Section 2.1 Sample Spaces and Events

Section 2.2 Axioms, Interpretations, and Properties of Probability
Introduction

To model a random phenomenon (such as flipping a coin, rolling a die), we need to specify the following components:

- Sample space
- Events
- Probability

Collectively, they define a probability space.

We’ll go through the above concepts one by one.
A little bit set terminology and notation:

- **A set** is an unordered collection of objects: \( A = \{1, 2, 3\} = \{3, 1, 2\} \), \( B = (1, 3) = \{x \mid 1 < x < 3\} \), \( C = [0, \infty) = \{x \mid x \geq 0\} \)

- **Notation**: To indicate an object is in or not in the set: \( 1 \in A \), \( 4 \notin A \)

- **A subset** is a subcollection of the objects: \( \{1\} \subset A \subset C \)

- **The size** (cardinality) of a set is the number of objects in it: \(|A| = 3\)

- **Special sets**: \( \emptyset \) (empty set), \( \mathbb{N} = \{1, 2, 3, \ldots\} \), \( \mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\} \), \( \mathbb{R} = (-\infty, \infty) \)
Sample space

Def 0.1. The set of all possible outcomes of a random phenomenon is called the sample space for that experiment.

- We denote a sample space by $S$ (some books use $\Omega$ instead).

- A sample space is typically represented by a rectangle, and the outcomes of the sample space are denoted by points within the rectangle.
Example 0.1. Write down the sample space of each of the following experiments:

- Tossing a coin: \( S = \{H, T\} \).
- Rolling a die: \( S = \{1, 2, 3, 4, 5, 6\} \).
- Drawing a card from an ordinary deck of 52: \( S = \{\text{All 52 cards}\} \).
Example 0.2. Write down the sample space of each of the following experiments:

- Throw a coin twice:
  \[ S = \{(H, H), (H, T), (T, H), (T, T)\}. \]

- Roll two dice:
  \[ S = \{(1, 1), \ldots, (1, 6), (2, 1), (2, 2), \ldots, (6, 6)\} \quad \text{← by enumeration} \]
  \[ = \{(i, j) \mid 1 \leq i \leq 6, 1 \leq j \leq 6\} \quad \text{← by formula} \]

- Throw a coin repeatedly until a head first appears:
  \[ S = \{H, TH, TTH, TTTH, \ldots\} \]
The sample spaces in the previous example are **discrete** sets, which are **countable**. That is, the number of objects in the set must be

- **finite** (e.g., \{1, 2, \ldots, 6\}), or
- **countably infinite**: There is a 1-to-1 correspondence to the set of natural numbers, \( \mathbb{N} = \{1, 2, 3, \ldots\} \).

For example, the set of all integers \( \mathbb{Z} \) is countable:

\[
\begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \ldots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \ldots \\
0 & 1 & -1 & 2 & -2 & 3 & -3 & 4 & -4 & 5 & \ldots \\
\end{array}
\]

In contrast, the set of all real numbers \( \mathbb{R} \) is uncountable.
In the following example, the sample spaces are continuous intervals (which are uncountable).

**Example 0.3.**

- Life time of a new light bulb. The sample space is an interval $S = (0, \infty)$.
- Waiting time (in minutes) to talk to a customer service representative: $S = (0, \infty)$
- Throw a dart to a unit disk and measure its distance to center: $S = [0, 1]$
Events

Consider the following probability questions about events:

- (Toss two fair dice) What is the probability of getting a sum of 8?
- (Toss two fair dice) What is the probability of getting two even numbers?
- (Toss two fair dice) What is the probability of getting two identical numbers?
- (Toss a fair coin repeatedly until a head first appears) What is the probability that at most 3 tails are observed?
Def 0.2. Mathematically, an event is just a subset $E$ of outcomes in the sample space $S$.

- In particular, $S, \emptyset$ are events.

- We say an event $E$ occurred if the actual outcome of the experiment lies in $E$.

- It is called a simple event if it contains only one outcome. Otherwise, it is called a compound event.

