Thick Lenses and the ABCD Formalism

Thursday, 10/12/2006

Physics 158

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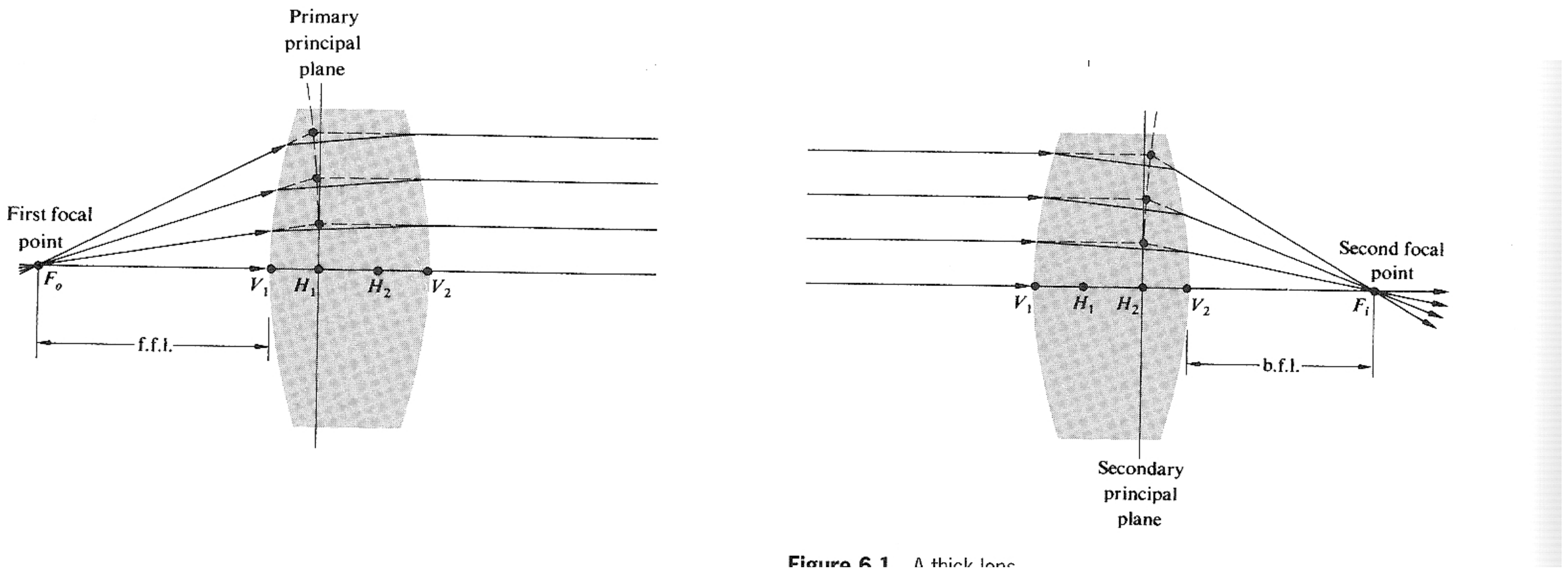
Class Outline

- Properties of Thick Lenses
- Paraxial Ray Matrices
- General Imaging Systems

Thick Lenses

When the thickness of a lens is not negligible compared to the object and image distances we cannot make the approximations that led to the "thin lens formula", and requires a few additional parameters to describe it

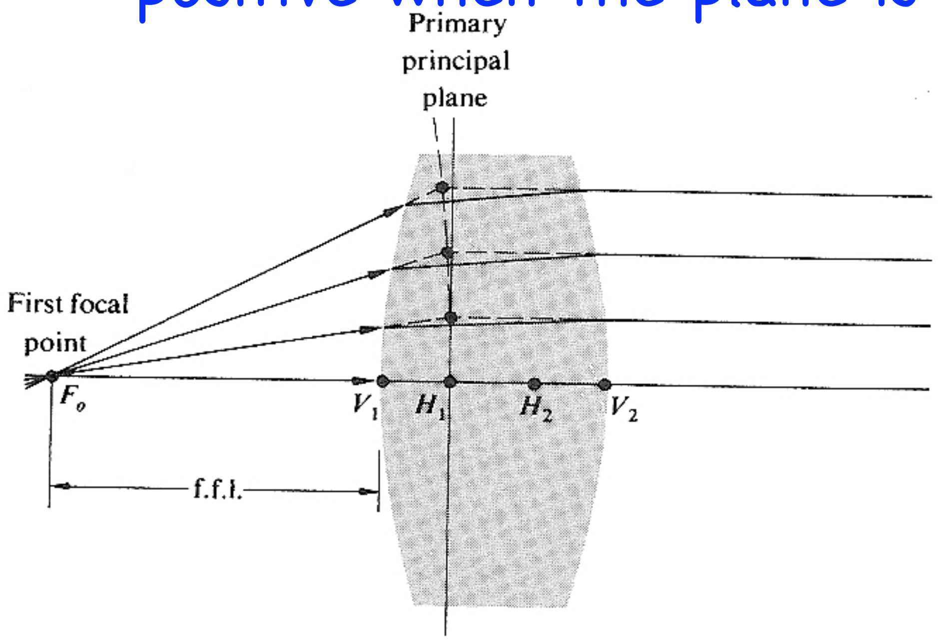
- Front and Back focal lengths
- Primary and secondary Principle planes

