Electro-optics

Chapter 7

Physics 208, Electro-optics

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Impermeability Tensor

It is convenient to consider the inverse of the relative permitivity, which we call the impermeability

$$\eta = \left(\frac{\epsilon}{\epsilon_0}\right)^{-1}$$

this allows the equation for the index ellipsoid in the principle coordinate system

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1$$

to be written as

$$\eta_{ii}(\vec{E})x_i^2 = 1$$

Electro-Optic Effect

The impermeability depends on the distribution of charges in the crystal, which can be modified by an externally applied electric field

$$\eta_{ij}(\vec{E}) = \eta_{ij}(0) + \left(\frac{\partial \eta_{ij}}{\partial E_k}\right)_{E=0} E_k + \frac{1}{2}\left(\frac{\partial^2 \eta_{ij}}{\partial E_k \partial E_l}\right)_{E=0} E_k E_l + \ldots$$

$$\Delta\eta_{ij} \equiv \eta_{ij}(\vec{E}) - \eta_{ij}(0) = r_{ijk}E_k + s_{ijkl}E_kE_l$$

with $r_{ijk} = \left(\frac{\partial \eta_{ij}}{\partial E_k}\right)_{E=0}$ and $s_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 \eta_{ij}}{\partial E_k \partial E_l}\right)_{E=0}$

Pockel's effect (r_{ijk}) Kerr effect (s_{ijkl})

resulting in an index ellipsoid that depends on the applied electric field $\eta_{ij}(\vec{E})x_ix_j = 1$

Electro-optic coefficients

The tensors r_{ijk} and s_{ijkl} are difficult to represent in matrix form, but using transpose symmetry of η ($\eta_{ij}=\eta_{ji}$) and freedom in choosing the order of partial differentiation ($\partial_i\partial_j=\partial_j\partial_i$) allows us to say

$$r_{ijk} = \left(\frac{\partial \eta_{ij}}{\partial E_k}\right)_{E=0} = \left(\frac{\partial \eta_{ji}}{\partial E_k}\right)_{E=0} = r_{jik}$$

$$s_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 \eta_{ij}}{\partial E_k \partial E_l}\right)_{E=0} = \frac{1}{2}\left(\frac{\partial^2 \eta_{ji}}{\partial E_k \partial E_l}\right)_{E=0} = s_{jikl}$$

$$s_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 \eta_{ij}}{\partial E_k \partial E_l}\right)_{E=0} = \frac{1}{2}\left(\frac{\partial^2 \eta_{ij}}{\partial E_l \partial E_k}\right)_{E=0} = s_{ijlk}$$

greatly reducing the number of independent tensor elements that we need to represent these tensors

Electro-optic coefficients

Contracting the subscript notation for the elements of a 3x3 tensor as

$$\begin{bmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{bmatrix} \rightarrow \begin{bmatrix} \eta_1 & \eta_6 & \eta_5 \\ \eta_6 & \eta_2 & \eta_4 \\ \eta_5 & \eta_4 & \eta_3 \end{bmatrix} \rightarrow \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \end{bmatrix}$$

allows us to write a 6x3 matrix for the r-coefficient and a 6x6 matrix for the s-coefficient.

These contracted matrices are only shorthand for keeping track of the independent coefficients of the r and s tensors, but do not obey normal mathematical matrix operations.

Contracted Notation

For the linear electro-optic effect the 27 coefficients have 18 independent values that can be expressed in the contracted notation as

$$
\begin{aligned}
r_{1k} &= r_{11k} \\
r_{2k} &= r_{22k} \\
r_{3k} &= r_{33k} \\
r_{4k} &= r_{23k} = r_{32k} \\
r_{5k} &= r_{13k} = r_{31k} \\
r_{6k} &= r_{12k} = r_{21k}
\end{aligned}
$$

for k=1,2,3

Similar relations exist for the 36 independent values of the 81 terms for the quadratic electro-optic effect

Modified Index Ellipsoid

$$\sum_k \left[\left(\frac{1}{n_x^2} + r_{1k}E_k \right) x^2 + \left(\frac{1}{n_y^2} + r_{2k}E_k \right) y^2 + \left(\frac{1}{n_z^2} + r_{3k}E_k \right) z^2 \right.$$
$$\left. + 2yzr_{4k}E_k + 2xzr_{5k}E_k + 2xyr_{6k}E_k \right] = 1$$

which we write as

$$\left(\frac{1}{n_x^2} + r_{1k}E_k \right) x^2 + \left(\frac{1}{n_y^2} + r_{2k}E_k \right) y^2 + \left(\frac{1}{n_z^2} + r_{3k}E_k \right) z^2$$
$$+ 2yzr_{4k}E_k + 2xzr_{5k}E_k + 2xyr_{6k}E_k = 1$$

where summation over k is assumed because it is a repeated index (note equation 7.2-3 in book has error).

The index ellipsoid is distorted by the presence of an electric field. A coordinate rotation is necessary to produce a new set of principle axis for the modified index ellipsoid.

Symmetry Considerations

Symmetry properties in various crystal classes dictate that certain terms in the electrooptic coefficient are zero.

- In a centro-symmetric crystal, inversion around the point of symmetry should reproduce the original crystal lattice, giving $r_{ijk}=r'_{ijk}$ where r_{ijk} is the electrooptic coefficient in the original crystal and r'_{ijk} is that in the inverted crystal.

- Because linear electrooptic coefficient is related to the linear displacement of charges in the crystal, an inversion of the crystal lattice should invert the electrooptic coefficient $r_{ijk}=-r'_{ijk}$