(Event $E$ occurred, while $F$ did not)
Example 0.4 (Roll a single die). The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. The following are events:

- $A = \{1\} = \{\text{The smallest number}\} \quad \leftarrow \text{simple event}$
- $B = \{6\} = \{\text{The largest number}\} \quad \leftarrow \text{simple event}$
- $C = \{2, 4, 6\} = \{\text{An even number}\} \quad \leftarrow \text{compound event}$
- $D = \{1, 3, 5\} = \{\text{An odd number}\} \quad \leftarrow \text{compound event}$

If an outcome of 1 was observed when performing the experiment, then which of the above events occurred (and which of them did not occur)?
Example 0.5 (Throw two dice). The sample space is

\[ S = \{(1, 1), (1, 2), \ldots, (6, 6)\} = \{(i, j) \mid 1 \leq i, j \leq 6\}. \]

The following are events:

\[ A = \{\text{Sum equals 6}\} \]
\[ = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\} \]

\[ B = \{\text{Two identical numbers}\} \]
\[ = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\} \]

\[ C = \{\text{Two even numbers}\} \]
\[ = \{(2, 2), (2, 4), (2, 6), (4, 2), (4, 4), (4, 6), (6, 2), (6, 4), (6, 6)\}. \]
Example 0.6. Consider the experiment where you repeatedly toss a coin until you see the first head. The following is an event:

\[ E = \{ \text{At most 4 tails are obtained} \} = \{ H, TH, TTH, TTTTH, TTTTTTH \} \]
Event operations

Def 0.3. Let $A, B \subseteq S$ be two events. We define

- **Complement** $A^c$: set of all outcomes not in $A$
- **Union** $A \cup B$: set of all outcomes in $A$ or $B$ (or both)
- **Intersection** $A \cap B$: set of all outcomes in both $A$ and $B$
- **Difference** $A - B = A \cap B^c$: set of all outcomes in $A$ and not in $B$

They can be represented by the so-called Venn diagrams (see next slide).
Probability basics

$A \cup B$

$A \cap B$

$A^c$

$A - B$
Example 0.7 (Throw two dice). Let

- $A = \{\text{Sum equals 6}\}$
- $B = \{\text{Two identical numbers}\}$
- $C = \{\text{Two even numbers}\}$

Compute $|C|, A \cap B, A \cup B, B^c, A - C$
Two useful set laws:

- **Distributive law:**
  \[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]

- **De Morgan’s Laws**
  \[
  (A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c
  \]
Disjoint events

Def 0.4. Two events $A$, $B$ are said to be disjoint, or mutually exclusive, if their intersection is empty: $A \cap B = \emptyset$.

A sequence of events $E_1, E_2, \ldots$ are said to be pairwise disjoint, or mutually exclusive, if $E_i \cap E_j = \emptyset$ for all $i \neq j$. 
Example 0.8 (Toss two fair dice). Are the following two events disjoint?

- $A = \{\text{Sum equals 7}\}$.
- $B = \{\text{Two identical numbers}\}$.
Probability basics

Probability

Intuitively, probability is a number \( P(E) \) describing the chance of an event \( E \) occurring.

The larger the probability, the more likely for the event to occur.

And it needs to satisfy certain conditions in order to be valid/meaningful.
Below is the formal definition of probability.

**Def 0.5.** Probability is a function defined on the space of events that satisfies the following Kolmogorov Axioms of Probability:

1. $P(E) \geq 0$ for any $E \subseteq S$.
2. $P(S) = 1$.
3. For any infinite sequence of pairwise disjoint events $E_1, E_2, \ldots$,

$$P \left( \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} P(E_i).$$
Theorem 0.1. The Axioms of Probability imply the following are true:¹

- $P(\emptyset) = 0$.

- If $E_1, E_2, \ldots, E_k \subset S$ are pairwise disjoint, then

$$P\left(\bigcup_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} P(E_i)$$

- $P(E^c) = 1 - P(E)$, from which we obtain that $P(E) \leq 1$.

- $P(B - A) = P(B) - P(B \cap A)$: If $A \subseteq B$, then it simplifies to $P(B - A) = P(B) - P(A)$.

¹This is why we did not include these properties in the definition of probability.
**Countable sample spaces**

The following property implies that, to define the probability function over a **countable** sample space, it suffices to specify only the probabilities of simple events.

**Theorem 0.2.** If the sample space \( S \) is countable, then for any event \( A \subseteq S \),

\[
P(A) = \sum_{a \in A} P(\{a\}).
\]
Example 0.9 (Fair coin model). Let $S = \{H, T\}$ with $P(\{H\}) = P(\{T\}) = .5$.

Example 0.10 (Biased coin model). Let $S = \{H, T\}$ with $P(\{H\}) = .55$, $P(\{T\}) = .45$.