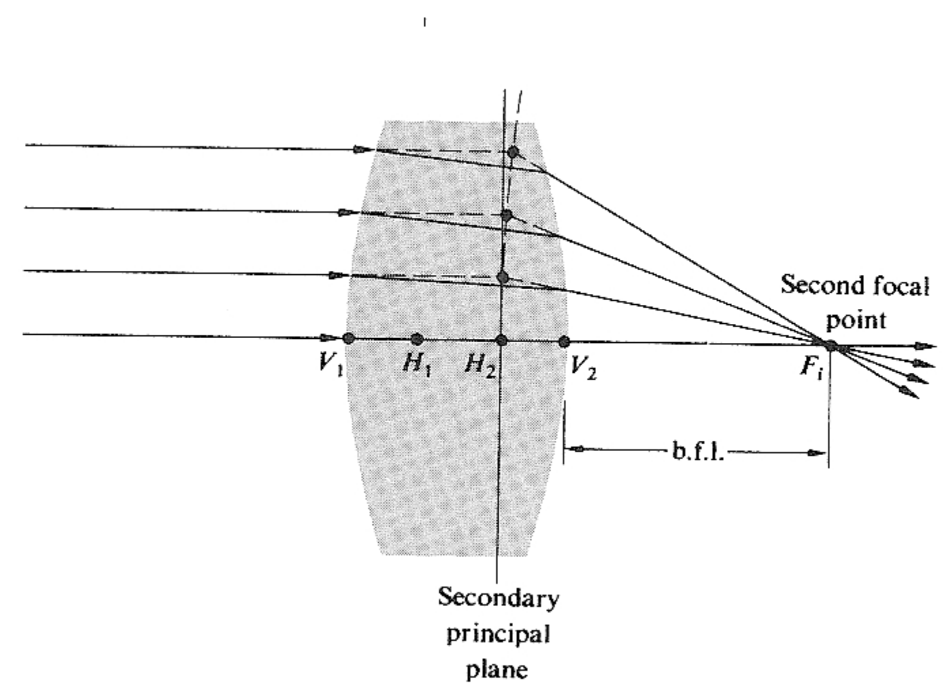


Terms used with Thick Lenses

Focal lengths are measured from the vertex of the lens (not the center) and are labeled as the front focal length and the back focal length. An effective focal length is also often used...

Principle Planes are the plane approximations to the locust of points where parallel incident rays would intersect converging exiting rays. There is a primary (on the front side) and a secondary (on the back side) principle plane. These are located a distance h_1 and h_2 from the vertices. These distances are positive when the plane is to the right of the vertex.





Imaging with a thick lens

The same derivation used for the thin lens equation can be used to show that for a thick lens

$$\frac{1}{s_i} + \frac{1}{s_o} = \frac{1}{f}$$

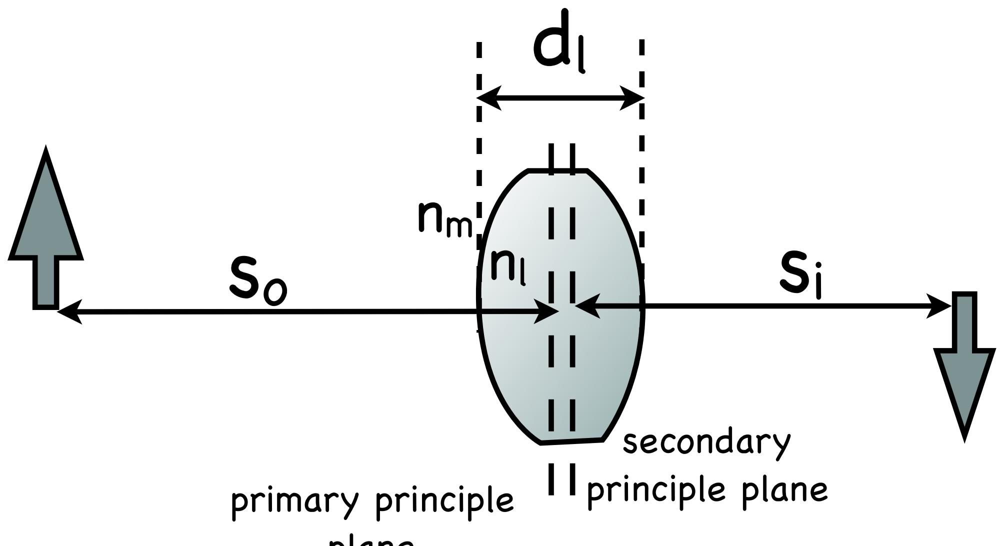
provide the effective focal length given by

$$\frac{1}{f} = (n_l - n_m)\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n_l - n_m)d_l}{n_l R_1 R_2}\right]$$

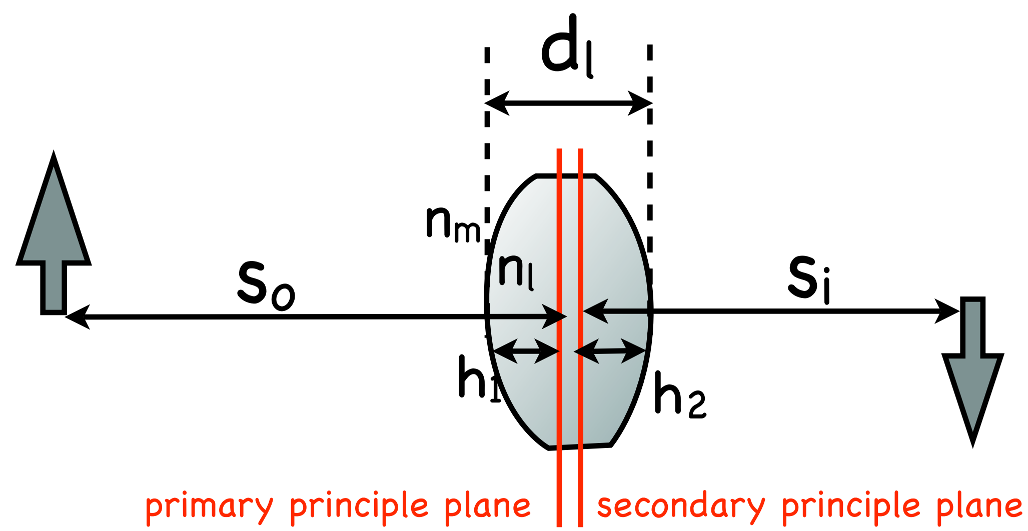
is used, and the distances s_o and s_i are measured from the principle points located at

$$h_1 = -\frac{f(n_l - n_m)d_l}{R_2 n_l}$$

$$h_2 = -\frac{f(n_l - n_m)d_l}{R_1 n_l}$$



Thick Lens Parameters



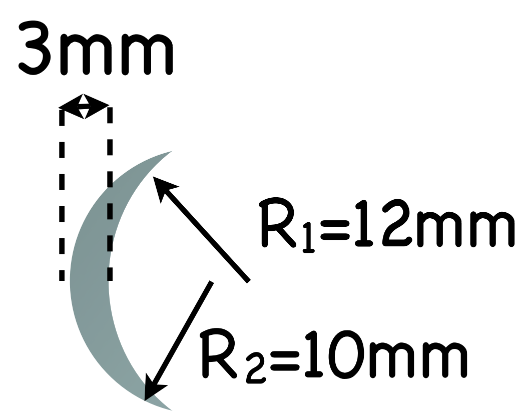
$$\frac{1}{f} = (n_l - n_m)\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n_l - n_m)d_l}{n_l R_1 R_2}\right]$$

$$h_1 = -\frac{f(n_l - n_m)d_l}{R_2 n_l} \qquad h_2 = -\frac{f(n_l - n_m)d_l}{R_2 n_l}$$

note h_1 and h_2 are positive when the principal point is to the right of the vertex

