

# Acousto-Optic Effect

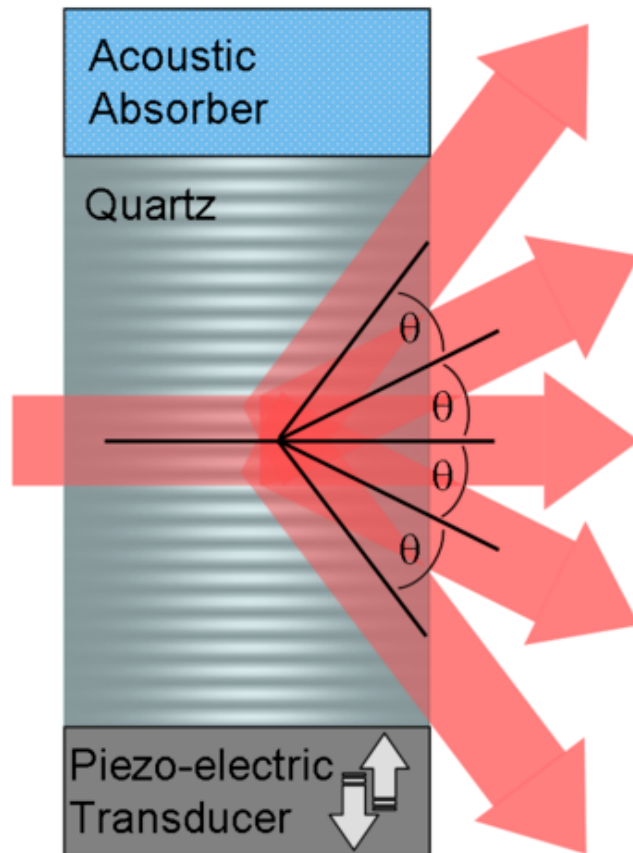
## Chapter 9

Physics 203, Electrodynamics

Peter Beyersdorf

# Acousto-Optics

Strain waves in a material produce a periodic structure that diffracts light. This can be used to modulate a beam.



# Photoelastic Effect

Photoelastic effect couples mechanical strain to a change in the optical index of refraction

$$\Delta\eta_{ij} = \Delta\left(\frac{1}{n^2}\right)_{ij} = P_{ijkl}S_{kl}$$

where  $S_{kl}$  is the strain tensor  $\Delta x_k = S_{kl}x_l$  and  $P_{ijkl}$  is called the strain-optic tensor. Diagonal terms of  $S$  are linear strains (volume deformation), off-diagonal terms are shear strains (shape deformation)

The index ellipsoid in the presence of a strain is

$$(\eta_{ij} + P_{ijkl}S_{kl})x_ix_j = 1$$

Using contracted notation we can write the effect of strain as

$$\Delta\left(\frac{1}{n^2}\right)_i = P_{ik}S_k, \quad i, k = 1, 2, \dots, 6,$$

# Strain-Optic Tensor

The strain-optic tensor  $P_{ijkl}$  can be written in contracted notation as  $P_{ik}$  and has a form that can be found from symmetry considerations for each crystal group. The form is identical to that of the Kerr electro-optic tensor  $S_{ik}$ .

For example in an isotropic material

$$P_{ik} = \begin{pmatrix} p_{11} & p_{12} & p_{12} & 0 & 0 & 0 \\ p_{12} & p_{11} & p_{12} & 0 & 0 & 0 \\ p_{12} & p_{12} & p_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(p_{11} - p_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(p_{11} - p_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(p_{11} - p_{12}) \end{pmatrix}$$

# AO effect in Water

Consider an acoustic plane wave in water propagating in the z-direction

$$\vec{u}(z, t) = A\hat{z} \cos(\Omega t - Kz)$$

producing a strain field,  $S_{ij} = \frac{\partial u_i}{\partial x_j}$ , of

$$S_3 = KA \sin(\Omega t - Kz) \equiv S \sin(\Omega t - Kz)$$

Using the  $P_{ik}$  tensor for an isotropic medium

$$P_{ik}S_i = \begin{pmatrix} p_{11} & p_{12} & p_{12} & 0 & 0 & 0 \\ p_{12} & p_{11} & p_{12} & 0 & 0 & 0 \\ p_{12} & p_{12} & p_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(p_{11} - p_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(p_{11} - p_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(p_{11} - p_{12}) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ S \sin(\Omega t - Kz) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

warning - this is not matrix multiplication

# AO effect in Water

$$\Delta\eta_i = P_{ik}S_i = \begin{pmatrix} p_{11} & p_{12} & p_{12} & 0 & 0 & 0 \\ p_{12} & p_{11} & p_{12} & 0 & 0 & 0 \\ p_{12} & p_{12} & p_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(p_{11} - p_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(p_{11} - p_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(p_{11} - p_{12}) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ S \sin(\Omega t - Kz) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

giving for  $\Delta\eta$

$$\Delta\eta_{11} = \Delta \left( \frac{1}{n^2} \right)_1 = p_{12}S \sin(\Omega t - Kz),$$

$$\Delta\eta_{22} = \Delta \left( \frac{1}{n^2} \right)_2 = p_{12}S \sin(\Omega t - Kz),$$

$$\Delta\eta_{33} = \Delta \left( \frac{1}{n^2} \right)_3 = p_{11}S \sin(\Omega t - Kz),$$

$$\Delta\eta_{ij} = 0 \quad \text{for } i \neq j$$

Resulting in an index ellipsoid of

$$(x^2 + y^2) \left[ \frac{1}{n^2} + p_{12}S \sin(\Omega t - Kz) \right] + z^2 \left[ \frac{1}{n^2} + p_{11}S \sin(\Omega t - Kz) \right] = 1$$

# AO effect in Water

$$(x^2 + y^2) \left[ \frac{1}{n^2} + p_{12} S \sin(\Omega t - Kz) \right] + z^2 \left[ \frac{1}{n^2} + p_{11} S \sin(\Omega t - Kz) \right] = 1$$

with new principle indices of refraction of

$$\begin{aligned} n_x = n_y &= n - \frac{1}{2} n^3 p_{12} S \sin(\Omega t - Kz) \\ n_z &= n - \frac{1}{2} n^3 p_{11} S \sin(\Omega t - Kz) \end{aligned}$$

