Lesson Plan

Lesson: Parametric Surfaces and Their Areas

Timeframe: 1:50 min

10 min	10 min	40 min	10 min	30 min	10 min
Q/A	Quiz	Lecture	Break	Worksheet	Summary

Materials needed: Paper, Pencil, Textbook, Computer (to watch videos and to do homework).

Objectives:

Basic:(to be practiced prior to class)

1. Identify and describe special types of surfaces:

   Cylinders, Quadric surfaces, Graphs of functions of two variables (i.e. Plane, Paraboloid) (Ch. 10.6, Pg. 574).

2. Recall vector functions (or parametric functions) (Ch. 10.7, Pg. 580)
3. Classify two types of surface parameterizations
4. Express each surface as a parametric function

Advanced: (to be mastered during the class or after class)

1. Distinguish the type of parameterization for given surface
2. Rewrite each surface as a vector function
3. Evaluate normal vector for each surface
4. Compute surface area of each surface

Background:

We describe a space curve by a vector function r(t) of a single parameter t. Similarly, we can describe a surface by a vector function r(u, v) of two parameters u and v.

Introduction to Lesson: In this section, we will learn about: Various types of parametric surfaces and computing their areas using vector functions.

Procedure [Time needed, include additional steps if needed]:

Pre-Class Individual Space Activities and Resources:

Steps	Purpose	Estimated Time	Learning Objective
Step 1: Review section 10.6 on "Cylinders and Quadratic Surfaces" Pg. 574-579.	To list all surfaces with the equations and graphs for your reference.	15 min	Surfaces
Step 2: Review section 10.5 on "Vector Functions" Pg. 580-582.	To recall what we call vector functions	10 min	Vector functions
Step 3: Watch following videos: https://www.bing.com/videos/search?q=Surface+parametrization+you+tube&view=detail&mid=10251000323E4A33FDE910251000323E4A33FDE9&FORM=VIRE OR Parametric Surfaces.mp4 attached here.	To determine parametric equations for a surface	6 min 56 sec	Parameterize a surface
Step 4: Study section 13.6 on "Parametric Surfaces and Their Area" Pg. 797-804 of the textbook.	To learn how to parameterize surface and how to find normal vector for each surface	1 hour	Parametric Surfaces and Their Areas

Step 5: Study Parametric Surfaces.ppt attached here.	To learn how to parameterize surface and how to find normal vector for each surface	1 hour	Parametric Surfaces and Their Areas

In-Class Group Space Activities and Resources:

Steps	**Purpose**	**Estimated Time**	**Learning Objective**
Step 1:Q/A on what you must study before coming to class		10 min	
Step 2: Parametric Surface Pre-Quiz on "Step 3: watch the video"		10 min	
Step 3: Lecture on two types of parameterization for special surfaces. Find the normal vector for each case. Evaluate surface area for different examples (Paraboloid, Sphere).		40 min	

Step 4: Parametric Surface Worksheet.		30 min	
Step 5: Hints to complete Parametric Surface Summary		10 min	

Closure/Evaluation:

Analysis: Try Parametric Surface Post-Quiz (20 minutes).

Post-Class Individual Space Activities:

1. ***Complete the Summary page of section 13.6 (posted on blackboard) due the next session***
2. ***Webassign Homework Section 13.6 due next week***

Connections to Future Lesson Plan(s): In the next section, we will learn about: Integration of different types of surfaces. The relationship between surface integrals and surface area is much the same as the relationship between line integrals and arc length. Compare the line integral formula:

$$\int_C f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))\,|\,\mathbf{r}'(t)\,|\,dt$$

with surface integral of *f* over parameterized surface S by vector function r(*u*, *v*)

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))\,|\,\mathbf{r}_u \times \mathbf{r}_v\,|\,dA$$

Parametric Surfaces
Pre-Quiz

10 minutes

1. What does it mean for a parametrization $\mathbf{r}(u,v) =< x(u,v), y(u,v), z(u,v) >$ of a surface to be smooth?

2. What does it mean for a vector field $\mathbf{F}$ to be conservative?

3. Consider the parametric surface defined by $\mathbf{r}(u,v) =< u^2, 3u^2v, \frac{1}{v} >$. Find the equation of the plane tangent to this surface when (u,v) = (3,2).