Example

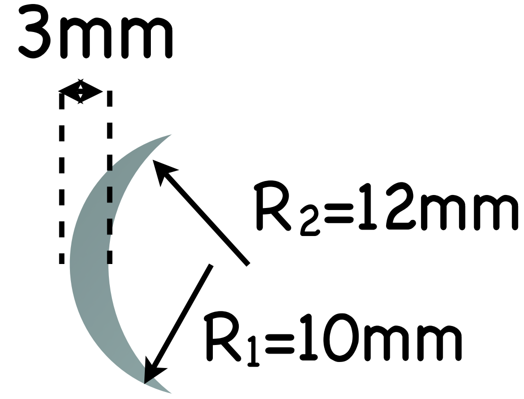
- For the lens shown, find the effective focal length and the principle points (points where the principle planes intersect the optical axis) if it is made of glass of index 1.5 and is in air



Note: Radius of curvature is considered positive if the vertex of the lens surface is to the left of its center of curvature

Example

- For the lens shown, find the effective focal length and the principle points (points where the principle planes intersect the optical axis) if it is made of glass of index 1.5 and is in air



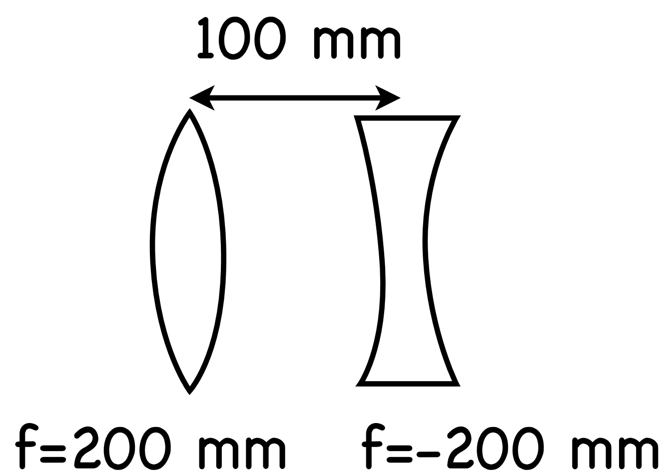
$$\frac{1}{f} = (1.5-1)\left[\frac{1}{10\text{ mm}} - \frac{1}{12\text{ mm}} + \frac{(1.5-1)3\text{ mm}}{1.5(10\text{ mm})(12\text{ mm})}\right]$$

f=80 mm

$$h_1 = -\frac{80\text{ mm}\,(1.5-1)\,3\text{ mm}}{12\text{ mm}\,(1.5)} = -6.7\text{ mm} \qquad h_2 = -\frac{80\text{ mm}\,(1.5-1)\,3\text{ mm}}{10\text{ mm}\,(1.5)} = -8\text{ mm}$$

Example

- Find the focal length and locations of the principle points for a thin lens system with two thin lenses of focal lengths 200mm and -200mm separated by 100mm



Example

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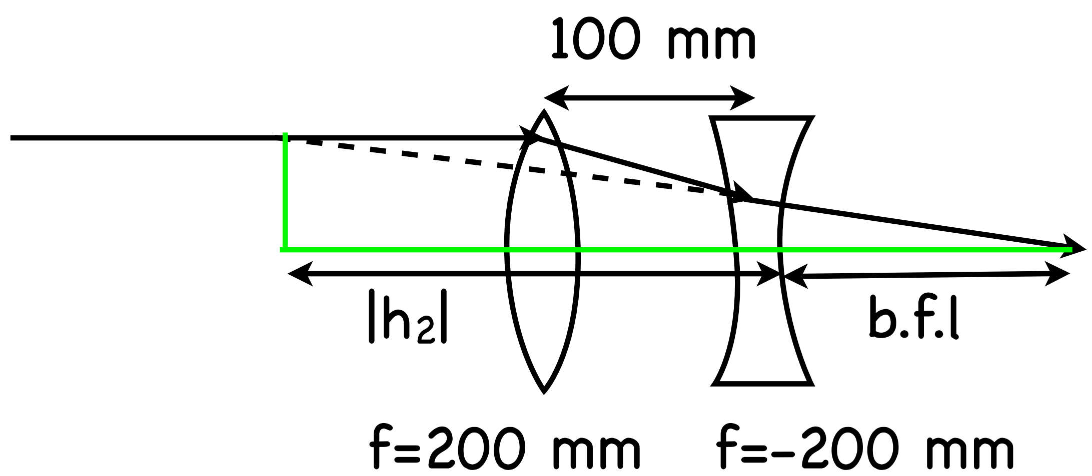