# AO effect in Gallium Arsenide

Consider a transverse (shear) wave in GaAs polarized in the  $y$  direction, traveling in the  $z$ -direction

$$\vec{u}(z, t) = A\hat{y} \cos(\Omega t - kz)$$

producing a strain wave of  $S_{ij} = \frac{\partial u_i}{\partial x_j}$  which is

$$S_4 = KA \sin(\Omega t - kz) = S \sin(\Omega t - Kz)$$

With the strain optic tensor for a cubic crystal

$$P_{ij} = \begin{pmatrix} p_{11} & p_{12} & p_{12} & 0 & 0 & 0 \\ p_{12} & p_{11} & p_{12} & 0 & 0 & 0 \\ p_{12} & p_{12} & p_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & p_{44} \end{pmatrix}$$

we get

$$\Delta\eta_{23} = \Delta\eta_{32} = p_{44}S \sin(\Omega t - Kz)$$

# AO effect in Gallium Arsenide

The modified index ellipsoid is

$$\frac{1}{n^2} (x^2 + y^2 + z^2) + 2yzp_{44}S \sin(\Omega t - kz) = 1$$

We can choose a new set of principle coordinates  $x'$ ,  $y'$  and  $z'$  that recast this equation into the form

$$\frac{x'^2}{n_{x'}^2} + \frac{y'^2}{n_{y'}^2} + \frac{z'^2}{n_{z'}^2} = 1$$

From inspection  $x=x'$ , so let  $z',y'$  and  $z,y$  differ by a rotation of  $\theta$  around the  $x$ -axis:

$$z = z' \cos \theta - y' \sin \theta$$

$$y = z' \sin \theta + y' \cos \theta$$

$$\frac{(z' \cos \theta - y' \sin \theta)^2}{n^2} + \frac{(z' \sin \theta + y' \cos \theta)^2}{n^2} + \frac{x^2}{n^2} + 2(z \cos \theta - y' \sin \theta)(z' \sin \theta + y' \cos \theta)p_{44}S \sin(\Omega t - Kz) = 1$$

# AO effect in Gallium Arsenide

## Evaluating

$$\frac{(z' \cos \theta - y' \sin \theta)^2}{n^2} + \frac{(z' \sin \theta + y' \cos \theta)^2}{n^2} + \frac{x^2}{n^2} + 2(z \cos \theta - y' \sin \theta)(z' \sin \theta + y' \cos \theta)p_{44}S \sin(\Omega t - Kz) = 1$$

gives

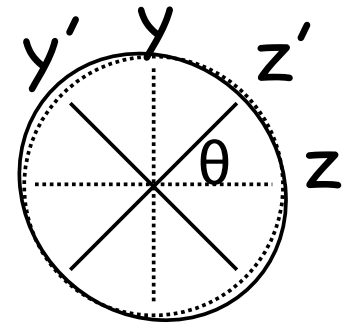
$$\frac{x'^2}{n^2} + \frac{y'^2}{n^2} + \frac{z'^2}{n^2} - \frac{z'y' \sin 2\theta}{n^2} + \frac{z'y' \sin 2\theta}{n^2} + (z'^2 \sin 2\theta + 2z'y' \cos 2\theta - y'^2 \sin 2\theta)p_{44}S \sin(\Omega t - Kz) = 1$$

cross term  $2z'y' \cos(2\theta) \rightarrow 0$  for  $\theta=45^\circ$  giving

$$\frac{z'^2}{n^2} + \frac{y'^2}{n^2} + \frac{z'^2}{n^2} + (z'^2 - y'^2)S \sin(\Omega t - Kz) = 1$$

or

$$\frac{x'^2}{n^2} + \left( \frac{1}{n^2} - S \sin(\Omega t - Kz) \right) y'^2 + \left( \frac{1}{n^2} + S \sin(\Omega t - Kz) \right) z'^2 = 1$$



# AO effect in Gallium Arsenide

Equating

$$\frac{x'^2}{n^2} + \left( \frac{1}{n^2} - S \sin(\Omega t - Kz) \right) y'^2 + \left( \frac{1}{n^2} + S \sin(\Omega t - Kz) \right) z'^2 = 1$$

to

$$\frac{x'^2}{n_{x'}^2} + \frac{y'^2}{n_{y'}^2} + \frac{z'^2}{n_{z'}^2} = 1$$

gives

$$\left( \frac{1}{n^2} + S \sin(\Omega t - Kz) \right) = \frac{1}{n_z'^2}$$

$$n_z'^2 = n^2 \left( \frac{1}{1 + n^2 S \sin(\Omega t - Kz)} \right)$$

$$n_z'^2 \approx n^2 (1 - n^2 S \sin(\Omega t - Kz))$$

so

$$n'_x = n \quad n'_z \approx n \left( 1 - \frac{1}{2} n^2 S \sin(\Omega t - Kz) \right) \quad n'_y \approx n \left( 1 + \frac{1}{2} n^2 S \sin(\Omega t - Kz) \right)$$

# Acousto-Optic Interactions

Light will diffract from the periodic medium produced by the acousto-optic effect. Since the acoustic wave is slow compared to the light wave ( $v_s/c \approx 10^{-5}$ ), the acousto-optic perturbation can be treated as a stationary volume grating

Strong coupling to the diffracted wave occurs when the Bragg condition is met:

$$2k \sin \theta = K$$

for typical parameters this occurs for  $\theta \approx 5^\circ$

# Acousto-Optic Interactions

The Bragg condition is equivalent to  $2\Lambda \sin \theta = \lambda/n$  found from geometrical consideration

The effect of motion of the acoustic wave is to Doppler shift the frequency of the diffracted wave according to

$$\Delta\omega = 2\omega \frac{v_s \sin \theta}{c/n}$$

where  $v_s \sin \theta$  is the component of the acoustic wave velocity in the direction of the incident beam. At the Bragg condition  $\Delta\omega = \frac{2\pi v_s}{\Lambda}$  for the diffracted beam

