Fraunhofer Diffraction

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Physics 158

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Class Outline

- Introduction
- Diffraction from a set of discrete sources
- Diffraction from a line source
- Diffraction of a slit
- Diffraction from a circular aperture
- Babinet's Principle

Introduction

from Latin "diffraction" means to break apart

Francesco Maria Grimaldi (1613 - 1663)

Discovered the diffraction of light from hard edges and gave it the name diffraction. He noticed that the shadow of small objects like pencil tips is wider than the computed geometrical shadow. Sometimes the shadow was encircled by rainbow-like colored bands.

He was one of the earliest physicists to suggest that light was wavelike in nature. He formulated a geometrical basis for a wave theory of light in his "Physico-mathesis de lumine" (1666).

Isaac Newton (1643-1727)

Repeated and improved on Grimaldi's experiments. Newton noticed that the bands of red lights were largest while the blue were least with the green of middle size. Newton used the term "inflection" but Grimaldi's word "diffraction" survived. Newton was in favor of the picture of light as a stream of ballistic particles.

Introduction

Thomas Young (1773-1829)

In 1800 he published his Experiments on Sound and Light in the Philosophical Transactions of the Royal Society

To explain the double refraction phenomenon observed in certain crystals he postulated that light must be a transverse wave.

He is best known for his double-slit experiment that was strongly in favor of the wave theory of light, allowing him to explain the phenomenon of interference of light. He also explained the color bands in soap films an the Newton rings between two glass plates.

He also showed that light exhibits a phase flip upon reflection from a denser medium.

Augustin Jean Fresnel (1788-1827)

He was able to calculate formulae which gave the position of the bright and dark lines based on where the vibrations were in phase and where they were out of phase. Using Fresnel's formulas Poisson predicted a bright spot in the center of the shadow behind a dark disk, a prediction that Arago found hard to believe before it was shown experimentally.

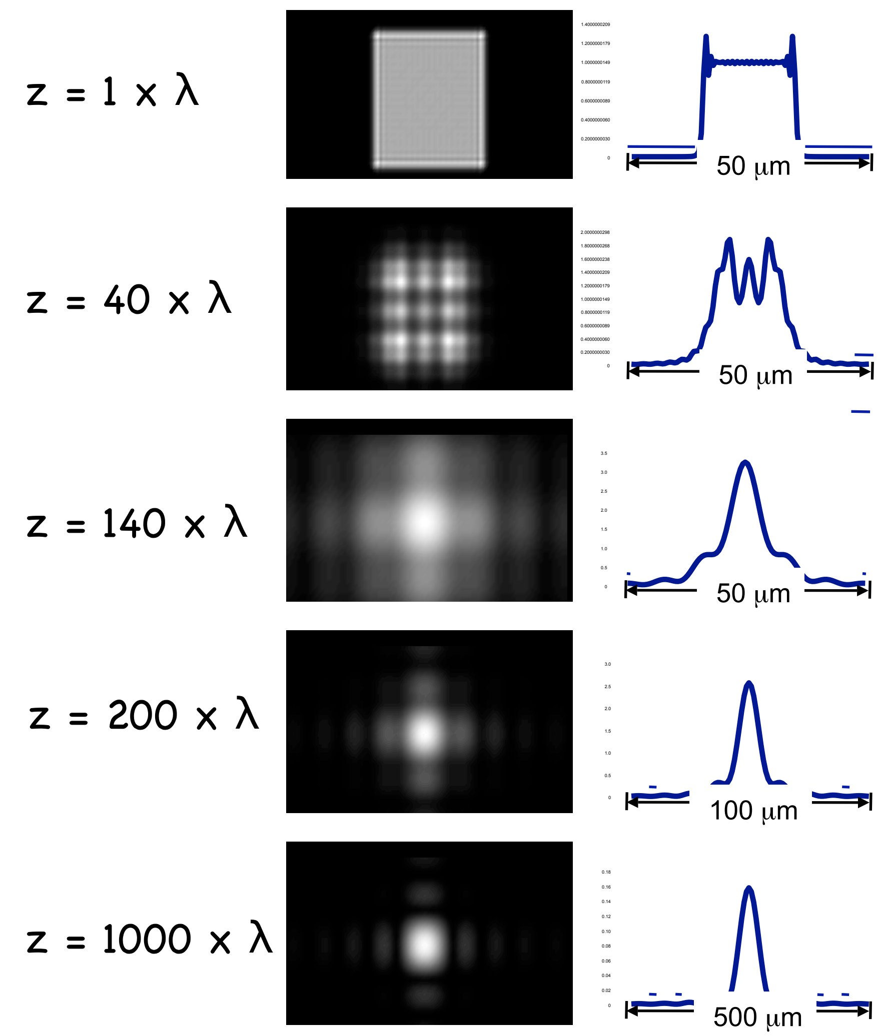
Joseph von Fraunhofer (1787 - 1826)

He re-discovered the dark lines in the spectra, that he labeled from A-Z (today known as the Franhofer lines).