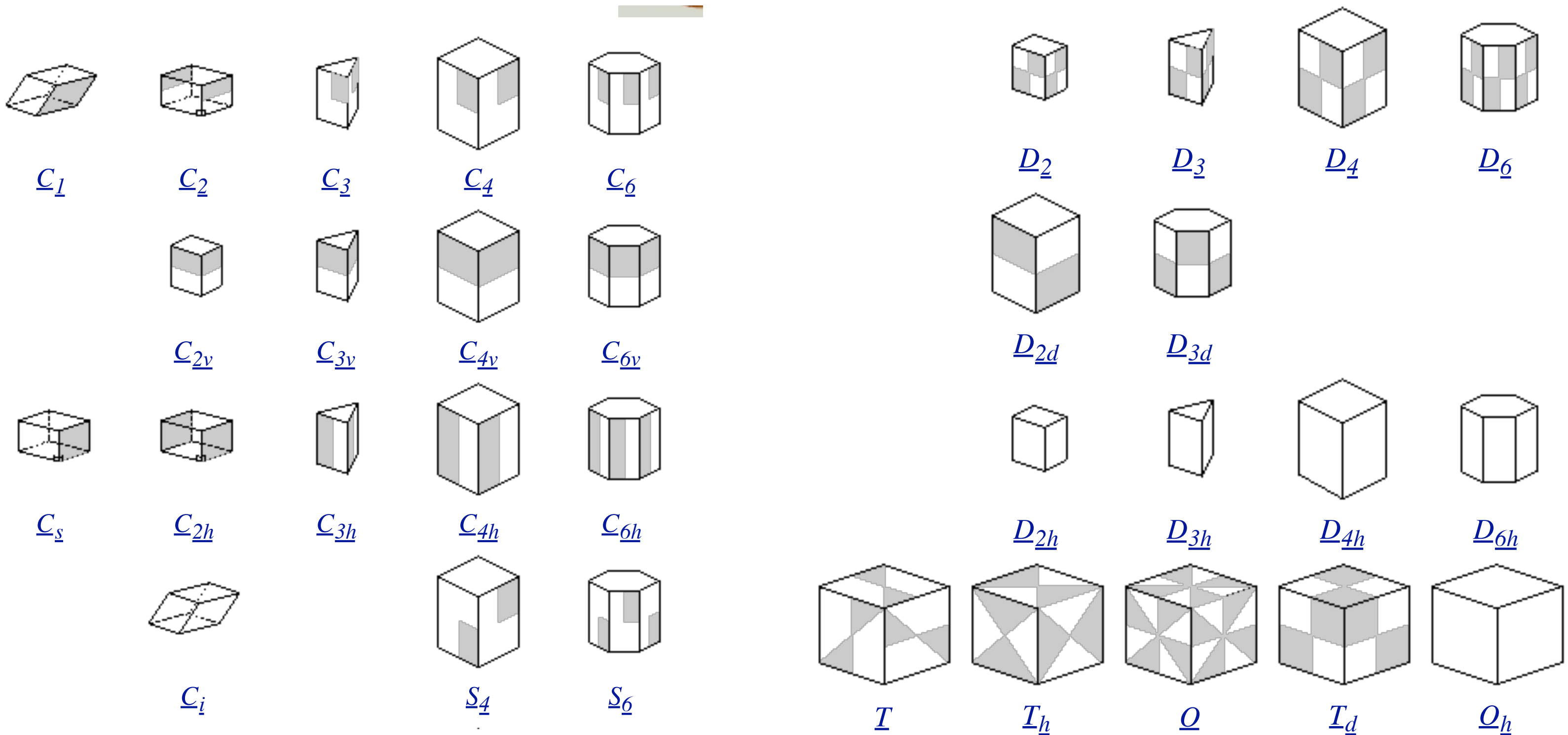
These two requirements dictate that $r_{ijk}=0$ in any centro-symmetric crystal

Crystal Point Groups

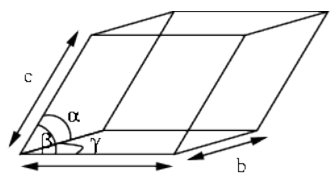
All crystals can be categorized into one of 32 "point groups" defined by seven symmetry properties including translational, rotational, inversion and mirror symmetries.

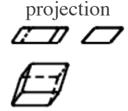
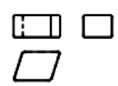
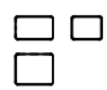
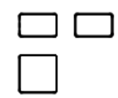
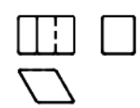
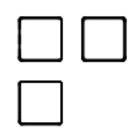
See for example http://www.phys.ncl.ac.uk/staff/njpg/symmetry/Solids.html for visualization of these 32 classes

Point Group Symmetries



Point Group Symmetries



Crystal system	Conditions	Point Groups	Order	Orthographic projection
Triclinic	$a \neq b \neq c$, $\alpha \neq \beta \neq \gamma$	C_1	1	
		C_i	2	
Monoclinic	$a \neq b \neq c$, $\alpha = \beta = \pi/2 \neq \gamma$	$C_s(C_{1h})$	2	
		C_2	2	
		C_{2h}	4	
Orthorhombic	$a \neq b \neq c$, $\alpha = \beta = \gamma = \pi/2$	C_{2v}	4	
		D_2	4	
		D_{2h}	8	
Tetragonal	$a = "b" \neq c$, $\alpha = \beta = \gamma = \pi/2$	C_4	4	
		S_4	4	
		C_{4h}	8	
		D_{2d}	8	
		C_{4v}	8	
		D_4	8	
		D_{4h}	16	
Trigonal	$a = b = "c"$, $\alpha = \beta < 2\pi/3$, $\gamma \neq \pi/2$	C_3	3	
		S_6	6	
		C_{3v}	6	
		D_3	6	
		D_{3d}	12	
Hexagonal	$a = "b" \neq c$, $\alpha = \beta = \pi/2$, $\gamma = 2\pi/3$	C_{3h}	6	
		C_6	6	
		C_{6h}	12	
		D_{3h}	12	
		C_{6v}	12	
		D_6	12	
		D_{6h}	24	
Cubic *	$a = b = "c"$, $\alpha = \beta = \gamma = \pi/2$	T	12	
		T_h	24	
		T_d	24	
		O	24	
		O_h	48	

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Form of Electro-Optic Tensor

Elements of electrooptic tensor r_{ik} which are zero by symmetry considerations are shown in table 7.2 of Yariv and Yeh

Table 7.2. Electro-optic Coefficients in Contracted Notation for All Crystal Symmetry Classes[a]

Centrosymmetric ($\bar{1}$, 2/m, mmm, 4/m, 4/mmm, $\bar{3}$, $\bar{3}$m 6/m, 6/mmm, m3, m3m):

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Triclinic:

1

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \\ r_{41} & r_{42} & r_{43} \\ r_{51} & r_{52} & r_{53} \\ r_{61} & r_{62} & r_{63} \end{pmatrix}$$

r_{61} r_{62} r_{63}

Monoclinic:

2 $(2 \parallel x_2)$

$$\begin{pmatrix} 0 & r_{12} & 0 \\ 0 & r_{22} & 0 \\ 0 & r_{32} & 0 \\ r_{41} & 0 & r_{43} \\ 0 & r_{52} & 0 \\ r_{61} & 0 & r_{63} \end{pmatrix}$$

2 $(2 \parallel x_3)$

$$\begin{pmatrix} 0 & 0 & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & r_{33} \\ r_{41} & r_{42} & 0 \\ r_{51} & r_{52} & 0 \\ 0 & 0 & r_{63} \end{pmatrix}$$

m $(m \perp x_2)$

$$\begin{pmatrix} r_{11} & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ r_{31} & 0 & r_{33} \\ 0 & r_{42} & 0 \\ r_{51} & 0 & r_{53} \\ 0 & r_{62} & 0 \end{pmatrix}$$

m $(m \perp x_3)$

$$\begin{pmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 0 \\ 0 & 0 & r_{43} \\ 0 & 0 & r_{53} \\ r_{61} & r_{62} & 0 \end{pmatrix}$$

Orthorhombic:

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$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{41} & 0 & 0 \\ 0 & r_{52} & 0 \\ 0 & 0 & r_{63} \end{pmatrix}$$

2mm

$$\begin{pmatrix} 0 & 0 & r_{13} \\ 0 & 0 & r_{23} \\ 0 & 0 & r_{33} \\ 0 & r_{42} & 0 \\ r_{51} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

See http://www.tulane.edu/~sanelson/eens211/32crystalclass.htm
For description of Hermann-Mauguin notation

KDP Example

Find the principle axes and E-field dependent indices of refraction for a KDP crystal with an electric field applied along the z-direction

KDP Example

- Potassium Dihidrogen Phosphate (KH_2PO_4)
- $\bar{4}2m$ crystal group (tetragonal) [table 7.3]
- Form of r_{ik} matrix is found from table 7.2

$$r_{ik} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{41} & 0 & 0 \\ 0 & r_{41} & 0 \\ 0 & 0 & r_{63} \end{pmatrix}$$