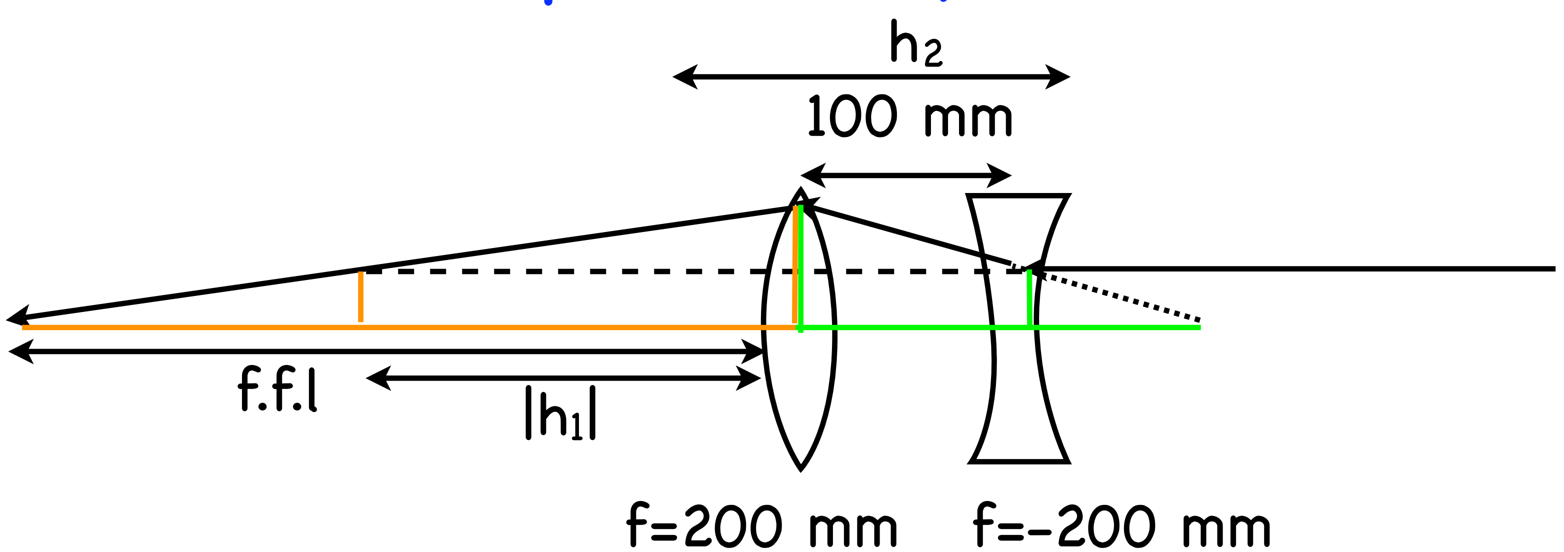
b.f.l=200 mm

h_2=-200 mm

f_{eff}=400mm

Example

- Find the focal length and locations of the principle points for a thin lens system with two thin lenses of focal lengths 200mm and -200mm separated by 100mm

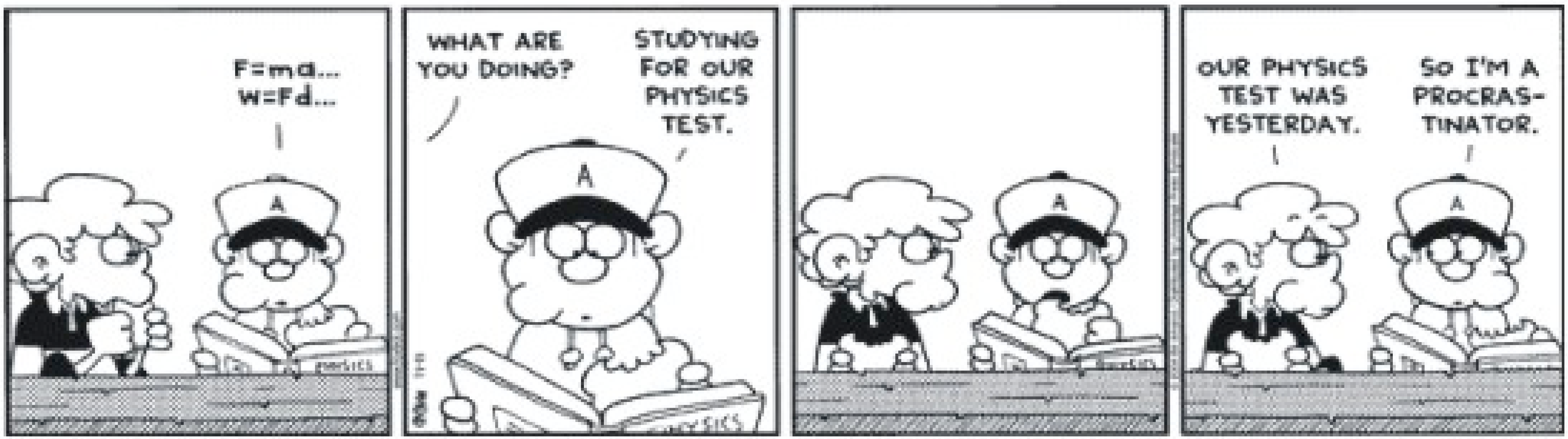


f.f.l=600 mm

h_1=-200 mm

f_{eff}=400 mm

Break


F=ma...
W=Fd...
WHAT ARE YOU DOING?
STUDYING FOR OUR PHYSICS TEST.
OUR PHYSICS TEST WAS YESTERDAY.
SO I'M A PROCRAS-TINATOR.

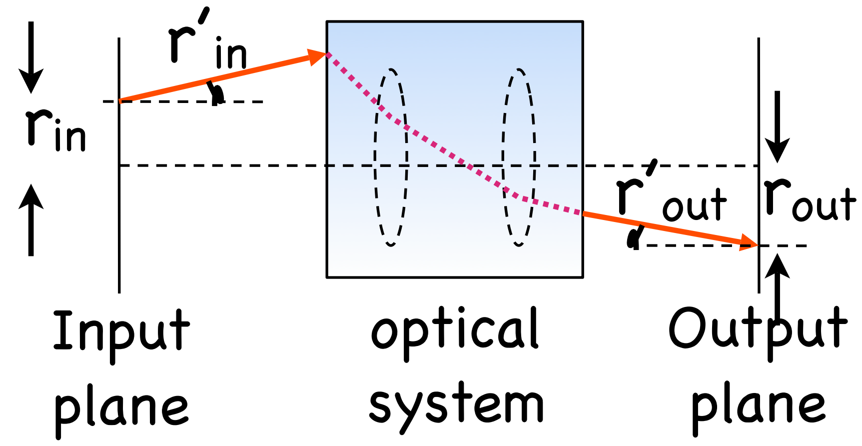
Compound Optical Systems

- Compound optical systems can be analyzed using ray tracing and the thin and thick lens equations
- A compound optical system can be described in terms of the parameters of a thick lens
- Is there an easier way?

Yes using paraxial ray matrices

ABCD matrices

- Consider the input and output rays of an optical system. They are lines and so they can be described by two quantities
  - Position (r)
  - Angle (r′)
- Any optical element must transform an input ray to an output ray, and therefor be describable as a 2x2 matrix