He investigated diffraction in the far-field, In the limit where the distance of the source is at infinity: a special case of the Fresnel diffraction formula, equivalent to having a "low Fresnel number" (a2/λR)

Diffraction

The intensity pattern of light transmitting through an aperture (or the shadow of a mask) can not be explained by geometrical optics. The further one gets from the aperture, the more the intensity pattern deviates from that predicted by geometrical optics. In the far field limit this phenomena is called Fraunhofer Diffraction



The intensity profile from a uniformly illumiaetd square aperture of size x by x at various distances from the aperture.

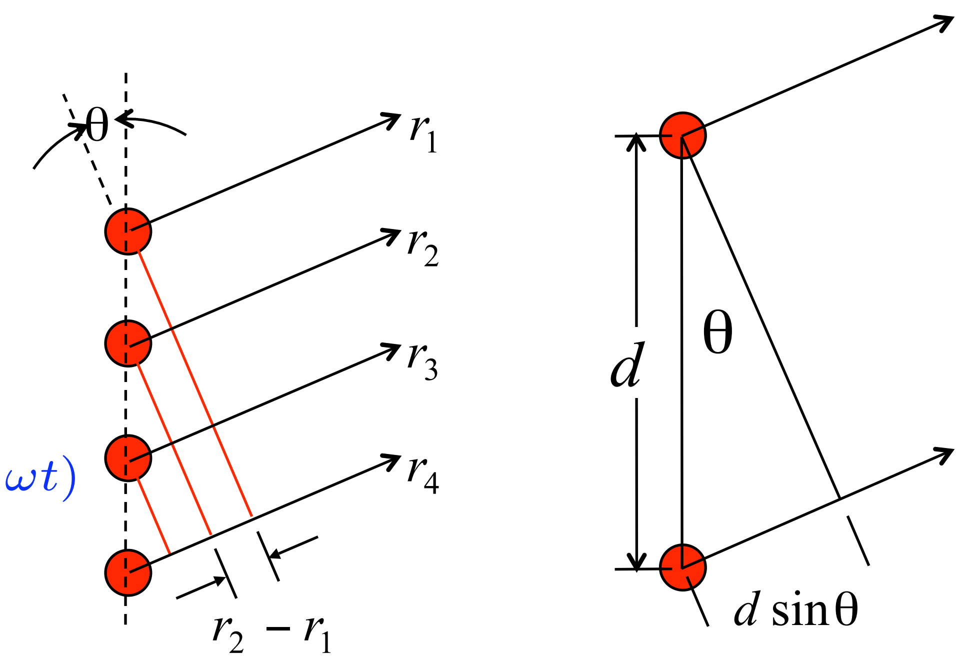
Diffraction effects from a discrete set of sources

Interference from coherent oscillators

- double slit
- gratings

The field at point p is

$$E_p = E_0 e^{i(kr_1 - \omega t)} + E_0 e^{i(kr_1 - \omega t)} + \ldots + E_0 e^{i(kr_n - \omega t)}$$
$$= E_0 e^{i(kr_1 - \omega t)} \left(1 + e^{ik(r_2 - r_1)} + \ldots + e^{ik(r_n - r_1)}\right)$$



If the sources are equally spaced along a line such that the phase difference from source n to point p (relative to source 1) is nδ=nkdsinθ

$$E_p = E_0 e^{i(kr_1 - \omega t)} \left(1 + e^{i\delta} + \ldots + e^{i(n-1)\delta}\right)$$

which is a geometric series

Assumptions:

- the size of the oscillators is small compared to the wavelength
- they act like point sources of equal amplitude
- The observer at point P is far from the array of sources such that all the rays from the point sources appear almost parallel.