Example 0.11 (Fair die model). Let $S = \{1, 2, \ldots, 6\}$ with $P(\{1\}) = P(\{2\}) = \cdots = P(\{6\}) = \frac{1}{6}$. The probability of getting an even number is

$$P(\{\text{An even number}\}) = P(\{2\}) + P(\{4\}) + P(\{6\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}.$$
Finite sample spaces with equally likely outcomes

*Theorem 0.3.* If $|S| < \infty$ (i.e., $S$ is a finite set) and all the outcomes are equally likely to occur, then for any event $A \subseteq S$,

$$P(A) = \frac{|A|}{|S|} = \frac{\# \text{ outcomes in } A}{\# \text{ outcomes in } S}.$$  

*Proof.* By the preceding theorem,

$$P(A) = \sum_{a \in A} P(\{a\}) = \sum_{a \in A} \frac{1}{|S|} = \frac{1}{|S|} \cdot |A| = \frac{|A|}{|S|}.$$
Joke: What is a probability to meet a dinosaur?

A: What is a probability to meet a dinosaur on the street?

B: Well, 50x50!

A: How, why???

B: You either meet it or not!

So, i met it!
Example 0.12 (Throw a fair die). Find the following probabilities:

\[ P(\{\text{An even number}\}) = \]

\[ P(\{\text{At least 5}\}) = \]

\[ P(\{\text{Not a 3}\}) = \]
**Example 0.13** (Throw two fair dice). Find the following probabilities:

\[ P(\{\text{Sum equals 6}\}) = \]

\[ P(\{\text{Two identical numbers}\}) = \]

\[ P(\{\text{Both even}\}) = \]
Example 0.14 (Toss a fair coin 5 times). What is the probability of getting at least one head?
**Theorem 0.4.** For any $A, B \subseteq S$,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

In particular, if $A \cap B = \emptyset$, then

$P(A \cup B) = P(A) + P(B)$.

**Proof.** By additivity for mutually exclusive events,

$P(A \cup B) = P(A - B) + P(A \cap B) + P(B - A)$

$= P(A) - P(A \cap B) + P(A \cap B) + P(B) - P(B \cap A)$

$= P(A) + P(B) - P(A \cap B)$
Example 0.15. In a large discrete math class, 55% of the students have a major in math, and 35% of the class have a major in CS. Among the two groups of students combined, 5% of them are dual majors (in math and CS). What is the probability that a randomly selected student from the class majors in

(a) at least one of math and CS,

(b) one and only one of math and CS,

(c) neither math nor CS?
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Inclusive-exclusive formula (3 events)

Theorem 0.5. For any three events $A, B, C \subseteq S$, we have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$
Example 0.16 (Select an integer from \{1, \ldots, 100\} at random). What is the probability that it is divisible by at least one of the three prime numbers 2, 3, 5? (Answer: .74)
Summary

We first introduced the concept of a \textbf{probability space} associated to a random phenomenon, which consists of the following:

- \textbf{Sample space} $S$ (set of all possible outcomes)
- \textbf{Events} $E \subseteq S$ (subsets of outcomes, often with a common trait)
- \textbf{Probability} (chance that an event occurs): a mapping from events to numbers, $P : E \subseteq S \mapsto P(E) \in \mathbb{R}$, that satisfies the three Axioms of Probability

1. $P(E) \geq 0$ for any $E \subseteq S$. 
2. \( P(S) = 1 \).

3. If an infinite sequence of events \( E_1, E_2, \ldots \) are pairwise disjoint, then

\[
P \left( \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} P(E_i).
\]

The Axioms imply more properties for the probability function:

- \( P(\emptyset) = 0 \).
- If \( E_1, E_2, \ldots, E_k \) are pairwise disjoint, then

\[
P \left( \bigcup_{i=1}^{k} E_i \right) = \sum_{i=1}^{k} P(E_i)
\]
• \( P(E^c) = 1 - P(E) \), from which we obtain that \( P(E) \leq 1 \).

• If \( A \subseteq B \), then \( P(A) \leq P(B) \). This is due to the property \( P(B - A) = P(B) - P(A \cap B) \).

• Inclusive-exclusive formula for any two events \( A, B \subseteq S \):

\[
P(A \cup B) = P(A) + P(B) - P(A \cap B)
\]

• Inclusive-exclusive formula for any three events \( A, B, C \subseteq S \):

\[
P(A \cup B \cup C') = P(A) + P(B) + P(C') - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C').
\]
Lastly, there are two special settings:

- If the sample space $S$ is countable, then for any event $A \subseteq S$,
  \[ P(A) = \sum_{a \in A} P(\{a\}) . \]

- If the sample space is finite and all the outcomes are equally likely to occur, then for any event $A \subseteq S$,
  \[ P(A) = \frac{|A|}{|S|} = \frac{\# \text{ outcomes in } A}{\# \text{ outcomes in } S} . \]