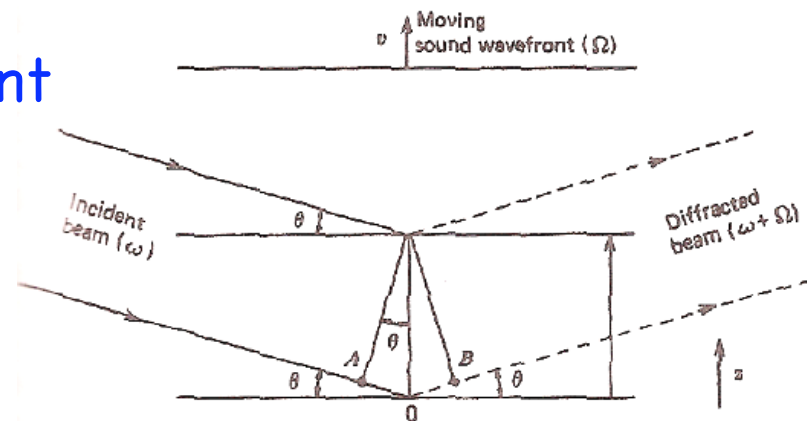


Figure 9.2

# Acousto-Optic Interactions

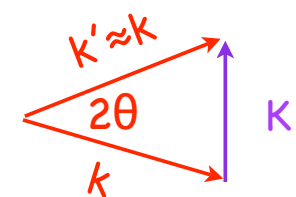
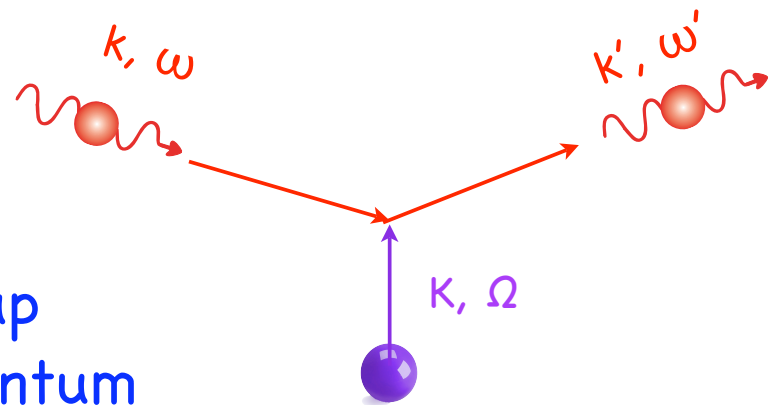
Alternatively consider the optical wave as made up of photons of energy  $\hbar\omega$  and momentum  $\hbar\vec{k}$

and the acoustic wave as made up of phonons of energy  $\hbar\Omega$  and momentum

$\hbar\vec{K}$ . For one photon absorbing one phonon

Conservation of energy requires  $\omega' = \omega + \Omega$

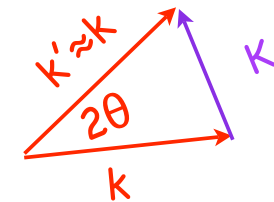
Conservation of momentum requires  $\vec{k}' = \vec{k} + \vec{K}$



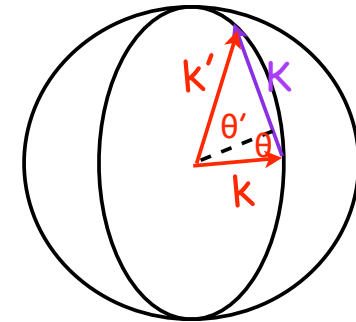
$$2k \sin \theta = K$$

# Bragg Diffraction in an Anisotropic Medium

Picture of momentum conservation for photons and phonons involved in diffraction in isotropic material is an isosceles triangle since  $k \approx k' = n\omega/c$



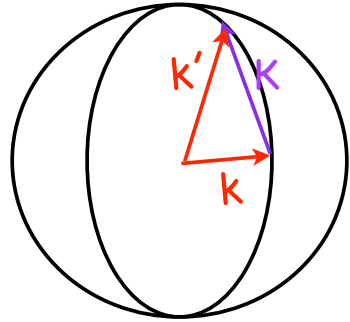
In anisotropic material  $k \neq k'$  since  $n$  is a function of propagation angle  $n \neq n'$ . We can use the intersection of the normal shells with the plane of diffraction to reconsider momentum diagram. The Bragg condition can be generalized to



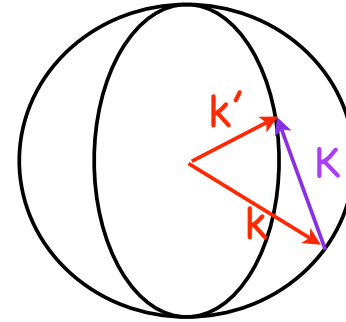
Propagation in the x-z plane of a uniaxial crystal, acoustic wave at  $30^\circ$  from z axis optical wave polarized as e-wave

$$k \sin \theta + k' \sin \theta' = K$$

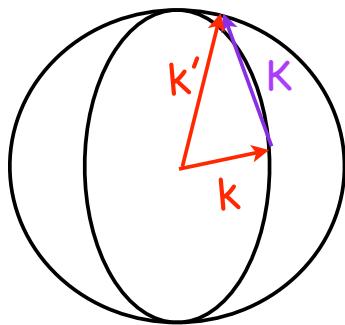
# Bragg Diffraction in an Anisotropic Medium