Diffraction effects from a discrete set of sources

Geometric series

$$S = 1 + a + a^2 + \ldots + a^{n-1}$$

multiply S by a and subtract S to get

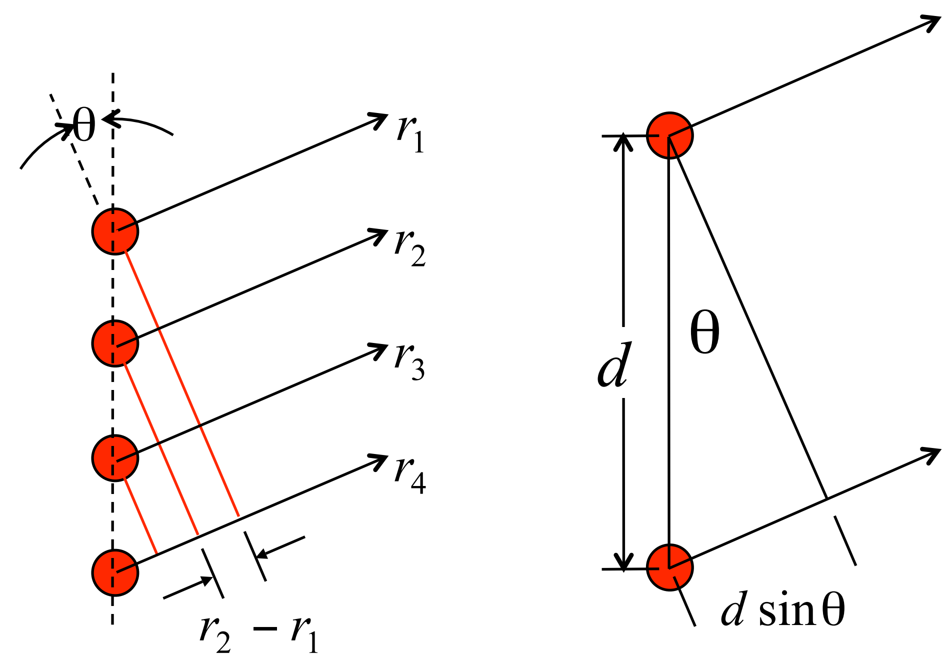
$$aS - S = a^n - 1$$

Allowing us to solve for S

$$S = \frac{a^n - 1}{a - 1}$$

so

$$E_p = E_0 e^{i(kr_1 - \omega t)} \left(1 + e^{i\delta} + \ldots + e^{in\delta}\right)$$
$$= E_0 e^{i(kr_1 - \omega t)} \left(\frac{e^{in\delta} - 1}{e^{i\delta} - 1}\right)$$



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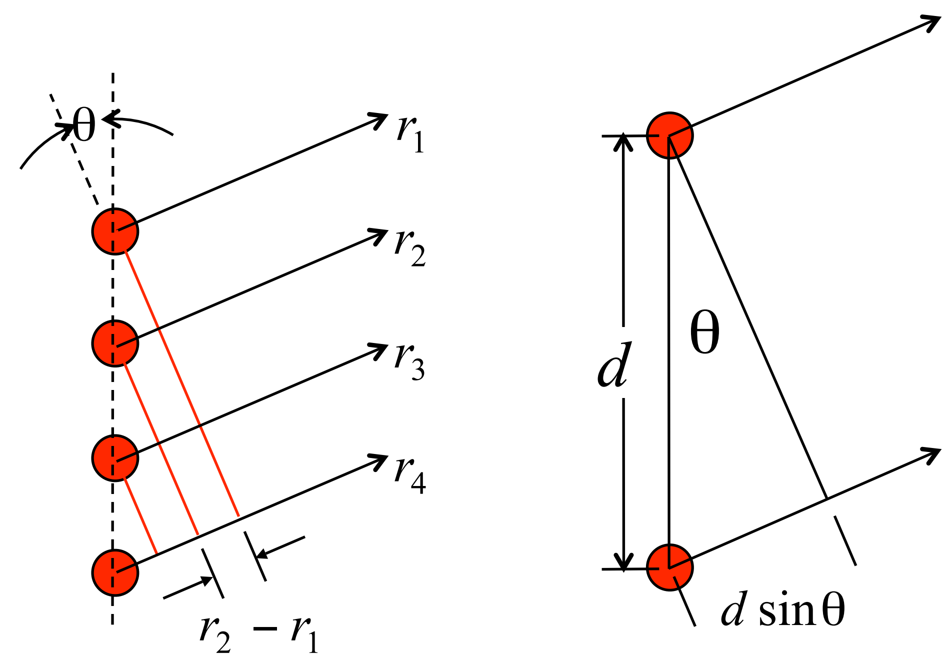
Diffraction effects from a discrete set of sources

$$E_p = E_0 e^{i(kr_1 - \omega t)} \left(\frac{e^{in\delta} - 1}{e^{i\delta} - 1} \right)$$

$$E_p = E_0 e^{i(kr_1 - \omega t)} \left(\frac{e^{in\delta/2} \left(e^{in\delta/2} - e^{-in\delta/2}\right)}{e^{i\delta/2} \left(e^{i\delta/2} - e^{-i\delta/2}\right)} \right)$$

$$E_p = E_0 e^{i(kr_1 - \omega t)} e^{i\delta(n-1)/2} \frac{\sin(n\delta/2)}{\sin(\delta/2)}$$



giving an intensity at point P of

$$I_p = I_0 \frac{\sin^2(n\delta/2)}{\sin^2(\delta/2)}$$

or

$$I_p = I_0 \frac{\sin^2((nkd/2)\sin\theta)}{\sin^2((kd/2)\sin\theta)}$$

rapid oscillation

slow envelope

Assumptions:

- the size of the oscillators is small compared to the wavelength
- they act like point sources of equal amplitude
- The observer at point P is far from the array of sources such that all the rays from the point sources appear almost parallel.

