, that the step response is the integral of the impulse response. Conversely, the derivative of the step response is the impulse response:

$$\frac{ds(t)}{dt} = \frac{d}{dt}\int_0^t h(\lambda)d\lambda = h(t) \qquad (2.5\text{-}12b)$$

This is quite useful because it is relatively easy to measure the step response of a system. The step response can then be numerically or graphically differentiated to obtain the impulse response.

Let f (t) be the input to a system having a step response w(t) and consider the convolution of w(t) and df(t)/dt. Note that:

$$x(t) = \frac{df(t)}{dt} * s(t) = f(t) * \frac{ds(t)}{dt} = f(t) * h(t) \qquad (2.5\text{-}13a)$$

and

$$\begin{aligned} x(t) &= \int_{0^-}^{t} \frac{df(\lambda)}{d\lambda} s(t-\lambda)d\lambda \\ &= \int_{0^-}^{0^+} \frac{df(\lambda)}{d\lambda} s(t-\lambda)d\lambda + \int_{0^-}^{t} \frac{df(\lambda)}{d\lambda} s(t-\lambda)d\lambda \qquad (2.5\text{-}13b) \\ &= f\left(0^+\right)s(t) + \int_{0^-}^{t} \frac{df(\lambda)}{d\lambda} s(t-\lambda)d\lambda \end{aligned}$$

where the first term involving $f(0^+)$ addresses discontinuities in f(t) at the origin (with the assumption that $f(0^-) = 0$.

This integral is called the superposition integral or DuHamel's Integral. It is an interesting alternative way to computing the output based on the step response rather than the impulse response.

In summary, we have now accomplished the following:

1. Demonstrated that the transfer function of a linear system H(s) and the impulse response of that system h(t) are a Laplace Transform pair;

2. Shown that convolution in the time domain and multiplication in the frequency domain are dual operations, each of them being an input-output relationship in a linear system;

3. Shown how to evaluate the convolution integral.

Consequently, we have now acquired the basic tools that will allow us to proceed.

2.6 The Laplace Tranform of a Periodic Function

We now turn to finding the Laplace Transform of a Semi-Periodic Signal. This transform and the surprising character of the inverse transform constitute a revealing transition to the Fourier Series representation of periodic functions, the subject of Chapter 4.

The definition of a semi-periodic signal is illustrated in **fig. 2.6-1.** A time limited signal f(t) is shown in fig. 2.6-1a. The semi – periodic version of f(t), called $f_T(t)$, is shown in fig. 2.6-1b. It is semi – periodic because it is equal to zero for t < 0. The fully periodic version of f(t) would extend for all values of t.

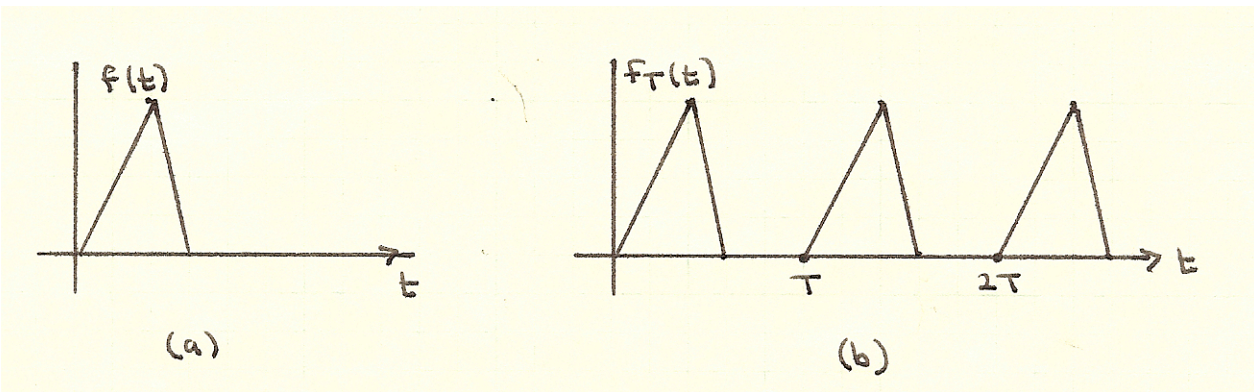


fig. 2.6-1

Theorem 2.10 The Laplace Transform of a Semi-Periodic Signal

Let f(t) have a Laplace Transform F(s). Note that **f(t) = 0 for t < 0**. For simplicity of conception (although not necessary for the Theorem) let **f(t) = 0 for t > T**. Define the semi-periodic signal $\mathbf{f_T(t) = \sum_{n=0}^{\infty} f(t - nT)}$. Then