- Electrooptic and optical properties [table 7.3]
  - $r_{41}=8\times10^{-12}$ m/V, $r_{63}=11\times10^{-12}$ m/V @633 nm
  - $n_x=n_y=n_o=1.5115$, $n_z=n_e=1.4698$

KDP Example

equation for index ellipsoid is

$$\left(\frac{1}{n_x^2}+r_1kE_k\right)x^2+\left(\frac{1}{n_y^2}+r_2kE_k\right)y^2+\left(\frac{1}{n_z^2}+r_3kE_k\right)z^2$$
$$+2yzr_{4k}E_k+2xzr_{5k}E_k+2xyr_{6k}E_k=1$$

which becomes

$$\left(\frac{1}{n_x^2}+r_1kE_k\right)x^2+\left(\frac{1}{n_y^2}+r_2kE_k\right)y^2+\left(\frac{1}{n_z^2}+r_3kE_k\right)z^2$$
$$+2yzr_{41}E_x+2xzr_{52}E_y+2xyr_{63}E_z=1$$

or

$$\frac{x^2}{n_o^2}+\frac{y^2}{n_o^2}+\frac{z^2}{n_e^2}+2yzr_{41}E_x+2xzr_{41}E_y+2xyr_{63}E_z=1$$

for a field applied entirely along z:

$$\frac{x^2}{n_o^2}+\frac{y^2}{n_o^2}+\frac{z^2}{n_e^2}+2xyr_{63}E_z=1$$

KDP Example

$$\frac{x^2}{n_o^2} + \frac{y^2}{n_o^2} + \frac{z^2}{n_e^2} + 2xyr_{63}E_z = 1$$

We can choose a new set of principle coordinates x′, y′ and z′ that recast this equation into the form

$$\frac{x'^2}{n_{x'}^2} + \frac{y'^2}{n_{y'}^2} + \frac{z'^2}{n_{z'}^2} = 1$$

From inspection z=z′, so let x′y′ and xy differ by a rotation of θ around the z-axis:

$x = x'\cos\theta - y'\sin\theta$

$y = x'\sin\theta + y'\cos\theta$

$$\frac{(x'\cos\theta - y'\sin\theta)^2}{n_o^2} + \frac{(x'\sin\theta + y'\cos\theta)^2}{n_o^2} + \frac{z^2}{n_e^2} + 2(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta)r_{63}E_z = 1$$

KDP Example

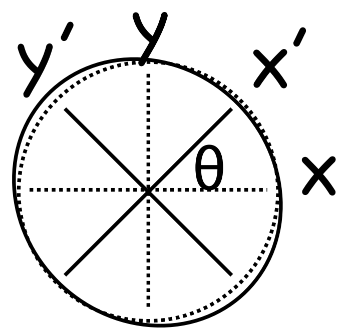
Evaluating gives

$$\frac{(x'\cos\theta - y'\sin\theta)^2}{n_o^2} + \frac{(x'\sin\theta + y'\cos\theta)^2}{n_o^2} + \frac{z^2}{n_e^2} + 2(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta)r_{63}E_z = 1$$

or

$$\frac{x'^2}{n_o^2} + \frac{y'^2}{n_o^2} + \frac{z'^2}{n_e^2} + \frac{z'^2}{n_e^2} - \frac{x'y'\sin 2\theta}{n_o^2} + \frac{x'y'\sin 2\theta}{n_o^2} + (x'^2\sin 2\theta + 2x'y'\cos 2\theta - y'^2\sin 2\theta)r_{63}$$

cross term 2x′y′cos(2θ)→0 for θ=45° giving



$$\frac{x'^2}{n_o^2} + \frac{y'^2}{n_o^2} + \frac{z'^2}{n_e^2} + (x'^2 - y'^2)r_{63}E_z = 1$$

or

$$\left(\frac{1}{n_o^2} + r_{63}E_z\right)x'^2 + \left(\frac{1}{n_o^2} - r_{63}E_z\right)y'^2 + \frac{z'^2}{n_e^2} = 1$$

KDP Example

Equating

$$\left(\frac{1}{n_o^2}+r_{63}E_z\right)x'^2+\left(\frac{1}{n_o^2}-r_{63}E_z\right)y'^2+\frac{z'^2}{n_e^2}=1$$

to

$$\frac{x'^2}{n_{x'}^2}+\frac{y'^2}{n_{y'}^2}+\frac{z'^2}{n_{z'}^2}=1$$

gives

$$\left(\frac{1}{n_o^2}+r_{63}E_z\right)=\frac{1}{n_x'^2}$$

$$n_x'^2=n_o^2\left(\frac{1}{1+n_o^2r_{63}E_z}\right)$$

$$n_x'^2\approx n_o^2(1-n_o^2r_{63}E_z)$$ for $E_z \ll 1/(n_o^2 r_{63}) = 4\times10^{10}$ V/m

so

$$n_x'\approx n_o\left(1-\frac{1}{2}n_o^2r_{63}E_z\right)\qquad n_y'\approx n_o\left(1+\frac{1}{2}n_o^2r_{63}E_z\right)\qquad n_z'=n_e$$

Quartz Example

Show that quartz remains uniaxial in the presence of an electric field applied along the z-axis

SiO_2

32 class trigonal crystal

$$r_{ij} = \begin{pmatrix} r_{11} & 0 & 0 \\ -r_{11} & 0 & 0 \\ 0 & 0 & 0 \\ r_{41} & 0 & 0 \\ 0 & -r_{41} & 0 \\ 0 & -r_{11} & 0 \end{pmatrix}$$

Quartz Example

For $E_x = E_y = 0$

$\Delta\eta = 0$

$$\Delta\eta_{ij} = \begin{pmatrix} r_{11} & 0 & 0 \\ -r_{11} & 0 & 0 \\ 0 & 0 & 0 \\ r_{41} & 0 & 0 \\ 0 & -r_{41} & 0 \\ 0 & -r_{11} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ E_z \end{pmatrix}$$

Thus the index ellipsod is not changed by the presence of an electric field in the z direction, so quartz , a positive uniaxial crystal, remains uniaxial

Modulation

Since an externally applied electric field can alter the index ellipsoid of a crystal, it alters the birefringence.

If the principle indices in the presence of an electric field are $n_x'(E)$, $n_y'(E)$ and $n_z'(E)$ then the birefringence (for a wave propagating in the z-direction) is $n_y'(E)-n_x'(E)$. If the material has a thickness d the relative phase retardation will be

$$\Gamma = \frac{\omega}{c}\left(n_y'(E) - n_x'(E)\right) d$$

Modulation

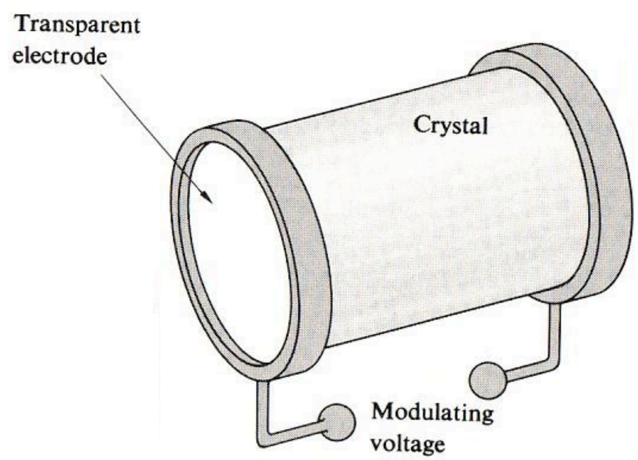
For KDP with an electric field applied along the z-axis (see example on page 14 of notes)

$$n_x' \approx n_o\left(1 - \frac{1}{2}n_o^2 r_{63} E_z\right) \qquad n_y' \approx n_o\left(1 + \frac{1}{2}n_o^2 r_{63} E_z\right) \qquad n_z' = n_e$$

thus $\Gamma = \frac{\omega}{c}\left(n_y'(E) - n_x'(E)\right) d$ becomes $\Gamma = \frac{\omega}{c} n_o^3 r_{63} E d = \frac{\omega}{c} n_o^3 r_{63} V$

where V is the voltage across d, the thickness of the crystal.