Gratings

$$I_p = I_0 \frac{\sin^2\left((nkd/2)\sin\theta\right)}{\sin^2\left((kd/2)\sin\theta\right)}$$

Maxima of interference pattern occur where denominator is zero

$$(kd/2)\sin\theta = m\pi$$

Using $k=2\pi/\lambda$ we get

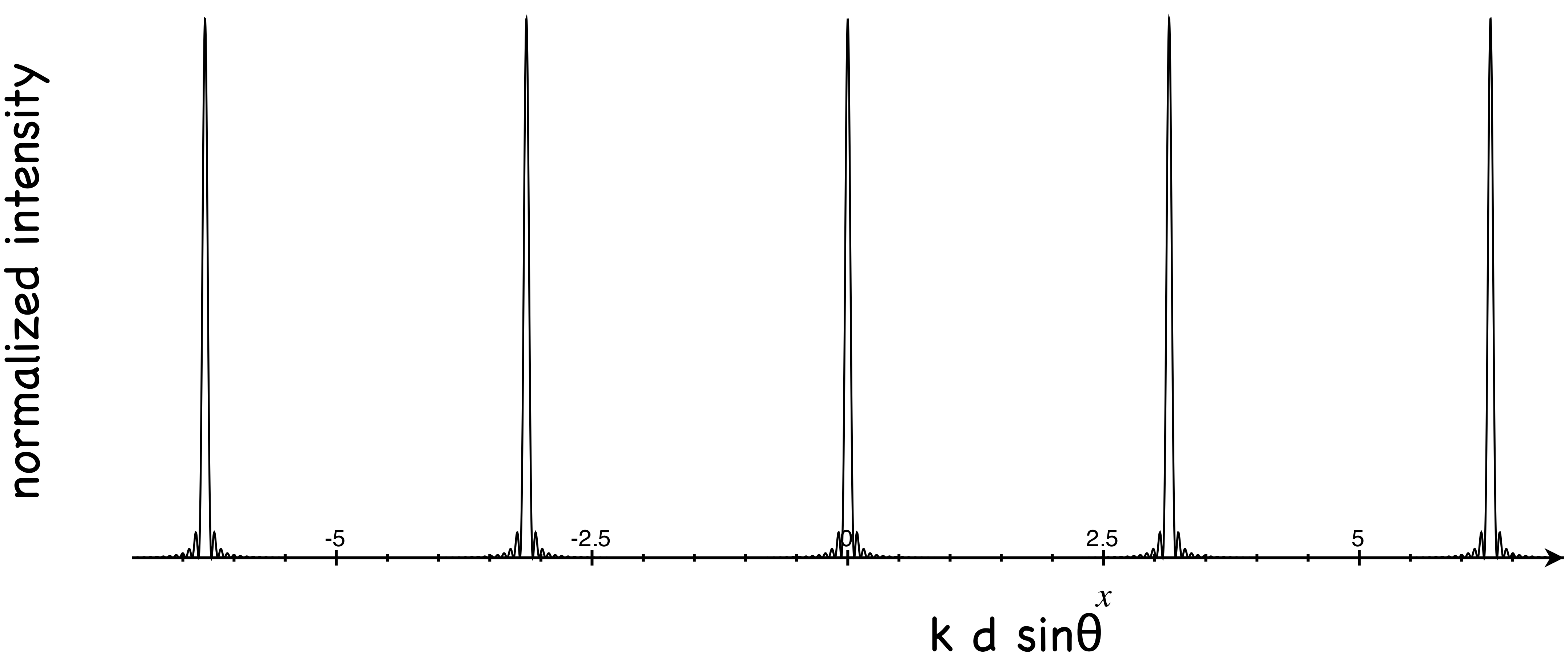
$$\sin\theta = m(\lambda/d)$$

If we generalize this to allow a phase shift between oscillators of $\epsilon = kd\sin\theta_i$ such that $\delta \rightarrow \delta + \epsilon$, the maxima occur at