$$\left\{ \mathbf{f_T(t) = \sum_{n=0}^{\infty} f(t - nT)} \right\} \Leftrightarrow \left\{ \mathbf{\frac{F(s)}{1 - e^{-sT}}} \right\} \qquad (2.6\text{-}1)$$

Proof:

According to the time shifting theorem

$$\left\{ \mathbf{\sum_{n=0}^{\infty} f(t - nT)} \right\} \Leftrightarrow \mathbf{F(s)} \left\{ 1 + \mathbf{e^{-sT}} + \mathbf{e^{-2sT}} + \mathbf{e^{-3sT}} + \mathbf{...} \right\}$$

Using the identity:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

provided that $|x| \le 1$, we can then argue that

$$\left\{ 1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \ldots \right\} = \frac{1}{1-e^{-sT}}$$

end of proof

An example will serve to illustrate this theorem. Consider the semi-periodic square wave $f_T(s)$ in **fig. 2.6-2a** and its first period f(t) in **fig. 2.6-2b.** The analytic expression for f(t) is

$$\mathbf{f(t) = E\left[u(t) - 2u\left(t - \frac{T}{2}\right) + u(t-T)\right]} \tag{2.6-2a}$$

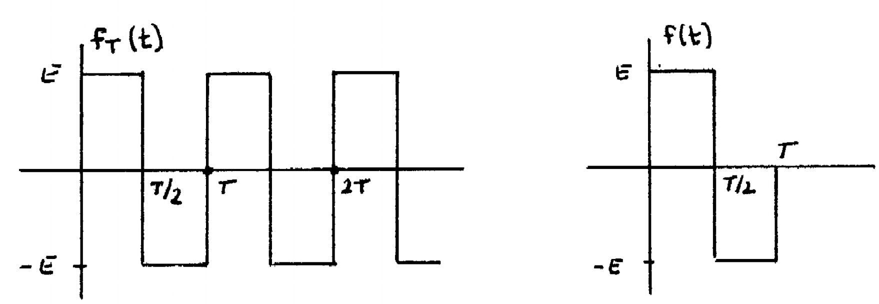


fig. 2.6-2 Laplace Transform of Periodic Function

having a Laplace Transform

$$\mathbf{F(s)} = \frac{\mathbf{E}}{\mathbf{s}}\left[1 - 2\mathbf{e}^{-sT/2} + \mathbf{e}^{-sT}\right] = \frac{\mathbf{E}}{\mathbf{s}}\left[1 - \mathbf{e}^{-sT/2}\right]^2 \tag{2.6-2b}$$

so that, according to eq. 2.6-1, the Laplace Transform of the semi-periodic version of f(t) is:

$$\mathbf{F_T(s)} = \frac{\mathbf{F(s)}}{1-\mathbf{e}^{-sT}} = \frac{\mathbf{E}\left[1-\mathbf{e}^{-sT/2}\right]^2}{\mathbf{s}\left(1-\mathbf{e}^{-sT}\right)} \tag{2.6-2c}$$

We want to find the inverse Laplace Transform of this function using partial fractions, but in so doing we will have to take a careful look at the denominator of eq. 2.6-2c in order to determine the poles. Recall that a pole is any value of 's' that causes the denominator to become zero. The factor 's' in the denominator causes s = 0 to be a pole. Other poles correspond to all values of 's' that cause

$e^{-sT} = 1$ yielding $s = jn\frac{2\pi}{T} = jn\omega_0$ for n = 0, ±1, ±2, ±3, …, i.e.- for all integer values of n. These poles are shown in **fig. 2.4-3.** Accordingly, we may rewrite eq. 2.6-2c as

$$F_T(s) = \frac{E\left[1 - e^{-sT/2}\right]^2}{s\left(1 - e^{-sT}\right)} = \frac{E\left[1 - e^{-sT/2}\right]^2}{s\prod_{n=-\infty}^{\infty}(s - jn\omega_0)} \tag{2.6-3}$$