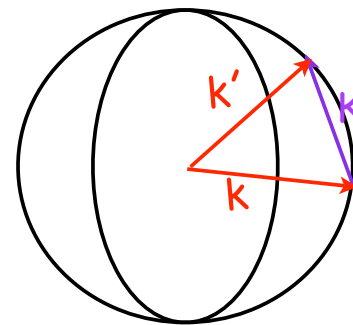
incident wave polarized as e-wave,  
diffracted wave polarized as e-wave



incident wave polarized as o-wave,  
diffracted wave polarized as e-wave



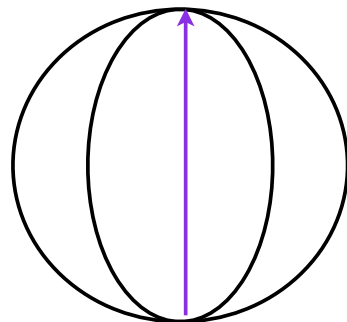
incident wave polarized as e-wave,  
diffracted wave polarized as o-wave



incident wave polarized as o-wave,  
diffracted wave polarized as o-wave

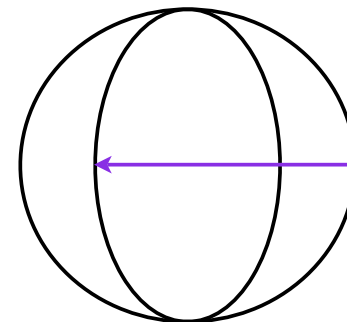
# Bragg Diffraction in an Anisotropic Medium

For a given direction of the acoustic wave, there is a maximum value for  $K$  that allows Bragg diffraction. The maximum value can easily be found from the geometry of the normal shells diagram



diffraction in the x-z plane of a uniaxial crystal with an acoustic wave propagating in the z-direction and incident wave o-polarized, diffracted wave e-polarized

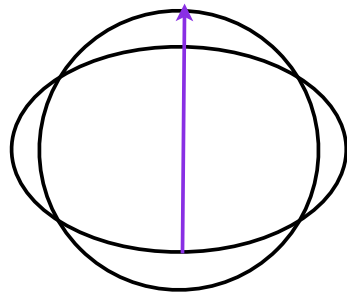
$$K_{\max} = 2k_0 n_o$$



diffraction in the x-z plane of a uniaxial crystal with an acoustic wave propagating in the x-direction and incident wave o-polarized, diffracted wave e-polarized

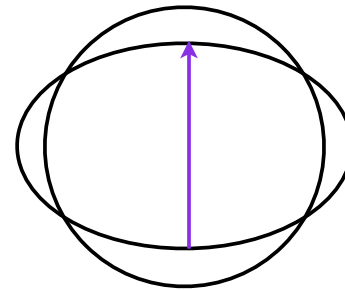
$$K_{\max} = k_0(n_o + n_e)$$

# Bragg Diffraction in an Anisotropic Medium



diffraction in the x-z plane of a biaxial crystal with an acoustic wave propagating in the z-direction, incident wave polarized in xz plane, diffracted beam polarized along y

$$K_{\max} = n_x + n_y$$



diffraction in the x-z plane of a biaxial crystal with an acoustic wave propagating in the z-direction, incident wave polarized in xz plane, diffracted beam polarized in xz plane

$$K_{\max} = 2n_x$$

# Coupled Mode Analysis of Bragg Diffraction

Geometrical arguments give a constraint on the Bragg angle for acousto-optic modulation, but determining the efficiency of conversion requires we analyze the coupling of the undiffracted and diffracted modes

Consider a total electric field

$$\vec{E} = A_1 \vec{E}_1 e^{i(\omega_1 t - k_1 \cdot r)} + A_2 \vec{E}_2 e^{i(\omega_2 t - k_2 \cdot r)}$$

where  $A_1$  and  $A_2$  are coupled by the effect of the acoustic wave

$$\Delta \eta_{ij} = \Delta \left( \frac{\epsilon_0}{\epsilon} \right)_{ij} = p_{ijkl} S_{kl} \cos(\Omega t - Kz)$$

# Coupled Mode Analysis of Bragg Diffraction

The perturbation to the impermeability tensor

$$\Delta\eta_{ij} = \Delta\left(\frac{\epsilon_0}{\epsilon}\right)_{ij} = p_{ijkl}S_{kl}\cos(\Omega t - Kz)$$

is equivalent to a perturbation to the dielectric tensor

$$\Delta\epsilon(z, t) = 2\epsilon_1 \cos(\Omega t - Kz) \equiv \Delta\epsilon \cos(\Omega t - Kz)$$

where  $\epsilon_1$  is a tensor of the first Fourier coefficients of the tensor  $\epsilon(z)$  and is given by

$$\epsilon_1 = -\frac{\epsilon(p_{ijkl}S_{kl})}{2\epsilon_0}$$

This perturbation  $\Delta\epsilon$  will couple the field amplitudes  $A_1$  and  $A_2$

# Coupled Mode Analysis of Bragg Diffraction

The wave equation in a perturbed system can be written as

$$(\nabla^2 + \omega^2 \mu \epsilon + \omega^2 \mu \Delta \epsilon) \vec{E} = 0$$

using the solutions in the unperturbed material

$$\vec{E} = \mathbf{A}_1 \vec{E}_1 e^{i(\omega_1 t - k_1 \cdot r)} + \mathbf{A}_2 \vec{E}_2 e^{i(\omega_2 t - k_2 \cdot r)}$$

along with  $k^2 = \omega^2 \mu \epsilon$  and assuming  $A$  is slowly varying ( $dA/dz \ll kA$ ) gives

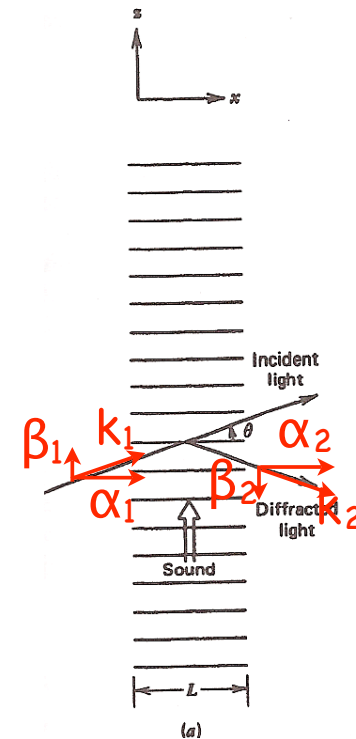
$$\sum_{m=1,2} \left\{ \left[ \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} - 2i\beta_m \frac{\partial}{\partial z} - 2i\alpha_m \frac{\partial}{\partial x} \right] A_m \right\} \vec{E}_m e^{i(\omega_m t - \alpha_m x - \beta_m z)} = -\omega^2 \mu \sum_{i=1,2} \Delta \epsilon \mathbf{A}_i \vec{E}_i e^{i(\omega_i t - \alpha_i x - \beta_i z)}$$