Parametric Surfaces
Post-Quiz

20 minutes

1. Find the surface are of the portion of the cone $z^2 = 4x^2 + 4y^2$ that is above the region in the first quadrant bounded by the line y = x and the parabola $y = x^2$.

2. Use an appropriate integral to calculate the volume of the solid in the first octant bounded by the coordinate planes and the plane $2x + y + z = 2$.

Parametric Surfaces
Worksheet

30 minutes

1. Let σ be the surface parametrized by

$$\mathbf{r}(u,v) = \langle \cos u, \quad \sin u, \quad v \rangle \; ; \quad \begin{array}{l} 0 \le u \le 2\pi \\ 0 \le v \le 1 \end{array}$$

   (a) Sketch σ in 3-space.

   (b) Evaluate $\iint_\sigma z \, dS$

2. Let σ be the unit sphere. Find the surface area of σ by appropriate parametrization.

Chapter 13.6

Tuesday, July 25, 2017 11:14 AM

Summary of 13.6

There are two different ways to parametrize a given surfaces. 1) Describe each of the following as a parametric equation. 2) Find the normal vector for each surface. 3) What will be the magnitude of this normal vector?

1. If the surface can be considered as a graph of two variables functions like:
   a. Plane
   b. Paraboloid

2. If the surface can be easily described in a new coordinate system like:
   a. Sphere in spherical system
   b. Cylinder in cylindrical system
   c. Cone in spherical system

So far, we have considered special types of surfaces:

- Cylinders
- Quadric surfaces
- Graphs of functions of two variables

Here, we use vector functions to describe more general surfaces, called parametric surfaces, and compute their areas.

13.6

Parametric Surfaces and their Areas

In this section, we will learn about:

Various types of parametric surfaces

and computing their areas using vector functions.

INTRODUCTION

We describe a space curve by a vector function **r**(t) of a **single parameter** $\boldsymbol{t}$.

Similarly, we can describe a **surface** by a vector function **r**(u, v) of **two parameters** u and v.

We suppose that

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z\,(u, v)\,\mathbf{k}$$

is a vector-valued function defined on a region *D* in the *uv*-plane.

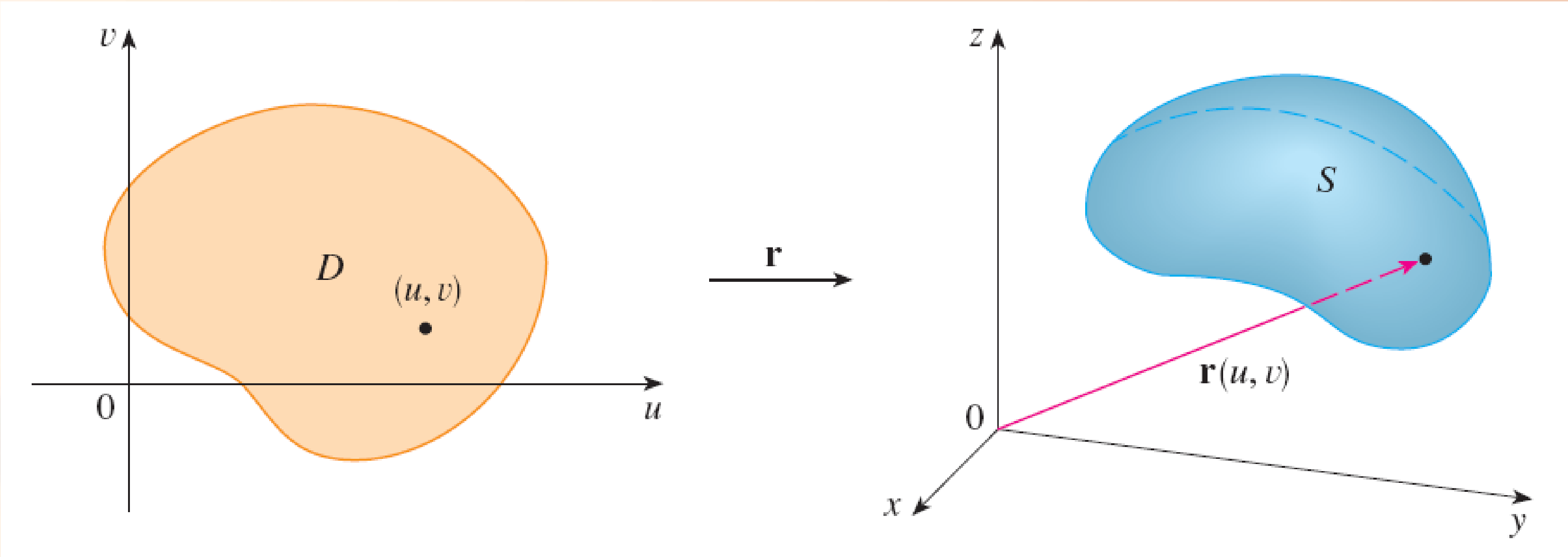
The set of all points (x, y, z) in R^3 such that

