Tensor Notation

For brevity, we often express tensor equations such as

$$\vec{D} = \epsilon_0 \left(\bar{I} + \bar{\chi}\right) \vec{E}$$

in the form

$$D_i = \epsilon_0(\delta_{ij} + \chi_{ij})E_j$$

where summation over repeated indices is assumed, so that this is equivalent to

$$D_i = \sum_j \epsilon_0(\delta_{ij} + \chi_{ij})E_j$$

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Eigenstates of a material

For any direction of propagation there exists two polarization directions in which the displacement vector is parallel to the transverse component of the electric field. These are the eigenstates of the material.

Requiring $\vec{D}$ and $\vec{E}$ be parallel in the matrix equation

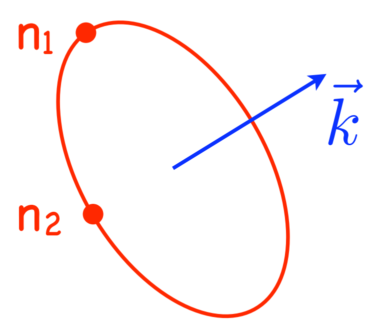
$$\vec{D} = \bar{\epsilon}\vec{E}$$

is equivalent to finding the eigenvalues and eigenvectors of ϵ satisfying

$$\lambda_i \vec{u_i} = \bar{\epsilon}\vec{u_i}$$

Index Ellipsoid

Index of refraction as a function of polarization angle



The polarization directions that have a max and min index of refraction form the major and minor axes of an ellipse defining n(θ) the index fora wave with the electric displacement vector at an angle of θ in the transverse plane.

Example

A wave is propagating in the [1,1,1] direction in Mica, what are the two principle indices of refraction and in which polarization directions do these correspond to?

plane of polarization is given by $\vec{k} \cdot \vec{r} = 0$ so x+y+z=0.
The intersection of this plane and the index ellipsoid, given by

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1$$

is,

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{(x+y)^2}{n_z^2} = 1$$

Find extremes of $r^2=x^2+y^2+z^2$ subject to the preceeding two constraints to find directions of eigenpolarizations. Plug directions back into $r=(x^2+y^2+z^2)^{1/2}$ to find index for each eigenpolarization

Example Solution in Mathematica

```
nx = 1.552;
ny = 1.582;
nz = 1.588;

In[148]:= eq1 = (x / nx) ^ 2 + (y / ny) ^ 2 + (z / nz) ^ 2 == 1;
          eq2 = x + y + z == 0;
          eq3 = r == Sqrt[x ^ 2 + y ^ 2 + z ^ 2];

In[151]:= rsol = r /. Solve[{eq1, eq2, eq3}, {x, y, r}][[1]][[1]]
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Out[151]= $\sqrt{\left(2.45481 + 0.\ z + 0.0269076\ z^2 - 0.0328601\ \sqrt{1.29082 - 1.\ z}\ z\ \sqrt{1.29082 + z}\right)}$

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In[208]:= rsolmax = Maximize[rsol, z]
          zmax = z /. rsolmax[[2]][[1]];
          rmax = rsolmax[[1]];
          rsolmin = Minimize[rsol, z]
          zmin = z /. rsolmin[[2]][[1]];
          rmin = rsolmin[[1]];

Out[208]= {1.58295, {z → -0.936293}}

Out[211]= {1.56375, {z → 0.280416}}

In[220]:= solmax = Solve[{eq1, eq2, eq3}, {x, y, r}] /. z → zmax
          xmax = x /. solmax[[1]];
          ymax = y /. solmax[[1]];
          solmin = Solve[{eq1, eq2, eq3}, {x, y, r}] /. z → zmin
          xmin = x /. solmin[[1]];
          ymin = y /. solmin[[1]];

Out[220]= {{r → 1.58295, x → -0.303467, y → 1.23976},
           {r → 1.56559, x → 1.22184, y → -0.285544}}

Out[223]= {{r → 1.56375, x → -1.21895, y → 0.938534},
           {r → 1.57116, x → 0.943902, y → -1.22432}}

In[228]:= p1 =
            {rmax, {xmax, ymax, zmax} /
              Sqrt[xmax ^ 2 + ymax ^ 2 + zmax ^ 2]}
          p2 =
            {rmin, {xmin, ymin, zmin} /
              Sqrt[xmin ^ 2 + ymin ^ 2 + zmin ^ 2]}

Out[228]= {1.58295, {-0.19171, 0.783194, -0.591485}}

Out[229]= {1.56375, {-0.779504, 0.600181, 0.179323}}
```

Index of extraordinary ray

The index of refraction of the ordinary ray is always equal to the index seen when the ray propagates along the optical axis

The extra-ordinary ray sees an index of refraction that is a function of the ray propagation angle from the optical axis. In a uniaxial crystal

$$\frac{1}{n_e^2(\theta)} = \frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2}$$

Optical Activity

The displacement vector thus has an additional contribution in a direction perpendicular to the electric field

$$\vec{D} = \epsilon \vec{E} + i\epsilon_0 \vec{G} \times \vec{E}$$

where $\vec{G} = (g_{ij} k_i k_j / k_0^2)\hat{k}$ is called the gyration vector and

$$\vec{G} \times \vec{E} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ G_x & G_y & G_z \\ E_x & E_y & E_z \end{bmatrix} = \begin{bmatrix} 0 & -G_z & G_y \\ G_z & 0 & -G_x \\ -G_y & G_x & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = [G]\vec{E}$$

so we can define an effective dielectric tensor

$\epsilon' = \epsilon + i\epsilon_0 [G]$ so that $\vec{D} = \epsilon' \vec{E}$

allowing the wave equation to be solved for eigenpolarizations of propagation and two corresponding indices of refraction that depend on [G]

Exercise

Given an isotropic material that is perturbed such that

$$\vec{D} = \epsilon \vec{E} + i \Delta\epsilon \vec{E}$$

where $\Delta\epsilon$ is

$$\Delta\epsilon = \epsilon_o \begin{bmatrix} 0 & -G_z & G_y \\ G_z & 0 & -G_x \\ -G_y & G_x & 0 \end{bmatrix}$$

find the eigenpolarizations for the electric field and associated indices of refraction. If a linearly polarized plane wave were to propagate along the direction of G through a thickness d of such material, what would the output wave look like?