# Interaction Configurations

Consider the cases of small angle diffraction, shown in figure 9.7a of the textbook

The interaction region is the width of the acoustic beam and the mode amplitudes are functions of only  $x$  giving

$$\begin{aligned}
 & -2i\alpha_1 \frac{dA_1}{dx} \vec{E}_1 e^{i(\omega_1 t - \alpha_1 x - \beta_1 z)} - 2i\alpha_2 \frac{dA_2}{dx} \vec{E}_2 e^{i(\omega_2 t - \alpha_2 x - \beta_2 z)} \\
 & = -\omega^2 \mu \left( \epsilon_1 e^{i(\Omega t - Kz)} + \epsilon_1 e^{-i(\Omega t - Kz)} \right) \\
 & \quad \times \left( \mathbf{A}_1 \vec{E}_1 e^{i(\omega_1 t - \alpha_1 x - \beta_1 z)} + \mathbf{A}_2 \vec{E}_2 e^{i(\omega_2 t - \alpha_2 x - \beta_2 z)} \right)
 \end{aligned}$$



# Small Angle Diffraction Efficiency

From

$$\begin{aligned}
 & -2i\alpha_1 \frac{dA_1}{dx} \vec{E}_1 e^{i(\omega_1 t - \alpha_1 x - \beta_1 z)} - 2i\alpha_2 \frac{dA_2}{dx} \vec{E}_2 e^{i(\omega_2 t - \alpha_2 x - \beta_2 z)} \\
 & = -\omega^2 \mu \left( \epsilon_1 e^{i(\Omega t - Kz)} + \epsilon_1 e^{-i(\Omega t - Kz)} \right) \\
 & \quad \times \left( \mathbf{A}_1 \vec{E}_1 e^{i(\omega_1 t - \alpha_1 x - \beta_1 z)} + \mathbf{A}_2 \vec{E}_2 e^{i(\omega_2 t - \alpha_2 x - \beta_2 z)} \right)
 \end{aligned}$$

we get

$$\begin{aligned}
 \frac{dA_1}{dx} & = -i\kappa_{12} A_2 e^{i\Delta\alpha x}, \\
 \frac{dA_2}{dx} & = -i\kappa_{12}^* A_1 e^{-i\Delta\alpha x},
 \end{aligned}$$

with  $\kappa_{12} = \frac{\omega^2 \mu}{2\sqrt{\alpha_1 \alpha_2}} \hat{p}_1^* \cdot \epsilon_1 \hat{p}_2$  where  $p_1$  and  $p_2$  are polarization unit vectors. This simplification requires the conditions for conservation of energy and momentum are met  $\beta_2 = \beta_1 \pm K$  and  $\omega_2 = \omega_1 \pm \Omega$

# Phase mismatch

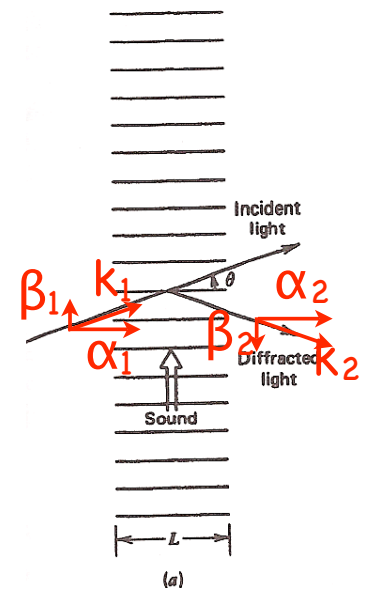
The term  $\Delta\alpha$  is related to the asymmetry of the incident and diffracted waves relative to the acoustic wave.

$$\frac{dA_1}{dx} = -i\kappa_{12}A_2e^{i\Delta\alpha x},$$

$$\frac{dA_2}{dx} = -i\kappa_{12}^*A_1e^{-i\Delta\alpha x},$$

The conversion from  $A_1$  to  $A_2$  occurs when  $A_1$  and  $A_2$  are in phase, but when they drift out of phase due to  $\Delta\alpha$  the acoustic wave converts field  $A_2$  into field  $A_1$

When the Bragg condition is satisfied  $\alpha_1=\alpha_2$  so  $\Delta\alpha=0$  and the conversion efficiency is maximized



# Bragg Diffraction Efficiency

At the Bragg condition  $\theta_b = \sin^{-1} \left( \frac{K}{2k} \right) = \sin^{-1} \left( \frac{\lambda}{2\Lambda} \right)$

$$\frac{dA_1}{dx} = -i\kappa_{12}A_2,$$

$$\frac{dA_2}{dx} = -i\kappa_{12}^*A_1$$

which has solutions

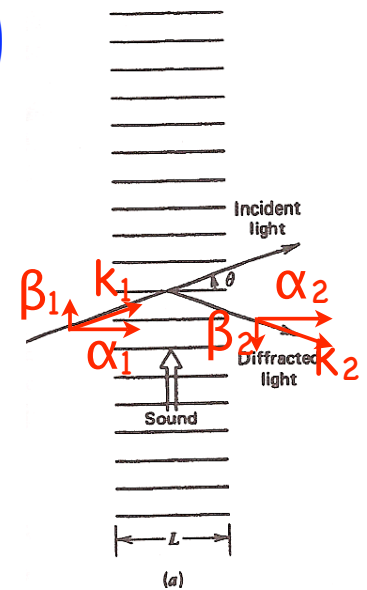
$$\vec{A}_1(x) = \vec{A}_1(0) \cos \kappa x - i \frac{\kappa_{12}}{\kappa} \vec{A}_2(0) \sin \kappa x,$$

$$\vec{A}_2(x) = \vec{A}_2(0) \cos \kappa x - i \frac{\kappa_{12}^*}{\kappa} \vec{A}_1(0) \sin \kappa x,$$