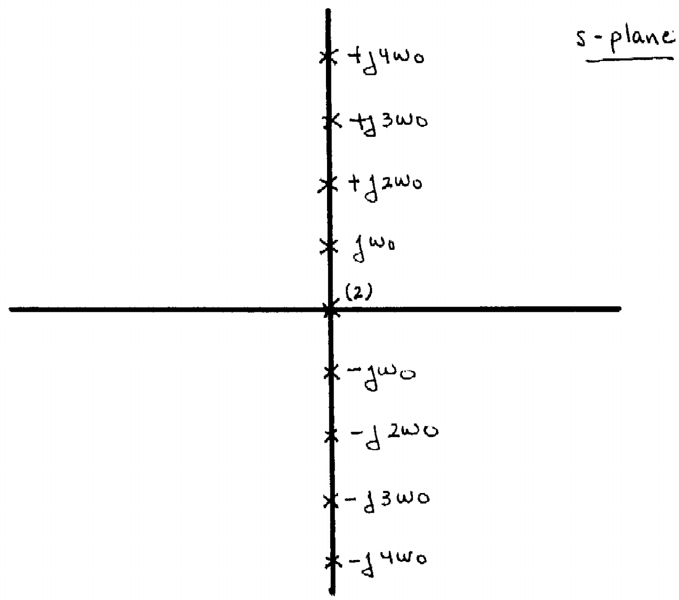


fig. 2.4-3 Poles of Laplace Transform of a Periodic Function

Taking into account that there are two poles at the origin, one from the 's' term in the denominator and one from the product term corresponding to n = 0, the partial fraction expansion of eq. 2.6-3 may be written as

$$F_T(s) = \frac{A_2}{s^2} + \frac{A_1}{s} + \sum_{n=1}^{\infty}\left[\frac{B_n}{(s - jn\omega_0)} + \frac{B_n^*}{(s + jn\omega_0)}\right] \tag{2.6-4}$$

Evaluating the partial fraction coefficients:

$$A_2 = s^2 F_T(s)\Big|_{s=0} = s^2 \frac{E\left(1 - e^{-sT/2}\right)^2}{s\left(1 - e^{-sT}\right)}\Bigg|_{s=0} = E\frac{s\left(1 - e^{-sT/2}\right)^2}{\left(1 + e^{-sT/2}\right)}\Bigg|_{s=0} = 0 \tag{2.6-5a}$$

This result is reassuring because there should not be a ramp function in the result.

$$\mathbf{A_1 = \frac{d}{ds}s^2F_T(s)\Big|_{s=0} = E\frac{d}{ds}\left.\frac{s\left(1-e^{-sT/2}\right)^2}{\left(1+e^{-sT/2}\right)}\right|_{s=0} = 0} \qquad (2.6\text{-}5b)$$

This result is equally reassuring because the signal has zero average value and consequently should not have a step function in the result either.

The confirmation of eq. 2.6-5b is referred to the Problems.

Continuing:

$$\mathbf{B_n = (s-jn\omega_0)F_T(s)\Big|_{s=jn\omega_0} = E\left.\frac{(s-jn\omega_0)\left[1-e^{-sT/2}\right]^2}{s\left(1-e^{-sT}\right)}\right|_{s=jn\omega_0}} \qquad (2.6\text{-}6a)$$

Since this expression yields 0/0, we must use L'Hospital's Rule and differentiate the numerator and the denominator separately. Taking into account that some of the resulting terms automatically will become zero, this results in

$$\mathbf{B_n = E\left.\frac{\left[1-e^{-sT/2}\right]^2}{sTe^{-sT}}\right|_{s=jn\omega_0} = E\frac{\left[1-e^{-jn\pi}\right]^2}{jn2\pi} = \begin{cases} \frac{E}{\pi n}e^{-j\pi/2}; & n = 1,3,5,.... \\ 0; & n \text{ even} \end{cases}} \qquad (2.6\text{-}6b)$$