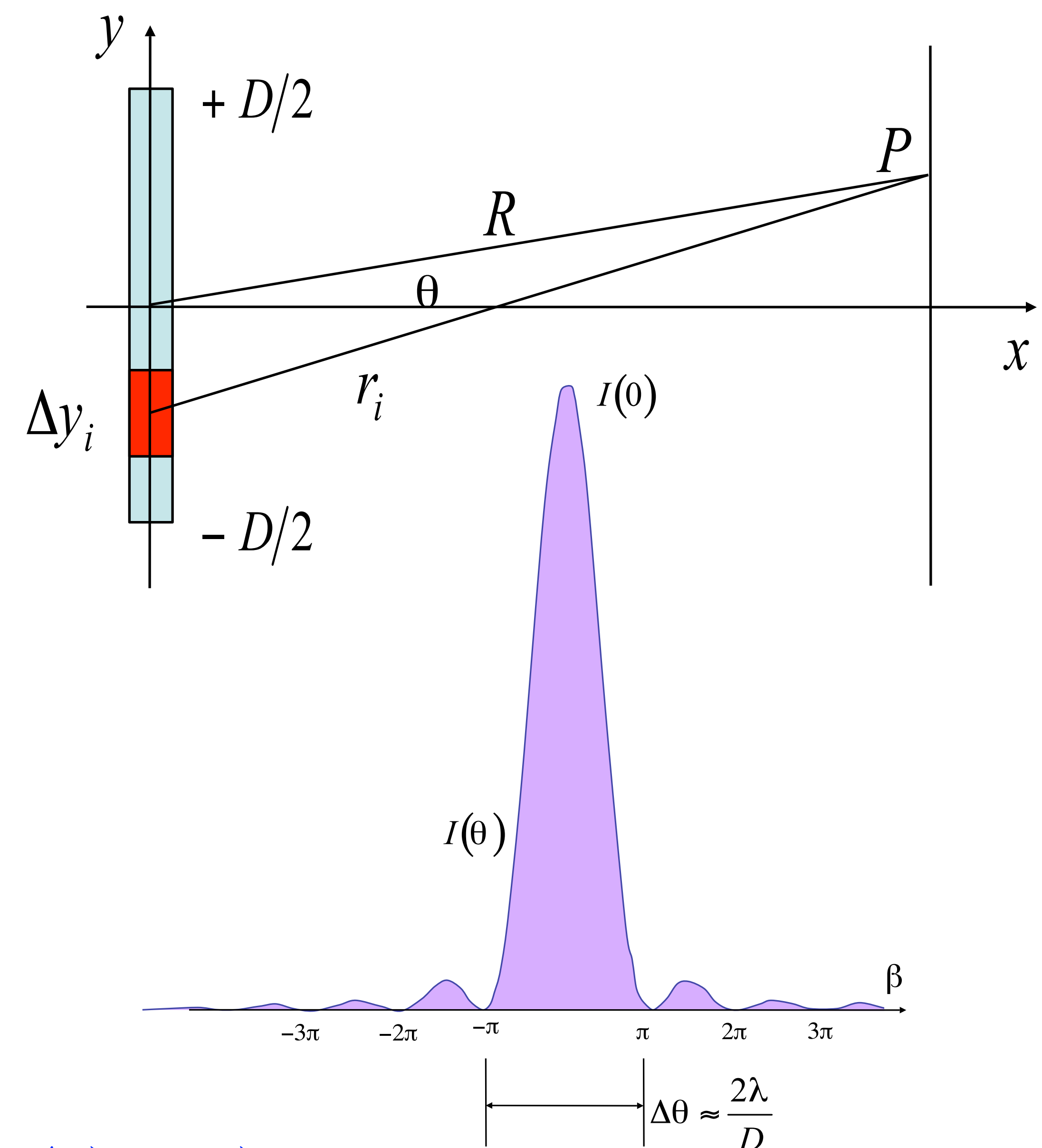
$$\sin\theta_i - \sin\theta = m(\lambda/d)$$

Which is the grating equation

Grating Diffraction Pattern



Diffraction from a Slit



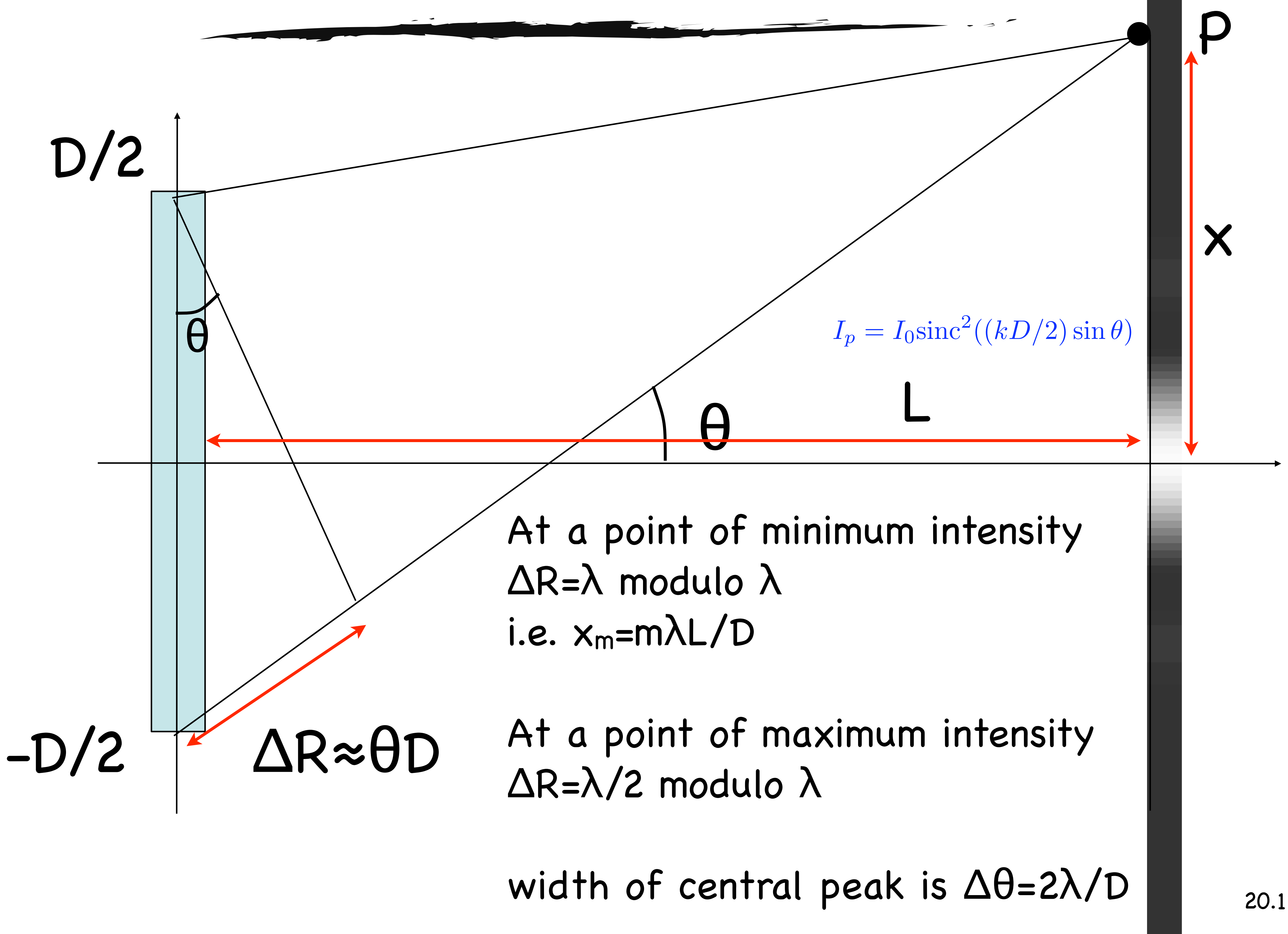
Consider the case where the point sources are really just secondary sources of Huygen's wavelets along an illuminated slit