where  $\kappa = |\kappa_{12}|$ . If  $A_2(0)=0$  this gives

$$\vec{A}_1(x) = \vec{A}_1(0) \cos \kappa x$$

$$\vec{A}_2(x) = -i \frac{\kappa_{12}^*}{\kappa} \vec{A}_1(0) \sin \kappa x$$



# Bragg Diffraction Efficiency

The fraction of the power transferred to the diffracted beam after an interaction distance  $x$  is

$$\frac{I_{\text{diffracted}}}{I_{\text{incident}}} = \frac{|\vec{A}_2(x)|^2}{|\vec{A}_1(0)|^2} = \sin^2 \kappa x,$$

with full power transfer occurring at  $x = \pi/2\kappa$ . Where  $\kappa$  can be expressed in terms of the original acousto-strain  $p_{ijkl}S_{kl}$  matrix

$$\kappa = \frac{\omega n^3}{4c \cos \theta_B} |\hat{p}_1^* \cdot (p_{ijkl} S_{kl}) \hat{p}_2|.$$

which depends on the intensity of the acoustic wave (via the strain tensor  $S_{kl}$ ), the material properties, and the geometry.

# Large Angle Diffraction

Consider the cases of large angle diffraction, shown in figure 9.7b of the textbook

The mode amplitudes only vary along  $z$  giving

$$\vec{E} = \left[ A_1(z) \vec{E}_1 e^{i(\omega_1 t - \beta_1 z)} + A_2(z) \vec{E}_2 e^{i(\omega_2 t - \beta_2 z)} \right] e^{-i\alpha x}$$

giving

$$\frac{dA_1}{dz} = -i \frac{\beta_1}{|\beta_2|} \kappa_{12} A_2 e^{i\Delta\beta z}$$

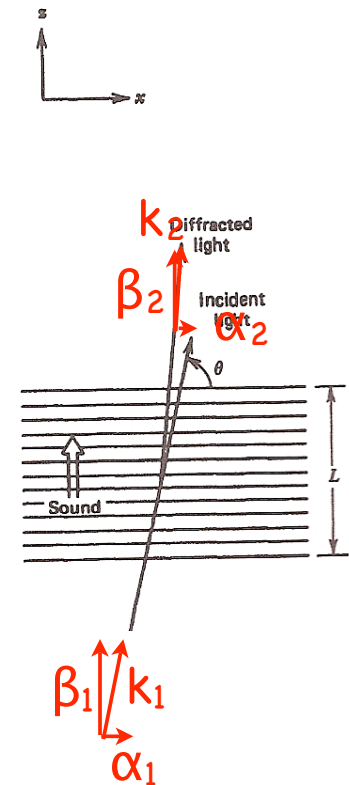
$$\frac{dA_2}{dz} = -i \frac{\beta_2}{|\beta_2|} \kappa_{12}^* A_1 e^{-i\Delta\beta z}$$

with

$$\Delta\beta = \beta_1 - \beta_2 \pm K$$

where

$$\kappa_{12} = \frac{\omega^2 \mu}{2\sqrt{|\beta_1 \beta_2|}} \hat{p}_1^* \cdot \epsilon_1 \hat{p}_2 \quad \text{ch 9.27}$$



# Codirectional Coupling

For the case where  $\beta_1\beta_2>0$  (i.e. forward diffraction)  
the solutions to

$$\begin{aligned}\frac{dA_1}{dz} &= -i\kappa_{12}A_2e^{i\Delta\beta z} \\ \frac{dA_2}{dz} &= -i\kappa_{12}^*A_1e^{-i\Delta\beta z}\end{aligned}$$

subject to the boundary conditions  $A_2(0)=0$  is

$$A_1(z) = A_1(0)e^{i\frac{1}{2}\Delta\beta z} \left[ \cos sz - i\frac{\Delta\beta}{2s} \sin sz \right]$$

$$A_2(z) = -iA_1(0)e^{-i\frac{1}{2}\Delta\beta z} \frac{\kappa_{12}^*}{s} \sin sz$$

$$\text{with } s^2 = |\kappa_{12}|^2 + \left(\frac{1}{2}\Delta\beta\right)^2$$

# Codirectional Coupling

The transfer of energy from  $A_1$  to  $A_2$  after interaction through a length  $L$  is given by

$$T = \frac{|A_2(L)|^2}{|A_1(0)|^2} = \frac{|\kappa_{12}|^2}{|\kappa_{12}|^2 + \left(\frac{1}{2}\Delta\beta\right)^2} \sin^2 sL.$$

which is maximized when  $\Delta\beta=0$  (no phase mismatch of incident and diffracted beams) and requires  $\Delta\beta=0$  ,  $L=\pi/2\kappa_{12}$  for 100% conversion.

# Contradirectional Coupling

For the case where  $\beta_1\beta_2 < 0$  (i.e. backwards diffraction) the solutions to

$$\begin{aligned}\frac{dA_1}{dz} &= i\kappa_{12}A_2e^{i\Delta\beta z} \\ \frac{dA_2}{dz} &= -i\kappa_{12}^*A_1e^{-i\Delta\beta z}\end{aligned}$$

subject to the boundary conditions  $A_2(0)=0$  are

$$A_1(z) = A_1(0)e^{i\frac{1}{2}\Delta\beta z} \left[ \frac{s \cosh s(L-z) + i\frac{1}{2}\Delta\beta \sinh s(L-z)}{s \cosh sL + i\frac{1}{2}\Delta\beta \sinh sL} \right]$$

$$A_2(z) = A_1(0)e^{-i\frac{1}{2}\Delta\beta z} \left[ \frac{i\kappa_{12}^* \sinh s(L-z)}{s \cosh sL + i\frac{1}{2}\Delta\beta \sinh sL} \right]$$

where  $s^2 = |\kappa_{12}|^2 - \left(\frac{1}{2}\Delta\beta\right)^2$

# Contradirectional Coupling

The transfer of energy from  $A_1$  to  $A_2$  after interaction through a length  $L$  is given by

$$R = \left| \frac{A_2(0)}{A_1(0)} \right|^2 = \frac{|\kappa_{12}|^2 \sinh^2 sL}{s^2 \cosh^2 sL + \left(\frac{1}{2}\Delta\beta\right)^2 \sinh^2 sL}$$

which is identical to that of a Bragg reflector. This configuration can be thought of as a periodic Bragg reflector, since the speed of the acoustic wave is so much smaller than the speed of light that it can be treated as equivalent to a static perturbation of the permittivity tensor.