This geometry is realized in a Pockel's cell (electrooptic modulator)



Modulation

The voltage for which the relative phase retardation increases by π is V_π and is called the half-wave voltage

with $$\Gamma = \frac{\omega}{c} n_o^3 r_{63} V$$

the half-wave voltage is

$$V_\pi = \frac{\lambda}{2 n_o^3 r_{63}}$$

this is independent of the crystal dimensions.

Mathematical Representation of Modulation

- To model the effects of birefringence it is necessary to have a method to keep track of polarization states
  - Jones Calculus
- To keep track of time dependent modulation it is helpful to describe modulated wave in terms of single frequency components
  - Carrier and sidebands

Describing Polarization Mathematically

- Since polarization, like vectors, can be described by the value of two orthogonal components, we use vectors to represent polarization, with each component a phasor
- Amplitude of the components can be complex to represent a time delay between the components of the waves
- Jones calculus allows us to keep track of the polarization of waves as they propagate through a system

Jones Vectors

Defining the phasor representation for the amplitude of a plane wave propagating in the z-direction as

$$\widetilde{E}(z,t) = \widetilde{E}_0 e^{i(kz-\omega t)}$$

the polarization can be included by defining the amplitude of the x and y components:

$$\vec{\widetilde{E}}(z,t) = \begin{pmatrix} \widetilde{E}_x(z,t) \\ \widetilde{E}_y(z,t) \end{pmatrix} = \begin{pmatrix} \widetilde{E}_{0x} \\ \widetilde{E}_{0y} \end{pmatrix} e^{i(kz-\omega t)}$$

the polarization of the wave is described by $\begin{pmatrix} \widetilde{E}_{0x} \\ \widetilde{E}_{0y} \end{pmatrix}$ called the “Jones vector”

Jones Vectors

Expressing polarization in terms of two orthogonal states with complex amplitude (i.e. amplitude and phase) of each component expressed in vector form

vertical	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
horizontal	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
linear at +45°	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
linear at θ	$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$
right circular *	$\frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$
left circular	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$

* Right (left) circular polarization has a CW (CCW) rotation at a fixed point in space as seen by an observer looking towards the source of the wave.

Jones Matrices

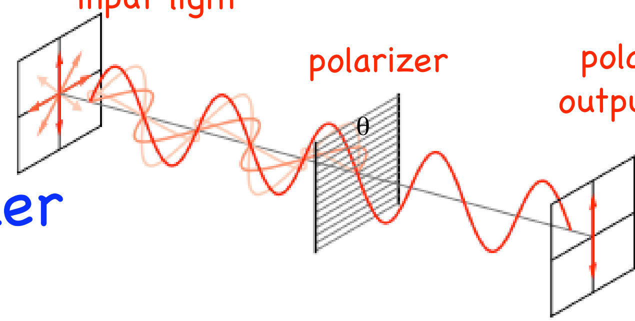
An optical element that transforms one polarization state into another can be treated as a 2x2 matrix acting on a jones vector

$$\begin{bmatrix} E_{x,out} \\ E_{y,out} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} E_{x,in} \\ E_{y,in} \end{bmatrix}$$

The Jones Matrices for a series of optical elements can be multiplied together to find how the optical system transforms the polarization of an input beam

Polarizers

Selectively attenuates one polarization

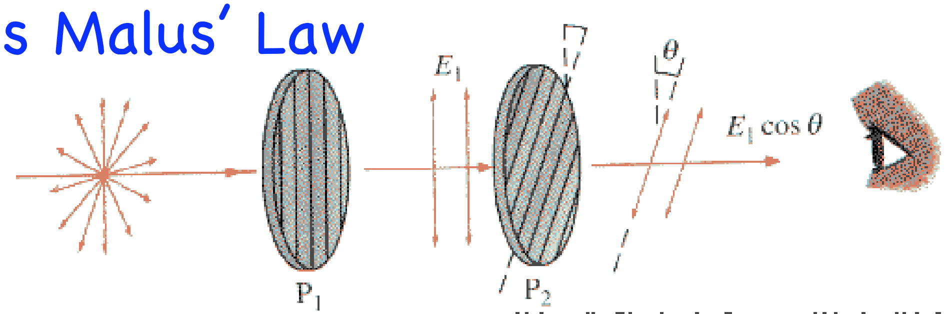


Wire Grid Polarizer

Malus discovered that an analyzer oriented at θ with respect to a polarizer would have a transmission of

$$I(\theta) = I_0 \cos^2 \theta$$

known as Malus' Law



University Physics by Sanny and Moebs, Vol. 2

Jones Matrix of a Polarizer

What is the Jones matrix for a polarizer that transmits horizontal polarization?

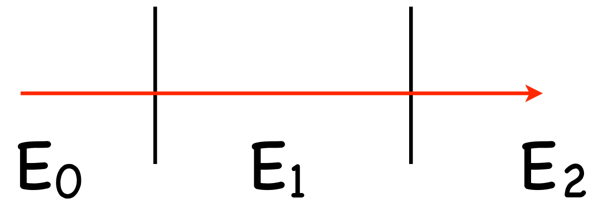
$$\begin{bmatrix} E_{x,out} \\ 0 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} E_{x,in} \\ E_{y,in} \end{bmatrix}$$

What if it is rotated at an angle θ?

$$R(\theta) M R(-\theta) \qquad R(\theta) \equiv \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Use this to verify Malus' Law

Malus' Law



$$\begin{bmatrix} E_{x,1} \\ E_{y,1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_{x,0} \\ E_{y,0} \end{bmatrix} = \begin{bmatrix} E_{x,0} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} E_{x,2} \\ E_{y,2} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} E_{x,1} \\ E_{y,1} \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} E_{x,2} \\ E_{y,2} \end{bmatrix} &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} E_{x,0} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta E_{x,0} \\ \sin\theta E_{x,0} \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta E_{x,0} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\theta E_{x,0} \\ -\sin\theta\cos\theta E_{x,0} \end{bmatrix} \end{aligned}$$

$$\frac{|E_2|}{E_{0,x}} = \cos\theta\sqrt{\cos^2\theta + \sin^2\theta} = \cos\theta \longrightarrow \frac{I_2}{I_1} = \cos^2\theta$$

Retarders

Devices which delays one polarization component with respect to the other

For a **birefringent** material of thickness d

$$\Delta\phi = (kn_y d - kn_x d) = \frac{2\pi}{\lambda}\Delta n d$$

Thus the Jones matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\Delta\phi} \end{bmatrix}$$

Quarter Wave Plate

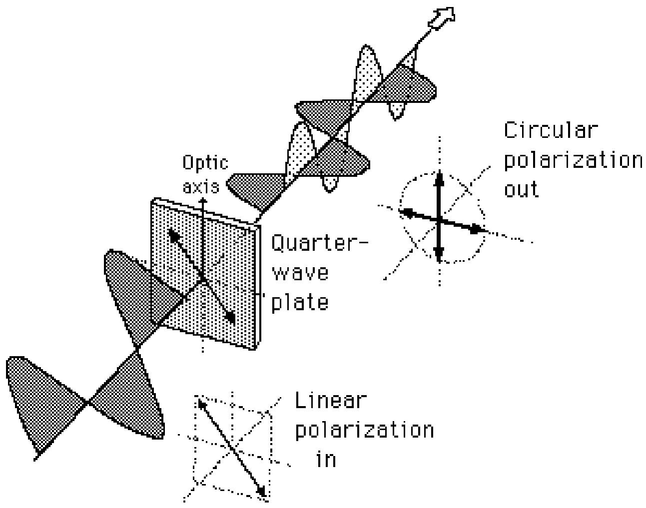
For a retarder with Δφ=π/2 (i.e. a retardation of λ/4)

$$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

And when a wave with linear polarization at ±45° passes through the retarder it gets converted to left (right) circular polarization

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}i}\begin{bmatrix} i \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



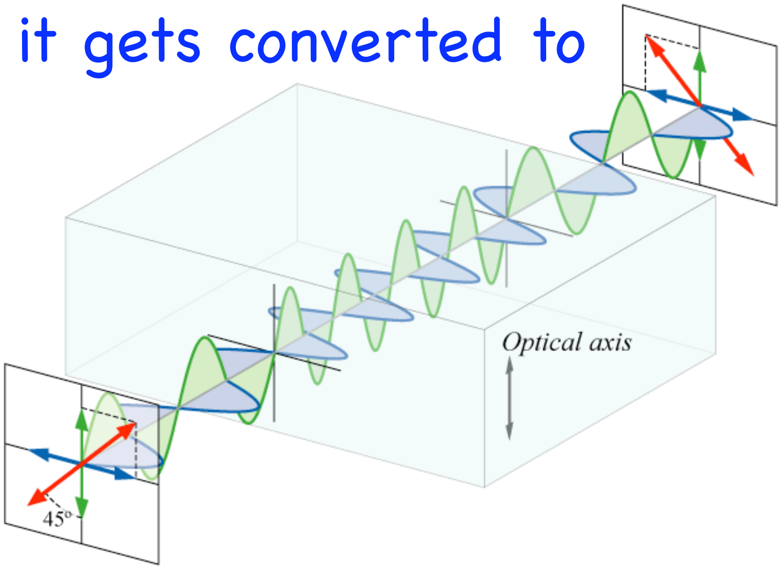
Half Wave Plate

For a retarder with $\Delta\varphi=\pi$ (i.e. a retardation of $\lambda/2$)

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

And when a wave with linear polarization at θ passes through the retarder it gets converted to linear polarization at $-\theta$

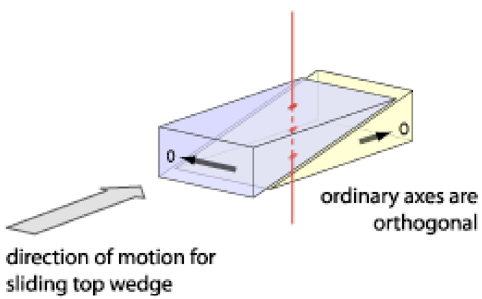
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} = \begin{bmatrix} \cos(-\theta) \\ \sin(-\theta) \end{bmatrix}$$



Compensator

Variable waveplate with Jones matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\Delta\phi} \end{bmatrix}$$



Babinet compensator

Berek compensator

Waveplate Order

Recall that the relative delay between the two polzrization states is

$$\Delta\phi = \frac{2\pi}{\lambda}\Delta n d$$

For a zero-order quartz waveplate if we wish $\Delta\varphi \approx \pi$ we need $d \approx 100 \mu m$

For a typical multi-order waveplate $\Delta\varphi \approx 11\pi$ so $d \approx 1mm$ which is easier to manufacture and has the same retardation (modulo 2π) as a zero order waveplate

Waveplate Order

dispersion of a waveplate is the sensitivity of the retardation to wavelength

$$\Delta\phi = \frac{2\pi}{\lambda}\Delta n d$$

$$\frac{d\Delta\phi}{d\lambda} = -\frac{2\pi}{\lambda^2}\Delta n d$$

which is minimized in a zero order waveplate.

Typical zero-order waveplates are actually made of two multiorder waveplates cemented together oriented at 90° so that $\Delta\varphi = \Delta\varphi_1 - \Delta\varphi_2 \approx \pi$ but the total thickness $d \approx 2$mm

Birefringence and Polarization

Consider light propagating in the z-direction of a z-cut KDP plate with an electric field applied along the z-direction. From the example on page 19 of these notes we have

$$n'_x \approx n_o\left(1 - \frac{1}{2}n_o^2 r_{63} E_z\right) \qquad n'_y \approx n_o\left(1 + \frac{1}{2}n_o^2 r_{63} E_z\right) \qquad n'_z = n_e$$

as the principle indices of refraction, thus the net birefringence is

$$\Gamma = \frac{\omega}{c}(n'_y - n'_x)d = \frac{2\pi}{\lambda} n_o^3 r_{63} V$$

and the output polarization is

$$\begin{bmatrix} e^{i\Gamma/2} & 0 \\ 0 & e^{-i\Gamma/2} \end{bmatrix} \begin{bmatrix} E_{0x'} \\ E_{0y'} \end{bmatrix}$$

Birefringence and Polarization

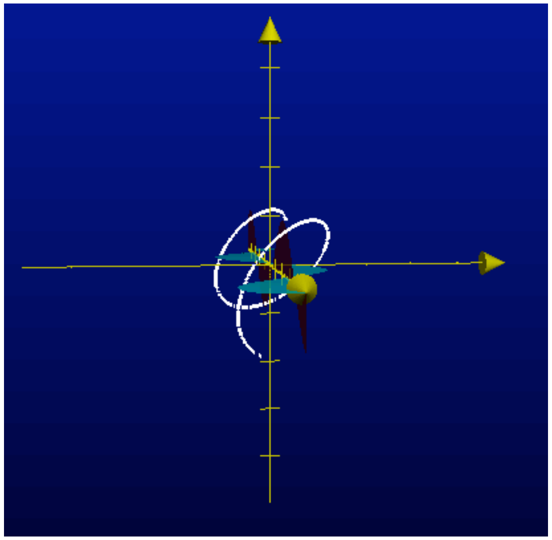
For linearly polarized light at the input with

$$\vec{E}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

the output state is

$$\vec{E}_{out} = \frac{1}{\sqrt{2}}\begin{bmatrix}e^{i\Gamma/2}\\e^{-i\Gamma/2}\end{bmatrix}$$

So external electric field affects the polarization state of the transmitted beam