Since there are infinitely many sources over the slit of width D we have n→∞ and d→D/n

then

$$I_p = I_0 \frac{\sin^2\left((nkd/2)\sin\theta\right)}{\sin^2\left((kd/2)\sin\theta\right)} \to I_0 \frac{\sin^2\left((kD/2)\sin\theta\right)}{\left((kd/2)\sin\theta\right)^2} = I_0 \text{sinc}^2\left((kD/2)\sin\theta\right)$$

Diffraction from a Slit



At a point of minimum intensity
$\Delta R = \lambda$ modulo λ
i.e. $x_m = m\lambda L/D$

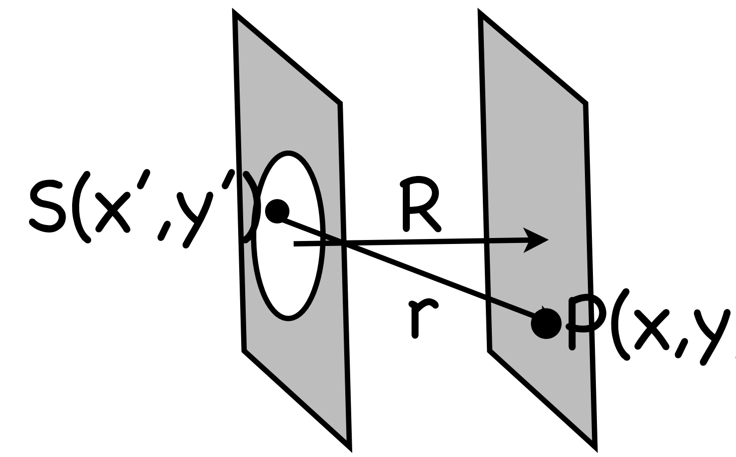
At a point of maximum intensity
$\Delta R = \lambda/2$ modulo λ

width of central peak is $\Delta\theta = 2\lambda/D$

Diffraction from a Circular aperture

For a circular aperture the far field intensity at a point P is

$$E = \int_{app} \frac{\mathcal{E}_0}{r} e^{i(\omega t - \vec{k}\cdot\vec{r})} dS$$



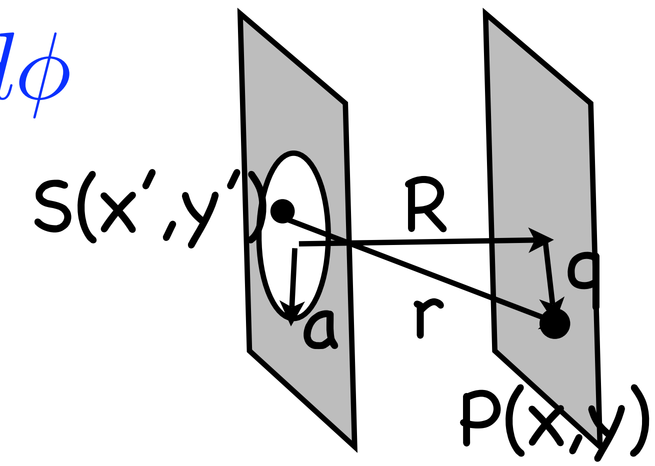
$$\begin{aligned}
\vec{k}\cdot\vec{r} &\approx k\sqrt{z^2 + (x-x')^2 + (y-y')^2} \\
&= k\sqrt{x^2 + y^2 + z^2 + 2xx' + 2yy' + x'^2 + y'^2} \\
&= k\sqrt{R^2(1 + (2xx' + 2yy')/R^2 + (x'^2 + y'^2)/R^2)} \\
&\approx kR + kxx'/R + kyy'/R
\end{aligned}$$