This means that the semi-periodic function can be written as:

$$\mathbf{f_T(t) = \sum_{n=1,3,5,...}\frac{E}{n\pi}Cos\left(n\omega_0 t - 90^0\right)u(t) = \sum_{n=1,3,5,...}\frac{E}{n\pi}Sin\left(n\omega_0 t\right)u(t)} \qquad (2.6\text{-}7)$$

This result is worth considering very carefully. The semi-periodic function $f_T(t)$ is expressed as an infinite series of cosines (or sines) having frequencies that are integer multiples of the fundamental frequency $\boldsymbol{\omega_0 = 2\pi/T}$. Since this is a consequence of the factor $\left(1-\mathbf{e^{-sT}}\right)$ that appears in the denominator of the Laplace Transform of every semi-periodic function having period T, it follows that every such function will have a representation of the form of eq. 2.6-8.

A more general expression for such a periodic function is:

$$\mathbf{f_T(t) = \left[f_0 + \sum_{n=1}^{\infty} 2|f_n| Cos\left(n\omega_0 t + \theta_n\right)\right]u(t)} \tag{2.6-8}$$

where f_0 is the dc component

Series of this kind are called Fourier Series. We looked at such series in Chapter 1. They will be considered in detail in Chapter 4.

Additional examples of such series are given in the Problems.

.

Table 2.2
Laplace Transforms

	f(t)	**F(s)**
1.	$\delta(t)$	1
2.	u(t)	$\frac{1}{s}$
1.	r(t) = tu(t)	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$\delta'(t) = \frac{d\delta(t)}{dt}$	s
6.	$\delta''(t) = \frac{d^2\delta(t)}{dt^2}$	s^2
7.	$e^{-at}u(t)$	$\frac{1}{s+a}$
8.	$t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$
9.	$\mathrm{Sin}(\beta t)u(t)$	$\frac{\beta}{s^2+\beta^2}$
10.	$\mathrm{Cos}(\beta t)u(t)$	$\frac{s}{s^2+\beta^2}$
11.	$e^{-at}\mathrm{Sin}(\beta t)u(t)$	$\frac{\beta}{(s+a)^2+\beta^2}$
12.	$e^{-at}\mathrm{Cos}(\beta t)u(t)$	$\frac{(s+a)}{(s+a)^2+\beta^2}$

Table 2.3

Laplace Transform Theorems

	f(t)	F(s)
1. Time Differentiation	$\frac{df(t)}{dt}$	$sF(s)-f(0^-)$
	$\frac{d^2f(t)}{dt^2}$	$s^2F(s)-sf(0^-)-f'(0^-)$
2. Time Integration	$\int_{-\infty}^{t} f(\lambda)d\lambda$	$\frac{F(s)}{s}+\frac{\int_{-\infty}^{0^-} f(\lambda)d\lambda}{s}$
3. Linearity and Superposition	$f(t)=af_1(t)+bf_2(t)$	$F(s)=aF_1(s)+bF_2(s)$
4. Convolution	$\int_{-\infty}^{\infty} f_1(\lambda)f_2(t-\lambda)d\lambda$	$F_1(s)F_2(s)$
5. Initial Value Theorem	$f(0^+)=\lim_{s\to\infty} sF(s)$	
6. Final Value Theorem	$f(\infty)=\lim_{s\to 0} sF(s)$	
7. Multiplication by an Exponential	$e^{-at}f(t)$	$F(s+a)$
	$e^{\pm j\beta t}f(t)$	$F(s\mp j\beta)$
8. Multiplication by t^n	$t^n f(t)$	$(-1)^n \frac{d^nF(s)}{ds^n}$
9. Time Shifting	$f(t-t_0)$	$F(s)e^{-st_0}$
9a. Periodicity	$\sum_{n=0}^{\infty} f(t-nT)$	$\frac{F(s)}{1-e^{-sT}}$
10. Scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
11. Division by t	$\frac{f(t)}{t}$	$\int_{s}^{\infty} F(u)du$
12. Frequency Convolution	$f_1(t)f_2(t)$	$\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F_1(u)F_2(s-u)du$