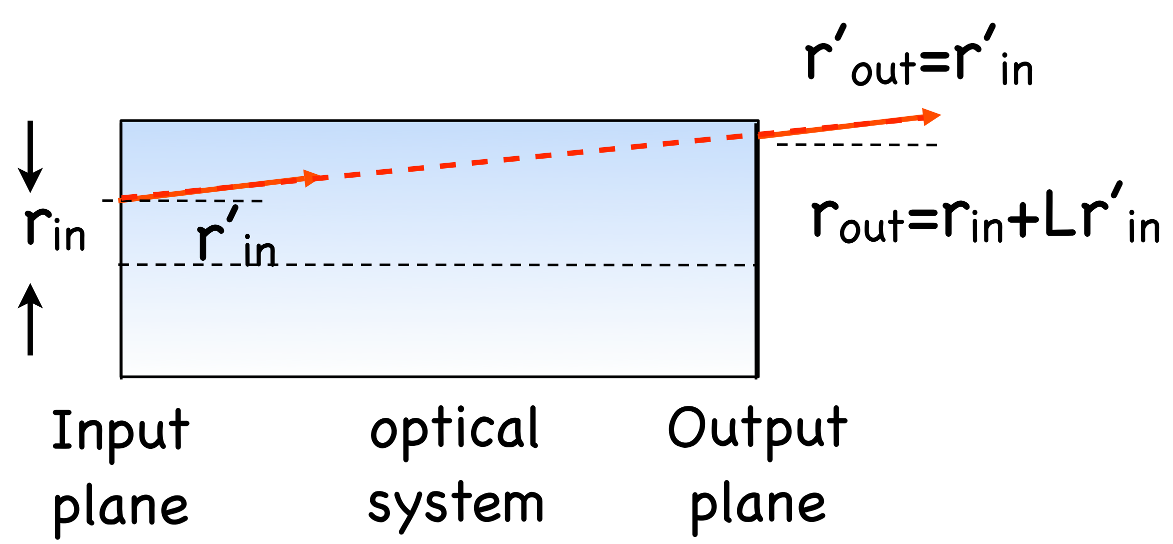
$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$\begin{bmatrix} r'_{out} \\ r_{out} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r'_{in} \\ r_{in} \end{bmatrix}$$

note this is the convention used by Hecht. Most other texts have [r r′]

Free Space Matrix

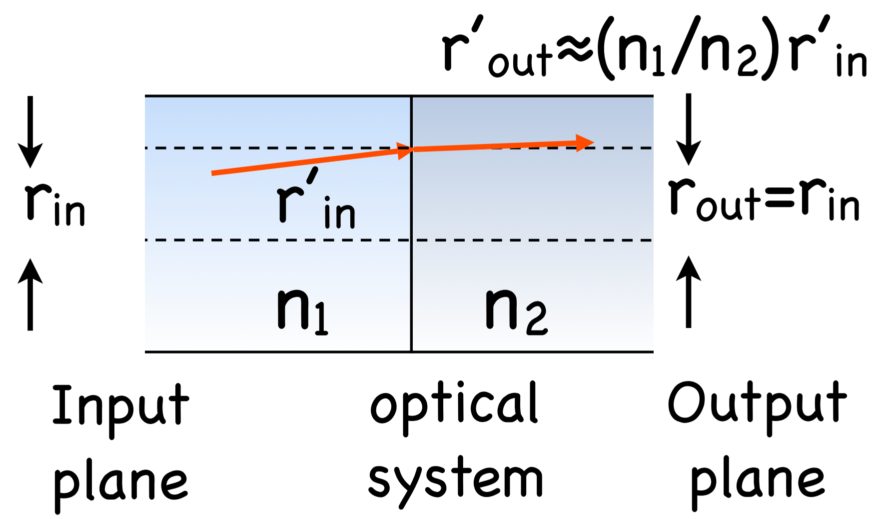
- Consider an optical system composed of only a length L. What is the ABCD matrix



$$M = \begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix}$$

Refraction Matrix

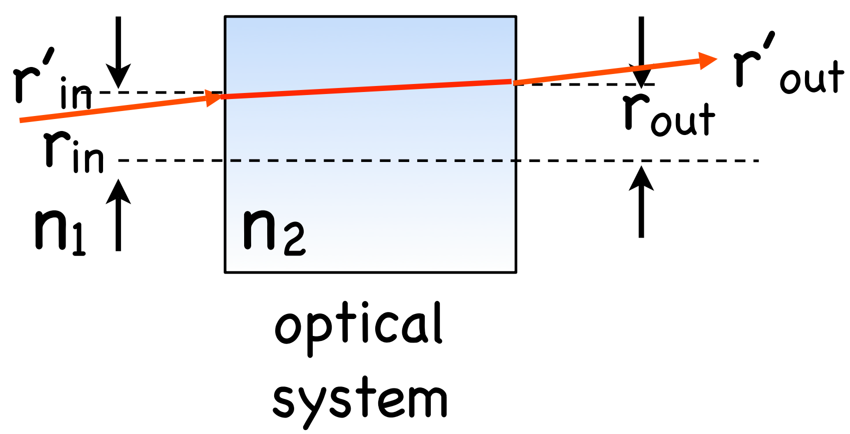
- Consider an optical system composed of only an interface from a material of index n_1 to one of index n_2 What is the ABCD matrix



$$M = \begin{bmatrix} n_1/n_2 & 0 \\ 0 & 1 \end{bmatrix}$$

Slab matrix

- What is the ABCD matrix for propagation through a slab of index n and thickness L?

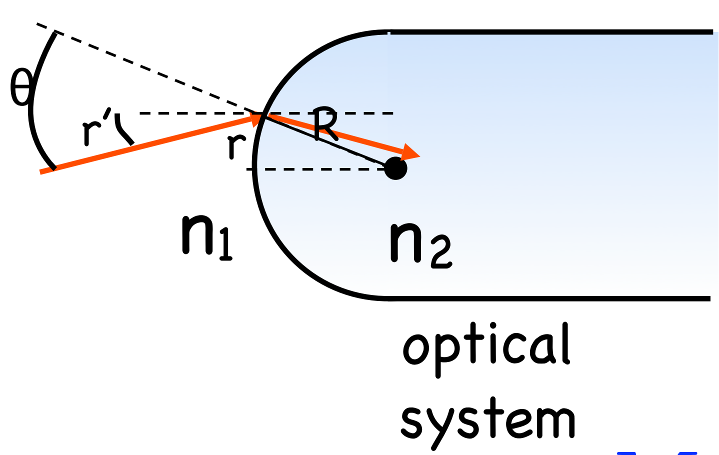


$$M = \begin{bmatrix} n_2/n_1 0 & \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix} \begin{bmatrix} n_1/n_2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 \\ \frac{n_1}{n_2} L & 1 \end{bmatrix}$$

Curved surface

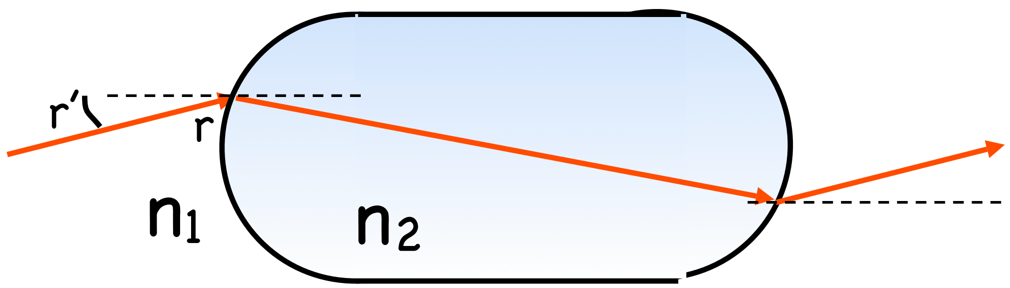
- Consider a ray displaced from the optical axis by an amount r. The normal to the surface at that point is at an angle r/R. The ray's angle with respect to the optical axis if r', so its angle with respect to the normal of the surface is θ=r'+r/R



$$M = \begin{bmatrix} n_1/n_2 & (n_1/n_2 - 1)/R \\ 0 & 1 \end{bmatrix}$$

Thick Lens

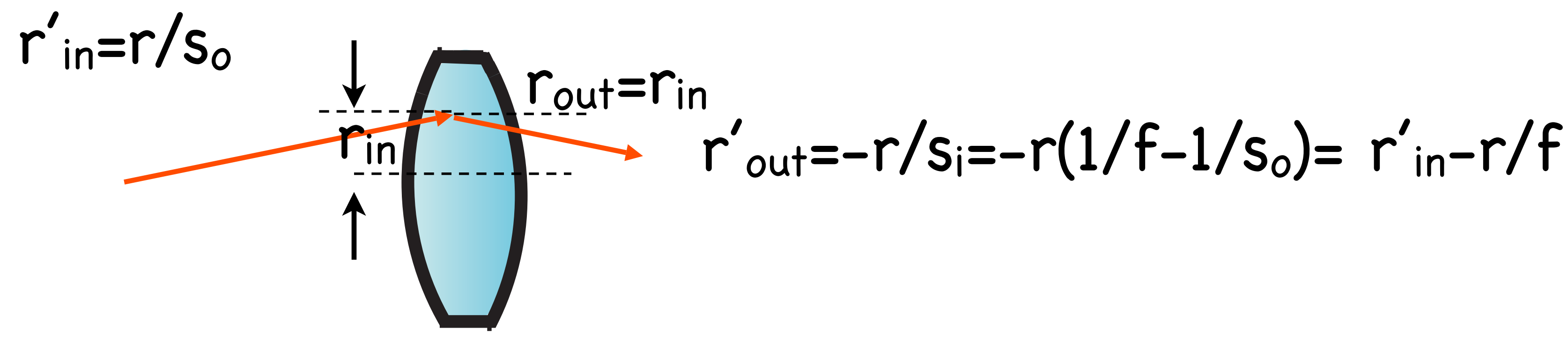
- A thick lens is two spherical surfaces separated by a slab of thickness d



$$M = \begin{bmatrix} n_2/n_1 & (n_2/n_1 - 1)/R_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix} \begin{bmatrix} n_1/n_2 & (n_1/n_2 - 1)/R_1 \\ 0 & 1 \end{bmatrix}$$

Thin Lens

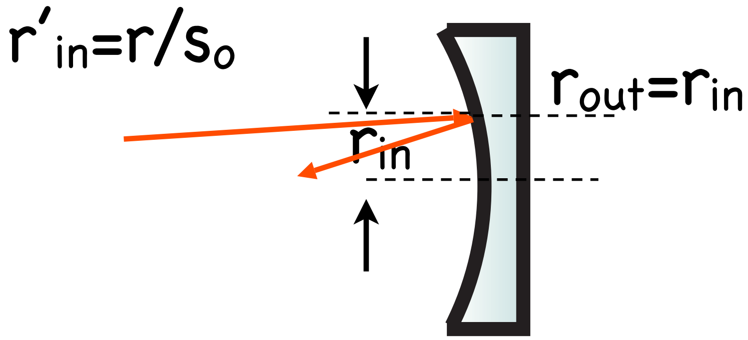
- For a thin lens we can use the matrix for a thick lens and set $d \rightarrow 0$
- Alternatively we can make geometrical arguments



$$M = \begin{bmatrix} 1 & -1/f \\ 0 & 1 \end{bmatrix}$$

Curved Mirrors

- For a thin lens we can use the matrix for a thin lens and set $f \to -R/2$
- Alternatively we can make geometrical arguments



$\theta_{in} = r'_{in} + r/R$

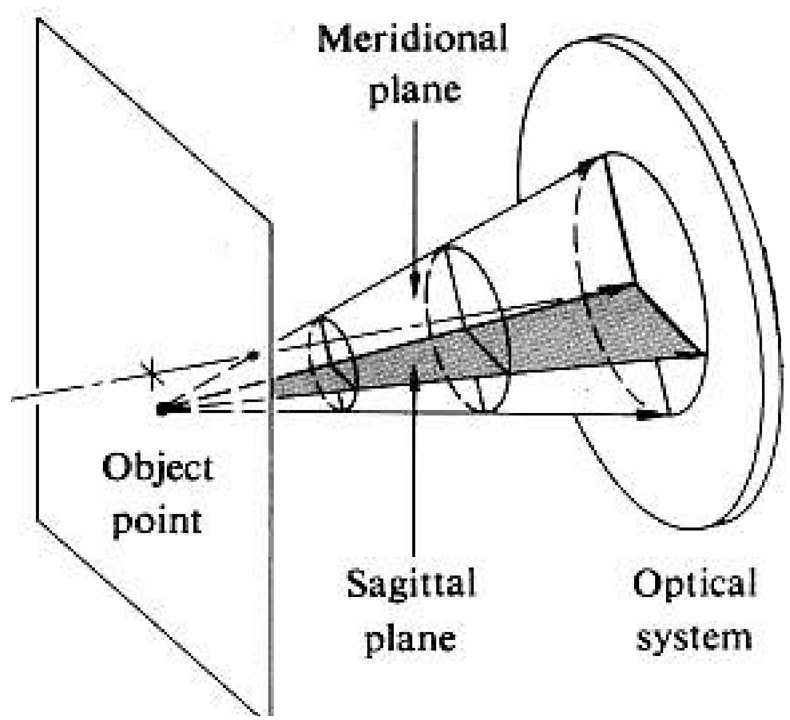
$\theta_{out} = r'_{out} + r/R = -\theta_{in}$

$$M = \begin{bmatrix} 1 & 2/R \\ 0 & 1 \end{bmatrix}$$

When you reflect off a mirror, the direction from which you measure r′ gets flipped.
Also note in this example R<0