Where x′ and y′ are coordinates on the aperture surface S and x and y are coordinates in the far field

Diffraction from a Circular aperture

In spherical coordinates this reduces to

$$E_p = \frac{\mathcal{E}_0}{r} e^{i(\omega t - kR)} \int_0^a \int_0^{2\pi} e^{i(k\rho q/R)\cos\phi} \rho d\rho d\phi$$



which evaluates to

$$I_p(\theta) = I(0) \left[\frac{2J_1(ka\sin\theta)}{ka\sin\theta} \right]^2$$

where $J_1(r)$ is called the first order Bessel function and is defined by

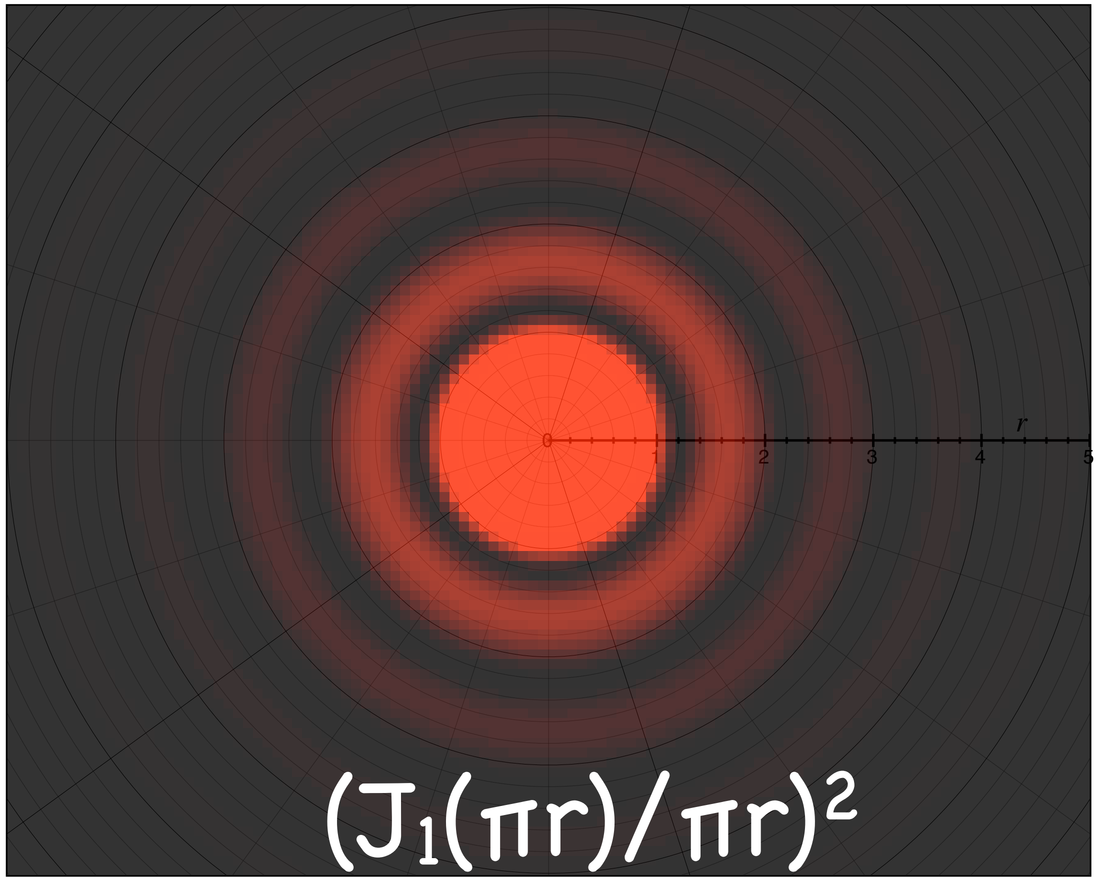
$$J_m(u) = \frac{i^{-m}}{2\pi} \int_0^{2\pi} e^{i(mv + u\cos v)} dv$$

$$\theta_1 = 1.22\frac{\lambda}{a}$$

Diffraction from a Circular aperture

The intensity distribution

$$I_p(\theta) = I(0)\left[\frac{2J_1(ka\sin\theta)}{ka\sin\theta}\right]^2$$



has a first minima at

$$\theta_1 = 1.22\frac{\lambda}{a}$$

The region inside of this is called the "Airy disk". The size of the airy disk is the minimum size a circular lens can focus light to. If this is larger than any other aberrations the lens or optical system is said to be "diffraction limited"

Example

A spy satellite has a 1m diameter circular mirror and is in low Earth orbit at an elevation of 145 km. What is its resolution of objects on the surface of the Earth?

A point source 145 km away would be imaged to an airy disk with an angular radius 1.22λ/a. Assuming λ=500nm, i.e. it is in the middle of the visible spectrum and with a=5m gives θ_1=1.22 μrad, thus two points that are to be clearly resolved by the satellite must be separated by more than this so the resolution

$$\delta\theta = \frac{\delta x}{145 \times 10^3 \text{ m}} \geq 1.22 \times 10^{-6}$$

so δx_{min}=18 cm

Example

Is your eye diffraction limited?

take a≈4mm, thus

$\Delta\theta_{min}$=1.22 500 nm/4mm=150 μrad

so at a distance of 10m you should be able to resolve objects that are

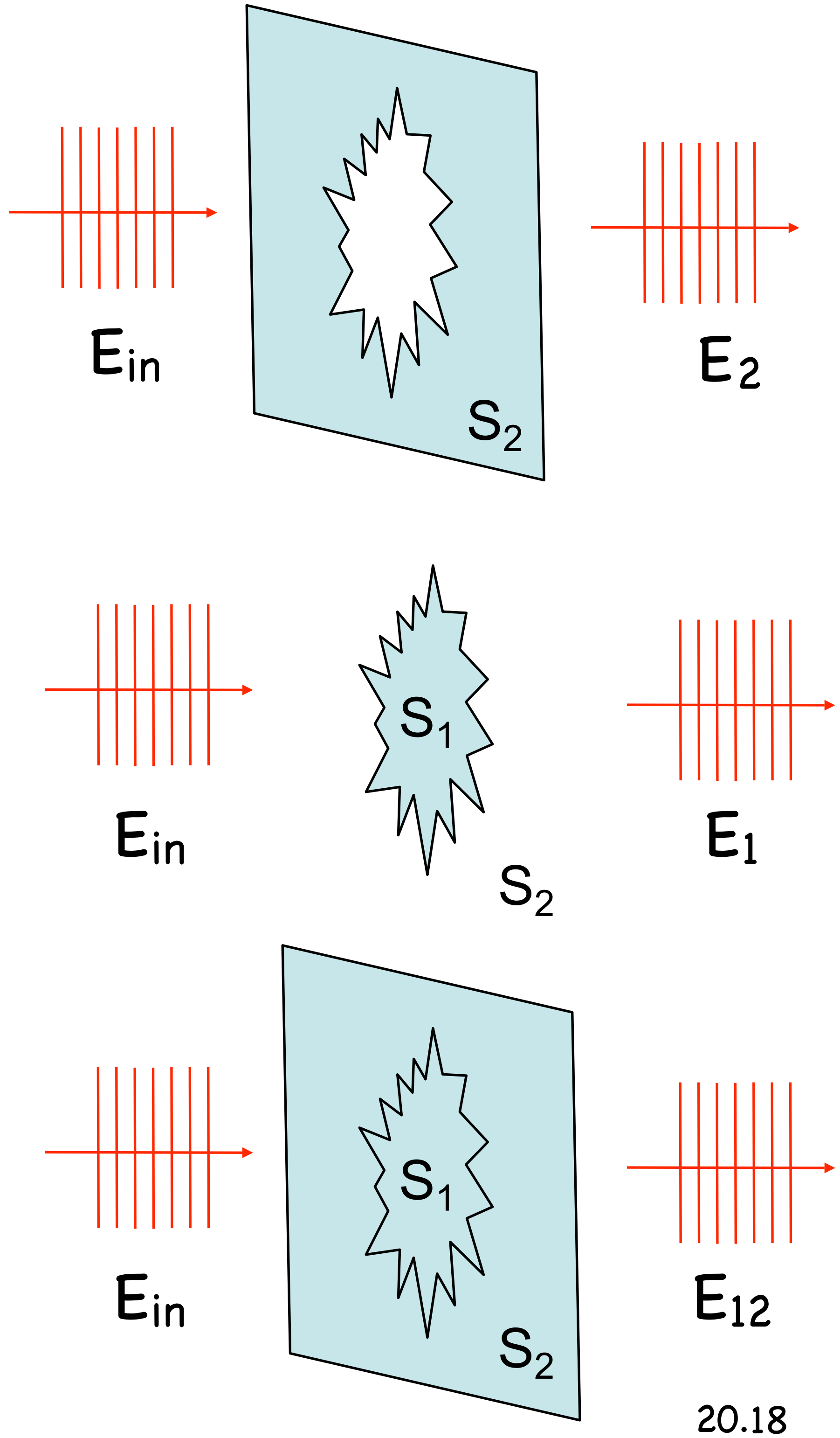
$$\delta\theta = \frac{\delta x}{10 \text{ m}} \geq 150 \times 10^{-6}$$

δx_{min}=1.5 mm apart. If you can't do this, then your eyes are not diffraction limited.

Babinet's Principle

The diffraction pattern of an aperture is the inverse of the diffraction pattern of the complementary aperture

$E_1 + E_2 = E_{12}$



Babinet's Principle

What does the shadow of a rectangular obstacle look like (in one dimension)

$$I_{12} = I_0$$

$$I_1 = I_0 \mathrm{sinc}^2((kD/2)\sin\theta)$$

$$I_2 = I_0 - I_1 = I_0(1 - \mathrm{sinc}^2((kD/2)\sin\theta))$$