# Example

$$S_{12} = S_6 = -iKu_0 e^{i(\Omega t - Ky)}$$

Which can be used with  $\Delta\eta_{ij} = p_{ijkl} S_{kl}$  or in contracted notation  $\Delta\eta_i = p_{ik} S_k$  to find the new index ellipsoid. Lithium Niobate is a trigonal crystal with  $3\bar{m}$  symmetry, so

$$p_{ik} = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} & 0 & 0 \\ p_{12} & p_{11} & p_{13} & -p_{14} & 0 & 0 \\ p_{13} & p_{13} & p_{33} & 0 & 0 & 0 \\ p_{41} & -p_{41} & 0 & p_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{44} & p_{41} \\ 0 & 0 & 0 & 0 & p_{14} & \frac{1}{2}(p_{11} - p_{12}) \end{pmatrix}$$

# Example

thus  $\Delta\eta_i = p_{i6}S_6$  giving

$$\left(\frac{1}{n_o^2}\right)x^2 + \left(\frac{1}{n_o^2}\right)y^2 + \left(\frac{1}{n_e^2}\right)z^2 - 2\left(ip_{41}Ku_0e^{i(\Omega t - Ky)}\right)xz - \left(i(p_{11} - p_{12})Ku_0e^{i(\Omega t - Ky)}\right)xy = 1$$

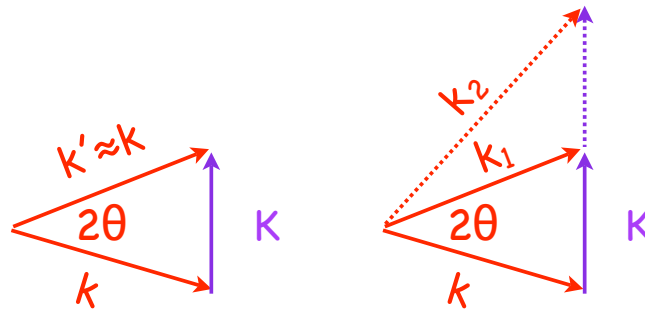
from which the index ellipsoid becomes

$$\left(\frac{1}{n_o^2}\right)x^2 + \left(\frac{1}{n_o^2}\right)y^2 + \left(\frac{1}{n_e^2}\right)z^2 - 2\left(ip_{41}Ku_0e^{i(\Omega t - Ky)}\right)xz - \left(i(p_{11} - p_{12})Ku_0e^{i(\Omega t - Ky)}\right)xy = 1$$

# Raman-Nath Diffraction

Our expressions for the optical field diffracted from the acoustic wave assume there is only one diffracted order ( $m=\pm 1$ ). Is it possible to have  $|m|>1$ , corresponding to more than 1 phonon absorbed?

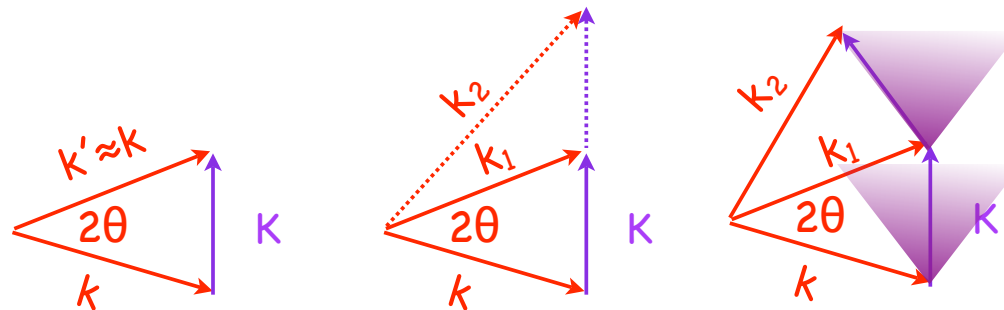
Conservation of energy requires  $k$  and  $k'$  be almost the same length (except for the relatively small increase of  $k'$  due to absorbed energy of phonon). This isn't possible for more than one order at a time, with a unique  $K$  vector.



# Acoustic Beam Spread

If the acoustic wave is a finite beam with width  $\Delta X$ , Heisenberg's uncertainty relation  $\Delta x \Delta p \geq \hbar/2$  tells us it will have a range of K-vectors  $\Delta K = 1/(2\Delta X)$ .

If the range of K-vectors is large compared to the Bragg angle, multiple phonons to be absorbed without changing length of  $k'$  relative to  $k$ .



# Raman-Nath Criterion

Acoustic beam spread is  $\Delta K = K \Delta \Theta$

$$\Delta \Theta = \frac{\Delta K}{K}$$

$$\Delta \Theta = \frac{\Lambda}{4\pi \Delta X}$$

Thus the Bragg angle ( $\theta_b = \lambda / (2n\Lambda)$ ) can be expressed in terms of the acoustic beam spread

$$Q = 2\pi \frac{\lambda \Delta X}{n\Lambda^2}$$

$Q > 1$  is called the "Bragg regime" and corresponds to single order diffraction

$Q < 1$  is called the "Raman-Nath regime" and corresponds to multiple order diffraction