Curved Mirror Correction

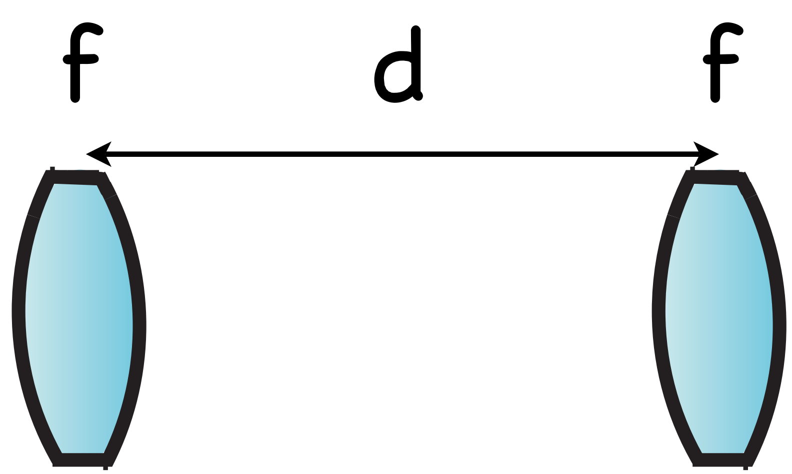
- For rays emanating from a point off the optical axis the curvature of the mirror that is seen by the rays is distorted
  - For tangential rays $R \rightarrow R\cos\theta$
  - For sagittal rays $R \rightarrow R/\cos\theta$



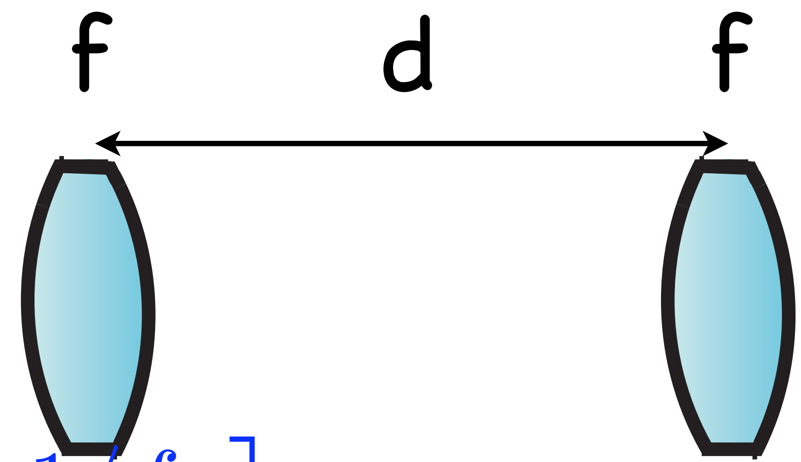
$$M = \begin{bmatrix} 1 & 2/R \\ 0 & 1 \end{bmatrix}$$

Example

- Find the back focal length of the following compound system



Example



Step 1, find the ABCD matrix for the system

$$M = \begin{bmatrix} 1 & -1/f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} 1 & -1/f \\ 0 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & -1/f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1/f \\ d & 1-d/f \end{bmatrix}$$

$$M = \begin{bmatrix} 1-d/f & -2/f+d/f^2 \\ d & 1-d/f \end{bmatrix}$$

$$\begin{bmatrix} -2r/f+dr/f^2 \\ r-rd/f \end{bmatrix} = \begin{bmatrix} 1-d/f & -2/f+d/f^2 \\ d & 1-d/f \end{bmatrix} \begin{bmatrix} 0 \\ r \end{bmatrix}$$

Example

Step 2. Require input rays parallel to the optical axis pass through the optical axis after the lens system and an additional path length equal to the back focal length

$$\begin{bmatrix} r'_{out} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b.f.l. & 1 \end{bmatrix} \begin{bmatrix} 1 - d/f & -2/f + d/f^2 \\ d & 1 - d/f \end{bmatrix} \begin{bmatrix} 0 \\ r_{in} \end{bmatrix}$$

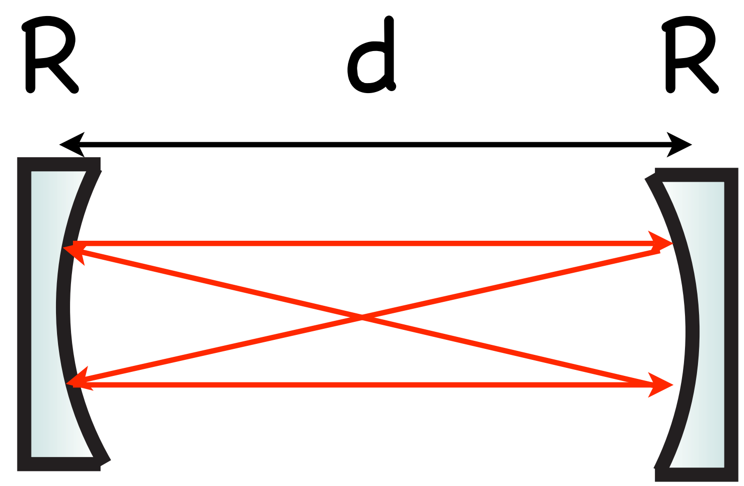
Step 3. solve for b.f.l.

$$0 = \left(b.f.l. \left(-2/f + d/f^2 \right) + 1 - d/f \right) r_{in}$$

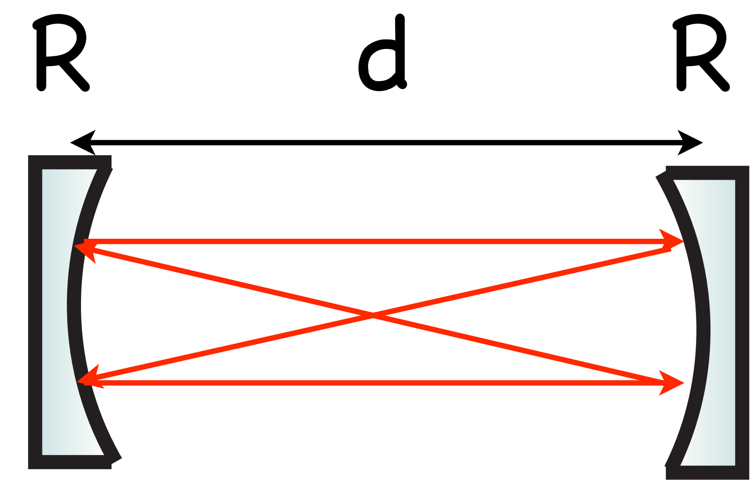
$$b.f.l. = \frac{1 - d/f}{-2/f + d/f^2} = \frac{f^2 - df}{d - 2f}$$