Birefringence and Polarization

For linearly polarized light at the input with

$$\vec{E}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ or } \vec{E}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

the output state is

$$\vec{E}_{out} = \begin{bmatrix} e^{i\Gamma/2} \\ 0 \end{bmatrix} \text{ or } \vec{E}_{out} = \begin{bmatrix} 0 \\ e^{-i\Gamma/2} \end{bmatrix} \quad \text{respectively}$$

So external electric field affects only the phase of input light polarized along the principle axes

Birefringence and Polarization

For linearly polarized light at the input with

$$\vec{E}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

and the crystal followed by a polarizer at -45°, the output state is

$$\frac{1}{2\sqrt{2}}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}e^{i\Gamma/2} & 0\\0 & e^{-i\Gamma/2}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{i}{\sqrt{2}}\sin(\Gamma/2)\begin{bmatrix}1\\-1\end{bmatrix} = i\sin(\Gamma/2)\vec{E}_0$$

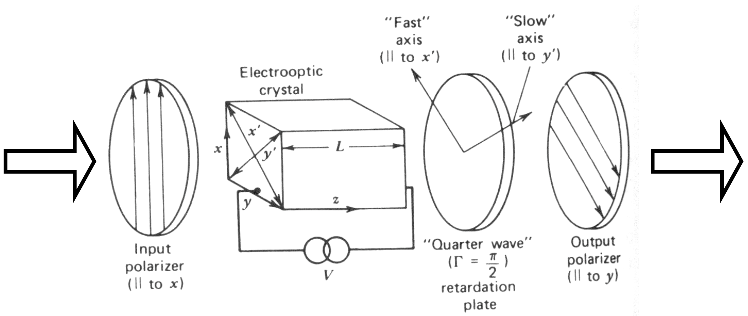
For a relative transmitted power of

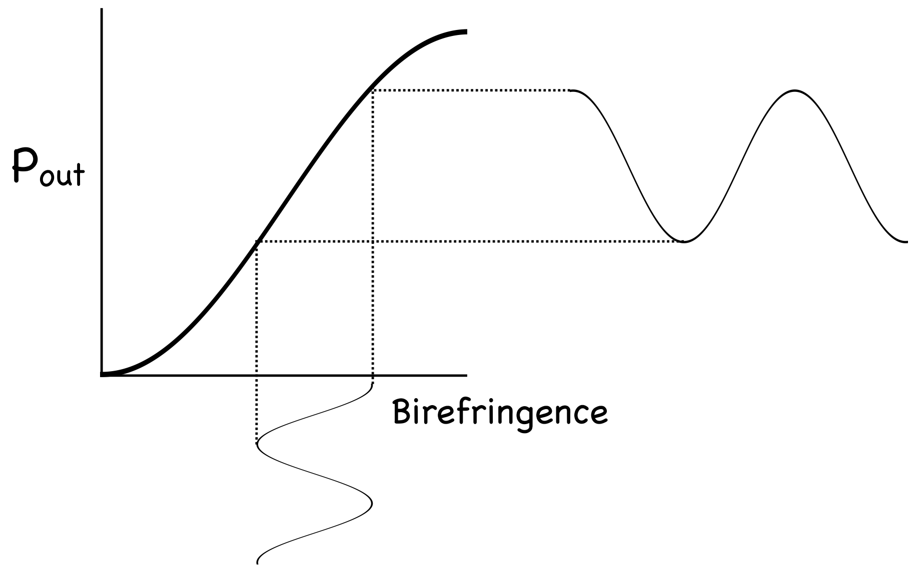
$$\frac{P_{out}}{P_o} = \sin^2(\Gamma/2)$$

An additional birefringence of π/2 can bias the output for a linear response

Birefringence and Polarization

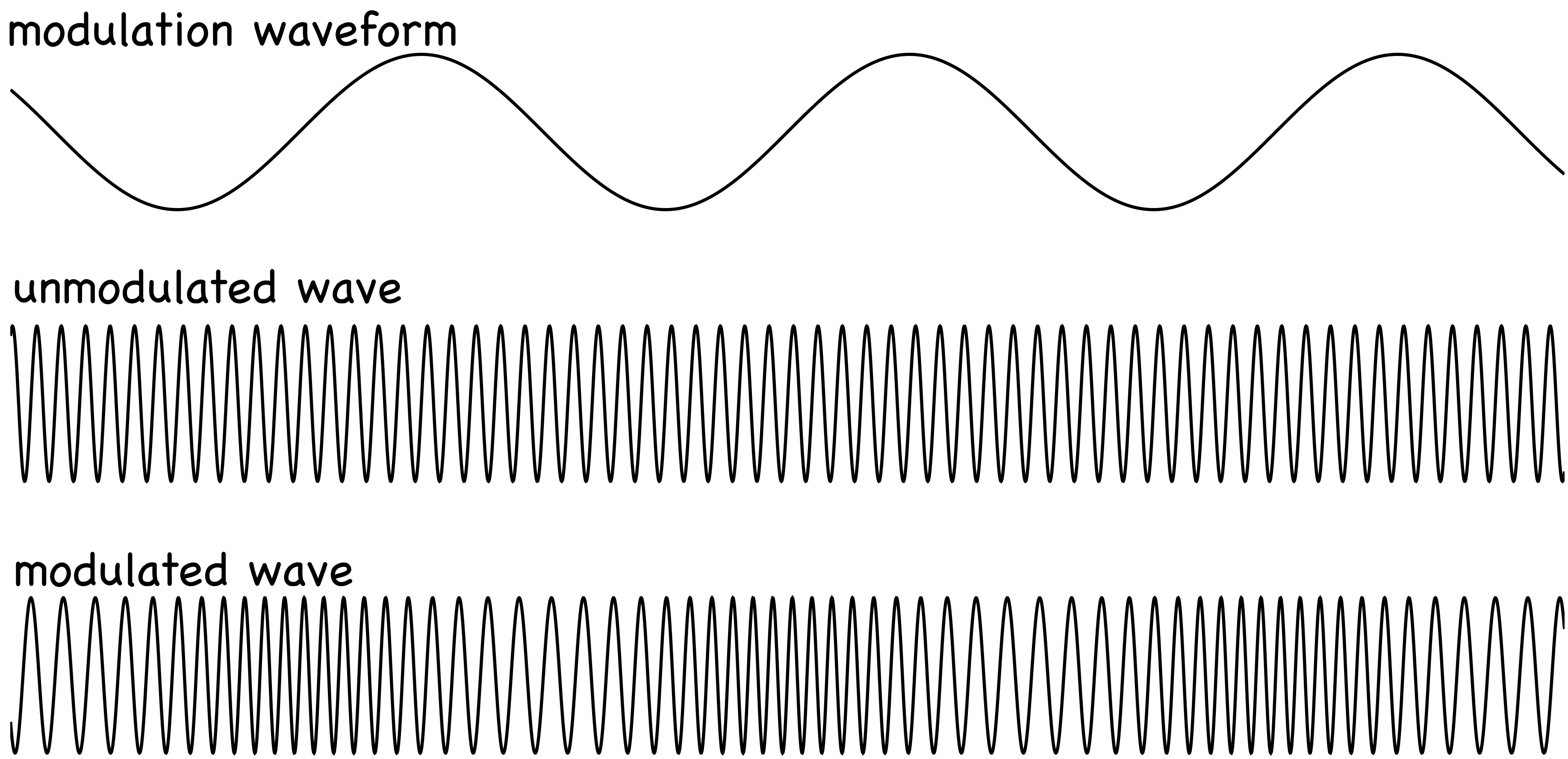
$$\frac{P_{out}}{P_o} = \sin^2(\Gamma/2 + \pi/4)$$