# Raman-Nath Diffraction

Consider a narrow acoustic beam introducing a change in the index of refraction in a material of

$$\Delta n(0 < x < \Delta X) = \Delta n_0 \sin(\Omega t - \vec{K} \cdot \vec{r})$$

an optical wave expressed by

$$\vec{E} = \vec{E}_0 e^{i(\omega t - \vec{k} \cdot \vec{r})}$$

is incident at  $x=0$ , upon exiting the acoustic beam at  $x=\Delta X$  it can be written as

$$\vec{E} = \vec{E}_0 e^{i(\omega t - \vec{k} \cdot \vec{r} - \phi)}$$

with

$$\phi = \int_0^{\Delta X} \frac{1}{\cos \theta} \frac{\omega}{c} \Delta n dx$$

where  $\theta$  is the direction of the optical beam relative to the  $x$ -axis

# Raman-Nath Diffraction

If  $\Delta X$  is small (i.e. in the Raman-Nath regime), index  $n(x,t)$  seen by optical beam can be considered constant across beam width

$$\phi = \frac{\Delta X}{\cos \theta} \frac{\omega}{c} \Delta n = \frac{\Delta X}{\cos \theta} \frac{\omega}{c} \Delta n_0 \sin(\Omega t - \vec{K} \cdot \vec{r})$$

giving

$$\vec{E} = \vec{E}_0 e^{i(\omega t - \vec{k} \cdot \vec{r} - \delta \sin(\Omega t - \vec{K} \cdot \vec{r}))}$$

where

$$\delta \equiv \frac{\Delta X}{\cos \theta} \frac{\omega}{c} \Delta n_0$$

is called the modulation index. This can be expanded using a form of the Jacobi-Anger identity

$$e^{-i\delta \sin x} = \sum_{m=-\infty}^{\infty} J_m(\delta) e^{im\phi}$$

# Bessel Function Expansion

Expanding

$$\vec{E} = \vec{E}_0 e^{i(\omega t - \vec{k} \cdot \vec{r} - \delta \sin(\Omega t - \vec{K} \cdot \vec{r}))}$$

into Bessel functions gives

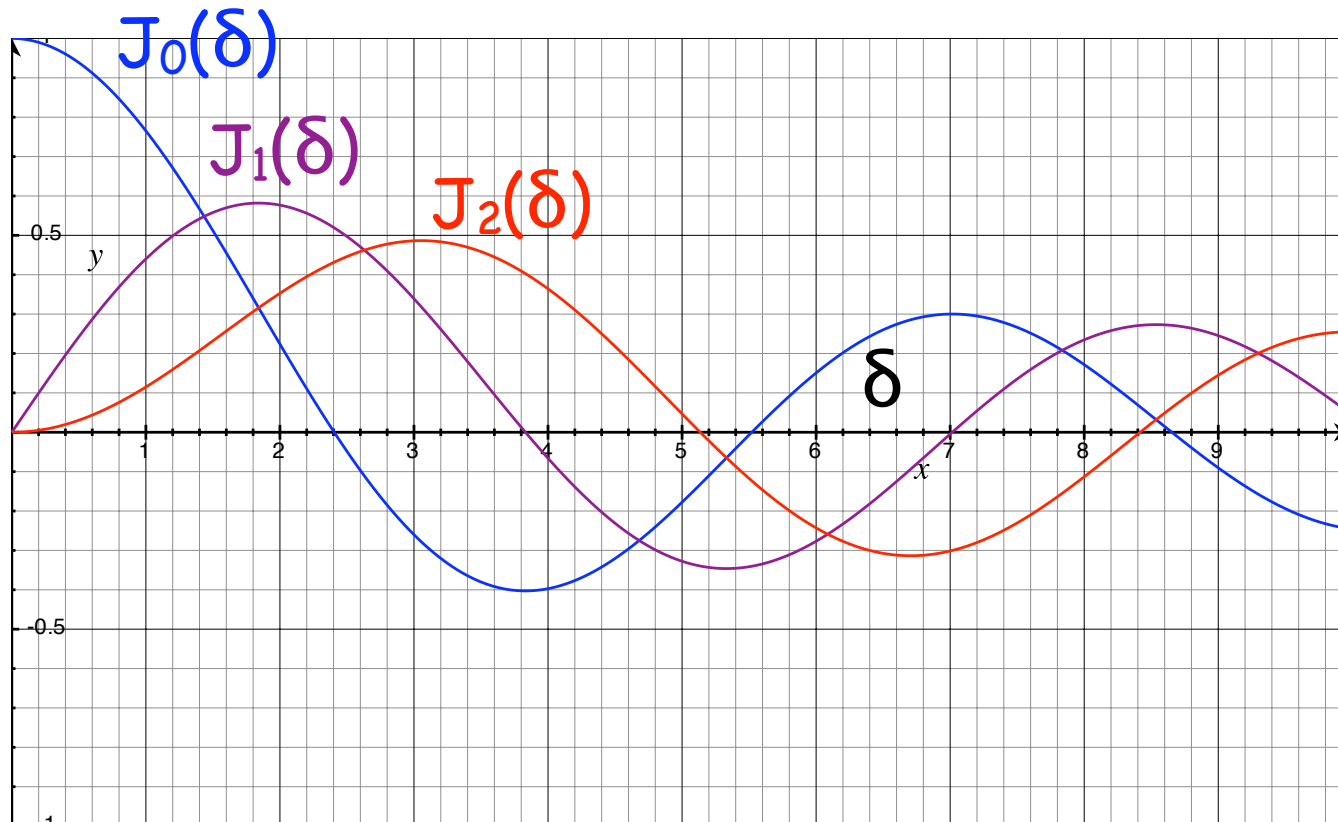
$$\vec{E} = \vec{E}_0 \sum_{m=-\infty}^{\infty} J_m(\delta) e^{i((\omega - m\Omega)t - (\vec{k} - m\vec{K}) \cdot \vec{r})}$$

and the diffraction efficiency for the  $m^{\text{th}}$  order diffracted beam is

$$\eta_m = J_m^2(\delta) = J_m^2 \left( \frac{\Delta X}{\cos \theta} \frac{\omega}{c} \Delta n_0 \right)$$

# Bessel Function Expansion

$$\eta_m = J_m^2(\delta) = J_m^2\left(\frac{\Delta X}{\cos\theta} \frac{\omega}{c} \Delta n_0\right)$$



# Example

Calculate the acoustic beam width  $\Delta X$  to maximize diffraction into the  $m=1$  order for  $\Delta n_0=0.0001$ ,  $\theta=0^\circ$  and  $\lambda=633$  nm. What fraction of the power gets diffracted into the  $m=1$  beam?

$J_1(\delta)$  is max at  $\delta \approx 1.85$ , solving for  $\Delta X$  we get

$\Delta X = 1.9$  mm

Evaluating  $\eta = J_1^2(1.85) \approx 0.33$

# References

---

- Yariv & Yeh "Optical Waves in Crystals" chapter 9