Example

After how many round trips will the beam be re-imaged onto itself? (this is called a "stable resonator")



Example



Step 1. Find the ABCD matrix for 1 round trip

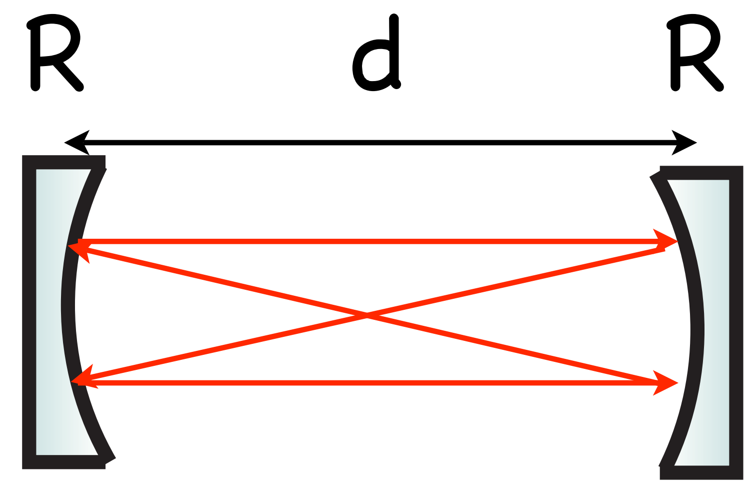
$$M_{rt} = \left(\begin{bmatrix} 1 & -2/R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \right)^2$$

$$M_{rt} = \begin{bmatrix} 1 - 2d/R & -2/R \\ d & 1 \end{bmatrix}^2$$

Step 2. Require that after N round trips the ray returns to its original state

$$M_{rt}^N = 1$$

Example



Relate the eigenvalues of M^N to M recalling that

$$\lambda\vec{v} = M\vec{v}$$

so

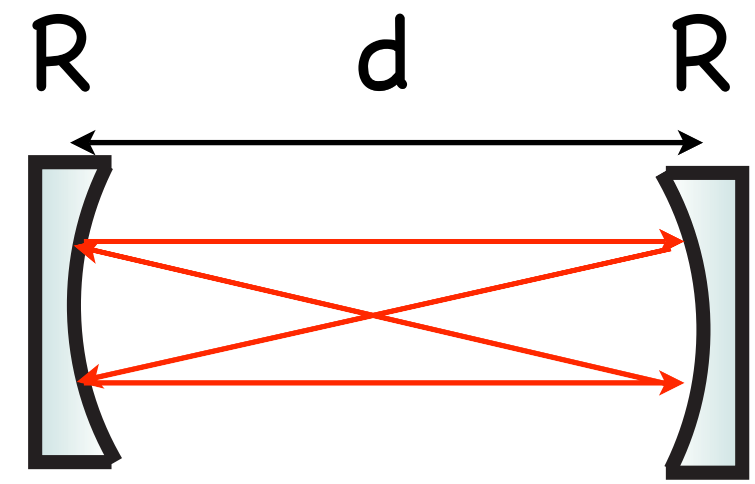
$$\lambda_{ow}^{2N} = \lambda_{rt}^{N} = \lambda_N = 1 \quad \text{from} \quad M_{rt}^{N} = 1$$

thus

$$\lambda_{ow} = e^{\pm i\theta} \qquad \text{where} \qquad 2N\theta = 2\pi m$$

$$M_{ow} - I\lambda_{ow} = \begin{bmatrix} 1 - 2d/R - \lambda_{ow} & -2/R \\ d & 1 - \lambda_{ow} \end{bmatrix} = 0$$

Example



Explicitly computing λ_{ow} gives

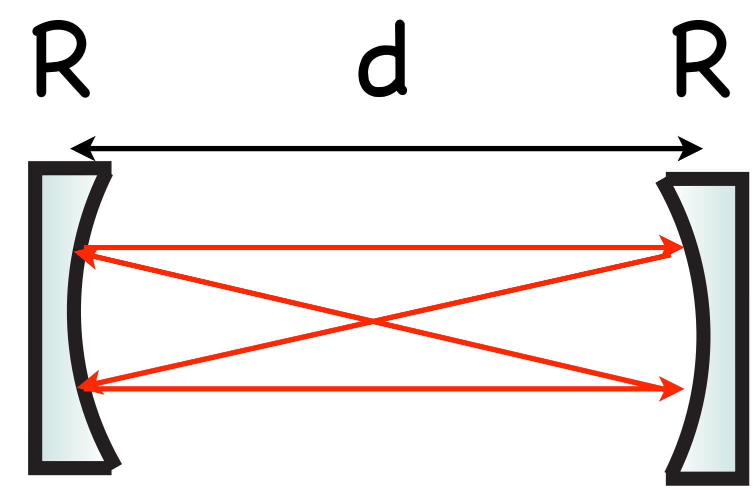
$$\det(M_{ow} - I\lambda_{ow}) = \begin{vmatrix} 1 - 2d/R - \lambda_{ow} & -2/R \\ d & 1 - \lambda_{ow} \end{vmatrix} = 0$$

$$(1 - 2d/R - \lambda_{ow})(1 - \lambda_{ow}) - 2d/R = 0$$

$$\lambda_{ow} = 1 - \frac{d}{R} \pm \sqrt{\left(1 - \frac{d}{R}\right) - 1} = 1 - \frac{d}{R} \pm i\sqrt{1 - \left(1 - \frac{d}{R}\right)^2}$$

$\lambda_{ow} = e^{\pm i\theta}$ with $\cos\theta = 1 - \dfrac{d}{R}$

Example



Relating the expressions for λ_{ow}

$$2N\theta = 2\pi m \quad \text{and} \quad \cos\theta = 1 - \frac{d}{R}$$

$$N = \frac{\pi m}{\cos^{-1}\left(1 - \frac{d}{R}\right)}$$

The so-called "g-factor" for a resonator is g=cosθ=1-d/R. Note the beam can only be re-imaged onto itself if d<2R. Another way of saying that is $0 \leq g^2 \leq 1$ the "stability criterion" for a resonator.

Summary

- Thick lenses require more parameters to describe than thin lenses
- If the parameters of a thick lens are properly described, its imaging behavior can be determined using the thin lens equation
- ABCD matrices are convenient ways to deal with the propagation of rays through an arbitrary optical system