for Γ<<1 $\quad \frac{P_{out}}{P_o} = \frac{1}{2} + \Gamma$

Phase Modulation



Phase Modulation Sidebands

Consider a phase modulation of $\Delta\varphi = m\cos(\Omega t)$ on a field of amplitude E_0:

$$E = E_0 e^{i(\omega t + m\cos\Omega t)}$$

using the Jacobi-Anger identity:

$$e^{iz\cos\phi} = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{in\phi}$$

with E in the form

$$E = E_0 e^{i\omega t} e^{im\cos\Omega t}$$

gives

$$E = E_0 e^{i\omega t} \sum_{n=-\infty}^{\infty} i^n J_n(m) e^{in\Omega t}$$

Phase Modulation Sidebands

$$E = E_0 e^{i\omega t} \sum_{n=-\infty}^{\infty} i^n J_n(m) e^{in\Omega t}$$

For m<<1 we can write

$$E \approx E_0 \left(-iJ_{-1}(m) e^{i(\omega t - \Omega t)} + J_0(m) e^{i\omega t} + iJ_1(m) e^{i(\omega t + \Omega t)} \right)$$

and with the relation

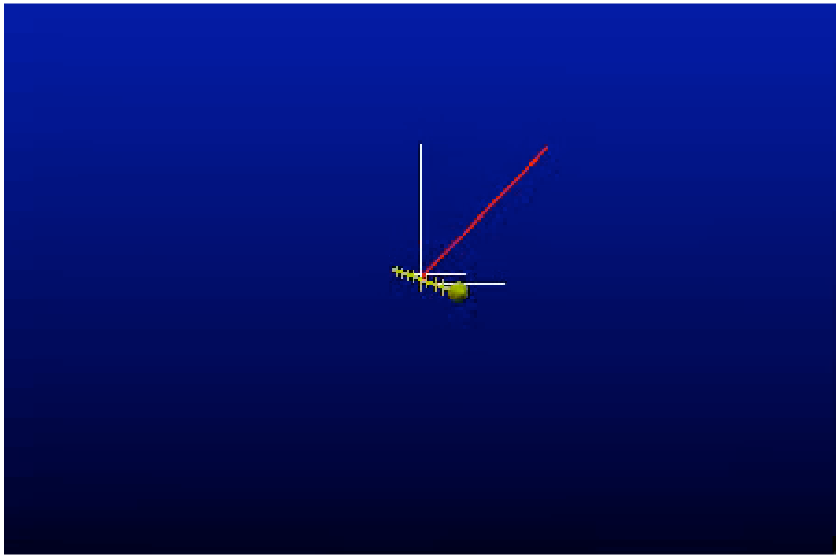
$$J_{-n}(z) = (-1)^n J_n(z)$$

this becomes

$$E \approx E_0 \left(J_0(m) e^{i\omega t} + iJ_1(m) \left(e^{i(\omega t - \Omega t)} + e^{i(\omega t + \Omega t)} \right) \right)$$

Phase Modulation Sidebands

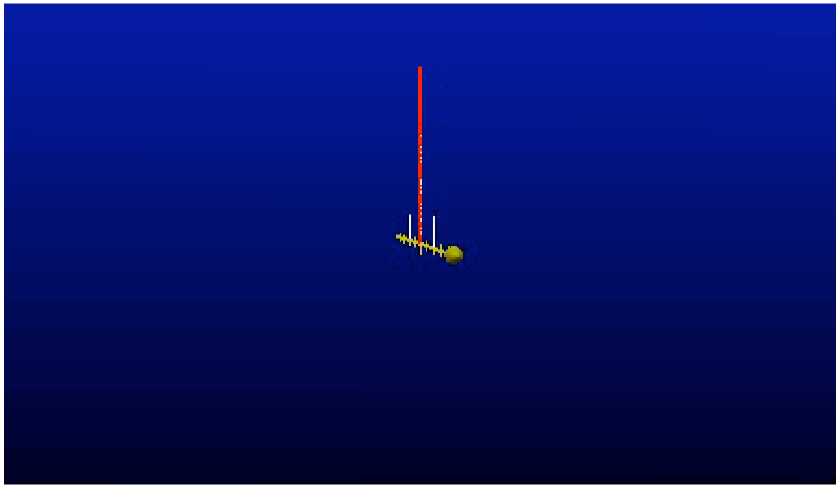
$$E \approx E_0 \left(J_0(m) e^{i\omega t} + i J_1(m) \left(e^{i(\omega t - \Omega t)} + e^{i(\omega t + \Omega t)} \right) \right)$$



Phasor representation of sidebands in a frame rotating at ω

Amplitude Modulation Sidebands

$$E \approx E_0 \left(J_0(m) e^{i\omega t} + J_1(m) \left(e^{i(\omega t - \Omega t)} + e^{i(\omega t + \Omega t)} \right) \right)$$



Phasor representation of sidebands in a frame rotating at ω

Modulation Depth

The phase acquired by a beam passing through an optic of thickness L is

$$\phi = k_0 n L$$

phase modulation comes from the index being a function of time n(t), in which case the phase acquired is

$$\phi = k_0 \bar{n}(t) L$$

where

$$\bar{n}(t) = \frac{1}{L/c} \int_{t-L/c}^{t} n(t')dt'$$

is the index of retraction of the optic averaged over the time the light is in the optic. If the index changes appreciably before the light traverses the optic the modulation depth will be reduced

Modulation Depth

$$\bar{n}(t) = \frac{1}{L/v_p}\int_0^{L/v_p} n(t, z(t))d\tau$$

with

$$n(t) = n + \delta n \sin\left(\Omega(t + \tau - \frac{z}{v_m})\right)$$ and $$z = v_p\tau$$

where the modulation is a traveling wave propagating at speed v_m

$$n(t) = n + \delta n \sin\left(\Omega t + \Omega\tau\left(1 - \frac{v_p}{v_m}\right)\right)$$

$$\bar{n} = n - \frac{\delta n v_m v_p}{\Omega(v_m - v_p)L}\left[\cos\left(\Omega\left(t - \frac{v_p}{v_m}\frac{L}{v_p} + \frac{L}{v_p}\right)\right) - \cos(\Omega t)\right]$$

Which can be evaluated using

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Modulation Depth

$$\bar{n} = n - \frac{\delta n v_m v_p}{\Omega(v_m - v_p)L}\left[\cos\left(\Omega\left(t - \frac{v_p}{v_m}\frac{L}{v_p} + \frac{L}{v_p}\right)\right) - \cos\left(\Omega t\right)\right]$$

$$\bar{n} = n + 2\frac{\delta n v_m v_p}{\Omega(v_m - v_p)L}\sin\left(\Omega t - \frac{\Omega L}{2}\left[\frac{1}{v_p} - \frac{1}{v_m}\right]\right)\sin\left(\frac{\Omega L}{2}\left[\frac{1}{v_p} - \frac{1}{v_m}\right]\right)$$

$$\bar{n} = n + \delta n \,\mathrm{sinc}\left(\frac{\Omega L}{2}\left[\frac{1}{v_p} - \frac{1}{v_m}\right]\right)\sin\left(\Omega t - \frac{\Omega L}{2}\left[\frac{1}{v_p} - \frac{1}{v_m}\right]\right)$$

