Worksheet 1b: system of linear equations

Example 0.5. Describe the solution sets of the homogeneous system (of only one equation)

\[ x_1 - 3x_2 + 2x_3 = 0 \]

and the nonhomogeneous system

\[ x_1 - 3x_2 + 2x_3 = 1 \]
as well as their relationship.

Example 0.6. Determine if the following vectors are linearly independent:

\[ v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \]

Example 0.7. Determine in each case if the vectors are linearly independent.

1. \[ v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \]
2. \[ v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]
3. \[ v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 7 \end{bmatrix} \]
4. \[ v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \]
Example 0.8. Consider the transformation

\[ T : \mathbb{R} \mapsto \mathbb{R}, \quad \text{with} \quad T(x) = x^2. \]

Determine the domain, co-domain (target space), and range of \( T \).

Example 0.9. Let \( A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \). Then the matrix \( A \) may be used to define a transformation

\[ T : \mathbb{R}^3 \mapsto \mathbb{R}^2, \quad \text{with} \quad T(x) = Ax. \]

Answer the following questions:

- What are the domain and co-domain of \( T \)?
- What is the image of \( x = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \)?
- Which points in \( \mathbb{R}^3 \) have an image of \( o \in \mathbb{R}^2 \)?
- What is the range of \( T \)?

Example 0.10. Determine the linear transformation that maps the points \((2, 0), (1, 1)\) in \( \mathbb{R}^2 \) to \((-1, 0), (0, -1)\) in \( \mathbb{R}^2 \), respectively.

Example 0.11. Determine in each case, if the linear transformation \( T(x) = Ax \) is one-to-one, or onto, or both.

- \( A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \)
- \( A = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \)
- \( A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \)