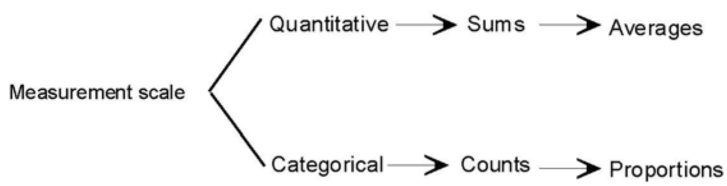
9: Inference about a Proportion

Binary response

The past series of chapters have focused on quantitative outcomes. This chapter addresses categorical outcomes with two possible values ("binary variables"). For example, classifying someone as a smoker or non-smokers is a binary variable.

Whereas quantitative variable were summarized with sums and averages, categorical variables are summarized with counts and proportions.



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The symbol $\hat{p}$ ("p-hat") is used to represent the **sample proportion**:

$$\hat{p} = \frac{\text{number of successes in the sample}}{n}$$

Illustrative example: *Smoking survey*. We select a SRS of 57 individuals. The sample has 17 smokers. Therefore, the sample proportion is $\hat{p} = 17 / 57 = 0.298$, or 29.8%. The goal of this chapter is to use this information to infer the proportion of people in the population who smoke.

Notes:

1. Proportions are a type of average in which "successes" are given a value of 1 and "failures" are given a value 0. For example, if we have 10 observations as follows $\{0, 0, 0, 1, 0, 0, 0, 0, 1, 0\}$, $n = 10$, $\sum x_i = 2$, and sample mean $\bar{x} = \frac{2}{10} = \hat{p}$.

   Principles applied in using $\bar{x}$ to infer population mean μ transfer to using sample proportion $\hat{p}$ in inferring population proportion p.

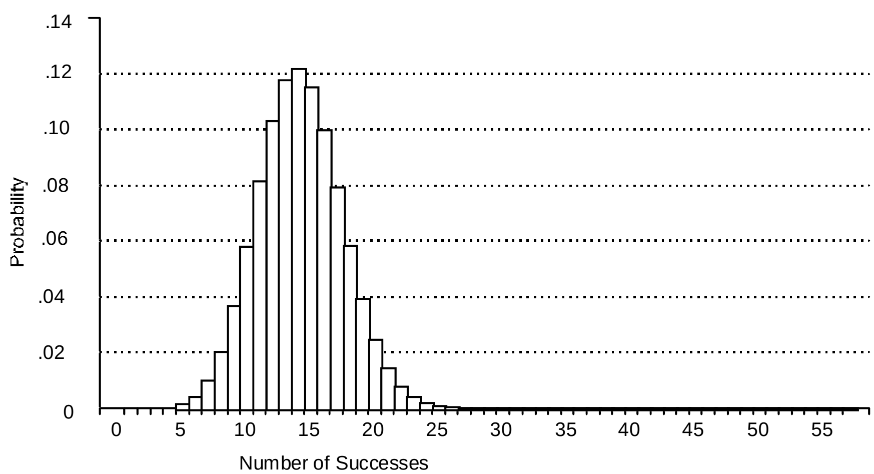
2. Sample proportions are used to estimate population prevalences and incidences. Prevalence ≡ the proportion in a cross-sectional sample and cumulative incidence ("risk") ≡ the proportion of susceptible in a cohort who develop a condition over a fixed period of time.

Inferring population proportion *p* (Normal approximation)

Let *p* represent the proportion in the population. Sample proportion $\hat{p}$ is an unbiased estimator of parameter *p*. Keep in mind that sample proportion $\hat{p}$ in any given sample will not be an exact replica of population proportion *p*; some of the $\hat{p}$ s will be less than *p*, and some will be more. That is the nature of sampling. Over the long run, with repeated independent samples, $\hat{p}$ is an unbiased estimator of *p*.

Inferences about parameter *p* rest on binomial distributions (Chapter 4). Binomial probabilities can be tedious to calculates so, when *n* is large, a Normal approximation to the binomial is used. The Normal approximation to the binomial says that the number of successes in a sample will have a Normal distribution with μ = *np* with standard deviation $\sigma = \sqrt{npq}$ where $q = 1 - p$. Equivalent, when *n* is large, the sample proportion $\hat{p}$ will vary according to a Normal distribution with expected value p and standard error $SE_{\hat{p}} = \sqrt{\frac{pq}{n}}$.

Here is the binomial sampling model of the number of successes for a binomial random variable *X* with *n* = 57 and *p* = 0.25:



This distribution is nearly Normal. The random number of success *X*~N(14.25, 3.27). It in addition, the sampling distribution of the proportion $\hat{p}$ ~*N*(0.25, 0.0574). This is pretty advances stuff, but for now please note these Normal approximation hold when $npq \geq 5$ (so-called *npq* rule). For the model above, *n* = 57 and *p* = .25, so *npq* = (57)(0.25)(1–0.25) = 10.6875. Since this exceeds 5, we can predict that the Normal approximation to the binomial can be trusted.

Confidence interval for *p*

A method called the "plus-four' method is used to calculate the confidence interval of *p*. This method is a modification of the standard Normal method, but is much more reliable, especially when *n* is small, providing reliable results even when *n* is as small as 10.