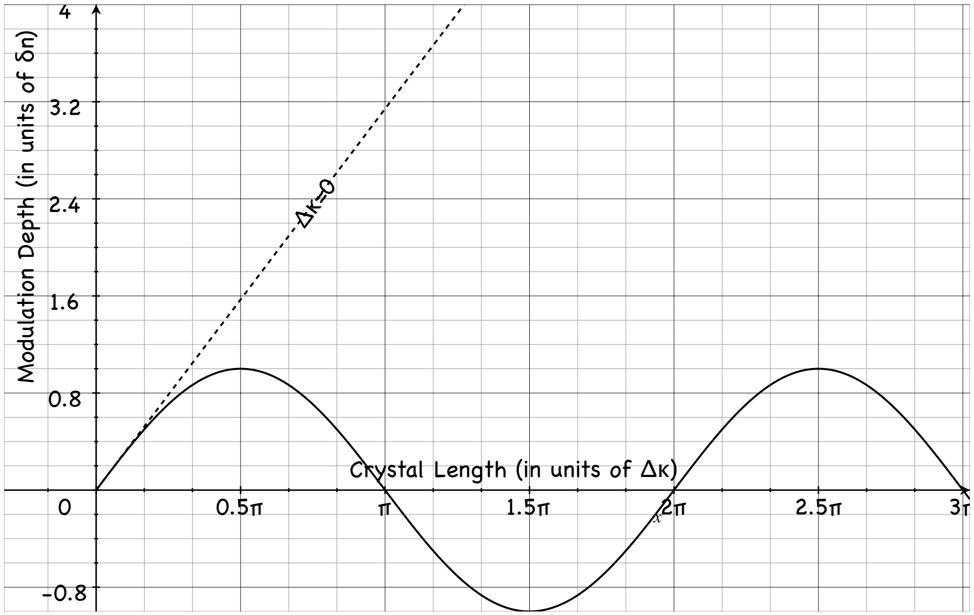
Modulation Depth

$$\bar{n} = n + \delta n \,\mathrm{sinc}\left(\frac{\Omega L}{2}\left[\frac{1}{v_p} - \frac{1}{v_m}\right]\right) \sin\left(\Omega t - \frac{\Omega L}{2}\left[\frac{1}{v_p} - \frac{1}{v_m}\right]\right)$$

δn_{eff}

$$\bar{n} = n + \delta n \,\mathrm{sinc}(\Delta\kappa L)\ \sin(\Omega t - \Delta\kappa L)$$

$$\Delta\kappa \equiv \frac{\Omega}{2}\left[\frac{1}{v_p} - \frac{1}{v_m}\right]$$

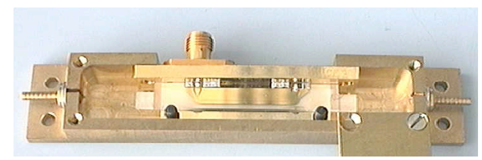


Thus the modulation depth is reduced by sinc(ΔκL).
At low modulation frequency Ω→0 the modulation depth can be increased by increasing the crystal length, but at frequencies Δκ>π/2L there is no improvement in modulation depth with increased crystal length

$LiNbO_3$ phase modualtor

Consider a Lithium Niobate phase modulator

$$r_{ik} = \begin{bmatrix} 0 & -r_{22} & r_{13} \\ 0 & r_{22} & r_{13} \\ 0 & 0 & r_{33} \\ 0 & r_{51} & 0 \\ r_{51} & 0 & 0 \\ -r_{22} & 0 & 0 \end{bmatrix}$$

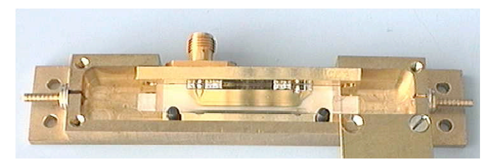


We want pure phase modulation, so need i=1,2 or 3. Since $r_{33}>r_{13}>r_{22}$ use an applied field along z to modulate light polarized in z-direction.

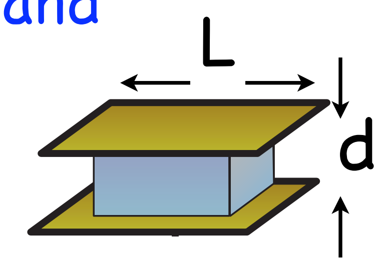
$LiNbO_3$ phase modualtor

Taking r_{33}=30.9 10^{-12} m/V, v_m=2 10^8 m/s, n_e=2.2 and L=40 mm and d=4 mm we have

$$\Delta\kappa \equiv \frac{\Omega}{2}\left[\frac{1}{v_p} - \frac{1}{v_m}\right] = 10^{-9}\ (\mathrm{s/m})\Omega$$



for V_{max}=15V we can find the modulation depth (in rad) for 100 MHz modulation and 100 GHz modulation



$$\Gamma = |k_0 \delta n_{eff} L|$$

$$\Gamma = \left| k_0 \left(\frac{1}{2} n_e^3 r_{33} \frac{V_{max}}{d}\ \mathrm{sinc}(\Delta\kappa L) \right) L \right|$$

for 100 MHz sinc(ΔκL)≈1, and Γ=0.15

for 100 GHz sinc(ΔκL)=-0.03 and Γ=0.04

Kerr Effect

In centro-symmetric and amorphous materials where the linear electro-optic effect vanishes, a second order effect can be observed

$$\Delta\eta_{ij} = s_{ijkl} E_k E_l$$

The second order electrooptic tensor s_{ijkl} can be written with contracted notation as s_{ik} and has 6x6=36 unique elements

Kerr Effect Example

Consider an electric field applied to a glass

$$s_{ik} = \begin{bmatrix} s_{11} & s_{12} & s_{12} & 0 & 0 & 0 \\ s_{12} & s_{11} & s_{12} & 0 & 0 & 0 \\ s_{12} & s_{12} & s_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(s_{11} - s_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(s_{11} - s_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(s_{11} - s_{12}) \end{bmatrix}$$

Taking the direction of the applied field as z gives an index ellipsoid defined by

$$x^2\left(\frac{1}{n^2} + s_{12}E^2\right) + y^2\left(\frac{1}{n^2} + s_{12}E^2\right) + z^2\left(\frac{1}{n^2} + s_{11}E^2\right) = 1$$

Kerr Effect Example

This is the index ellipsoid of a uniaxial crystals with

$$n_o = n - \frac{1}{2} n^3 s_{12} E^2$$

$$n_e = n - \frac{1}{2} n^3 s_{11} E^2$$

giving a birefringence of

$$n_e - n_o = \frac{1}{2} n^3 (s_{12} - s_{11}) E^2$$

which is often written as

$$n_e - n_o = K \lambda_0 E^2$$

where K is called the "Kerr constant" and λ_0 is the vacuum wavelength

References

- Yariv & Yeh "Optical Waves in Crystals" chapter 7