The general idea is to add two "successes" and two "failures" to the data before calculating the confidence interval. Then, the typical "estimate ± z · standard error" formula is applied. Let $\tilde{x}$ ≡ the observed number of success plus two = $x + 2$, $\tilde{n}$ ≡ the sample size plus four = $n + 4$, and $\tilde{p} = \frac{\tilde{x}}{\tilde{n}}$. The (1−α)100% confidence interval for *p* is

$$\tilde{p} + z_{1-\frac{\alpha}{2}} \cdot se_{\tilde{p}}$$

where $se_{\tilde{p}} = \sqrt{\frac{\tilde{p}\tilde{q}}{\tilde{n}}}$.

Use $z = 1.645$ for 90% confidence, $z = 1.96$ for 95% confidence, and $z = 2.576$ for 99% confidence.

Illustrative example: ***confidence interval for proportion p.*** In the smoking prevalence illustrative example $n = 57$ and $x = 17$. What is the 95% confidence interval for population prevalence *p*?

$\tilde{x} = x + 2 = 17 + 2 = 19$
$\tilde{n} = n + 4 = 57 + 4 = 61$
$\tilde{p} = \frac{\tilde{x}}{\tilde{n}} = \frac{19}{61} = 0.3115$
$\tilde{q} = 1 - \tilde{p} = 1 - 0.3115 = 0.6885$
$se_{\tilde{p}} = \sqrt{\frac{(0.3115)(0.6885)}{61}} = 0.0593$

The 95% confidence interval for *p*
= 0.3115 ± (1.96)(0.0593)
= 0.3115 ± 0.1162
= 0.1953 to 0.4277 or between 20% and 43%.

Sample Size Requirements to Limit Margin of Error

In planning a study, we want to collect enough data to estimate population proportion *p* with adequate precision. In an earlier chapter we had determined the sample size to determine population mean μ with margin of error *d*. We apply a similar method in determining sample size requirements to estimate population proportion *p*.

Let *d* represent the margin of error. This provides the "wiggle room" around $\hat{p}$; it is half the confidence interval width. To achieve margin of error *d* use

$$n = \frac{z^2_{1-\frac{\alpha}{2}} p^* q^*}{d^2}$$

where p^* represent the an educated guess for the proportion and $q^* = 1 - p^*$.

When no reasonable guess of *p* is available, use $p^* = 0.50$ to provide a "worst-case scenario" sample size (i.e., more than enough data).

Illustrative example: *Smoking survey, sample size requirements for confidence interval*. Recall the "Smoking survey" illustrative example presented earlier in the chapter. We want to re-sample the population and calculate a 95% confidence interval with greater precision. How large a sample is needed to shrink the margin of error in the "Smoking survey illustrative data" to 0.05? How large a sample is needed to shrink the margin of error to 0.03? The prior sample had $\hat{p} = 0.30$, so let's use this for p^*.

Solutions:

To achieve a margin of error of 0.05, $n = \frac{z^2_{1-\frac{\alpha}{2}} p^* q^*}{d^2} = \frac{1.96^2 \cdot 0.30 \cdot 0.70}{0.05^2} = 322.7$. Round this up to 323 to ensure adequate precision.

To achieve a margin of error of 0.05, $n = \frac{1.96^2 \cdot 0.30 \cdot 0.70}{0.03^2}$. = 896.4, so use 897 individuals.

The increased precision has the price of a larger sample size.

Hypothesis test (Normal approximation)

Let p_0 denote the value of population proportion p under the null hypothesis. Before beginning the test, check to see check whether a Normal approximation can be used by checking whether $np_0q_0 \geq 5$, where $q_0 = 1 - p_0$. If $np_0q_0 < 5$, an "exact" binomial test is required. We do not cover the exact binomial test.

(A) Hypotheses: The null hypothesis is H_0: $p = p_0$, where p represents the population proportion and p_0 is its expectation under the null hypothesis. The alternative hypothesis is either H_1: $p \neq p_0$ (two-sided), H_1: $p < p_0$ (one-sided to the left), or H_1: $p > p_0$ (one-sided to the right).

(B) Test statistic: The test statistic is $z_{stat} = \frac{\hat{p} - p_0}{SE_{\hat{p}}}$ where $\hat{p}$ represents the sample proportion, p_0 = the null value, and $SE_{\hat{p}} = \sqrt{\frac{p_0 q_0}{n}}$.

(C) *P*-value: The z_{stat} is converted to a *P*-value in the usual fashion. Small *P*-values provide strong evidence against H_0.

(D) Significance statement (optional). Reject H_0 when $P \leq \alpha$**.** in which case the difference is said to be significant.

Illustrative example. The prevalence of smoking in U. S. adults is approximately 25% (NCHS, 1995, Table 65). We observe 17 smokers in 57 individuals. Therefore, $\hat{p}$ = 29.8%. Does this provide significant evidence that the population from which the sample was drawn has a prevalence that exceeds the national average? Let's do a two-sided test.

Under the null hypothesis $p_0 = 0.25$. Before conducting the test we check whether the Normal approximation to the binomial holds by calculating $np_0q_0 = (57)(.25)(1 - .25) = 10.7$. We can proceed with a Normal approximation test.

(A) H_0: $p = .25$ versus H_1: $p \neq .25$.
(B) $SE_{\hat{p}} = \sqrt{\frac{(.25)(1-.25)}{57}} = .0574$ and $z_{stat} = \frac{.298 - .25}{.0574} \approx 0.84$.
(C) $P = 0.4010$. This does *not* provide strong evidence against H_0.
(D) $P > \alpha$; H_0 is retained. The difference is not significant.