$$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

and (u, v) varies throughout D, is called a parametric surface S.

- Equations 2 are called parametric equations of S.

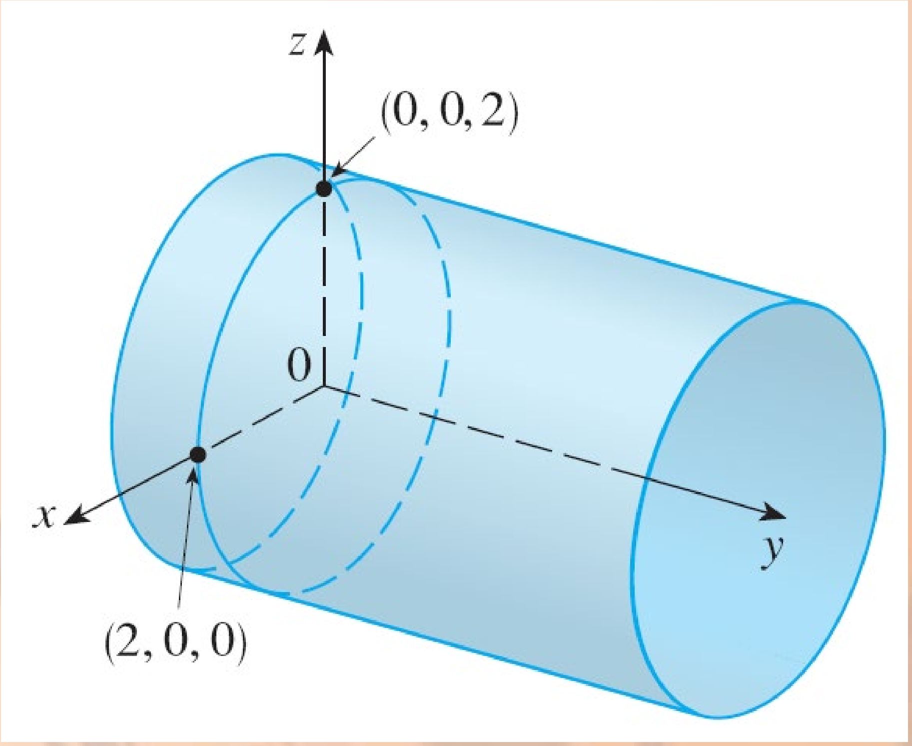
The surface S is traced out by the tip of the position vector $\mathbf{r}(u, v)$ as (u, v) moves throughout the region D.



Identify and sketch the surface with vector equation

$$\mathbf{r}(u, v) = 2 \cos u\, \mathbf{i} + v\, \mathbf{j} + 2 \sin u\, \mathbf{k}$$

Since $y = v$ and no restriction is placed on v, the surface is a circular cylinder with radius 2 whose axis is the y-axis.



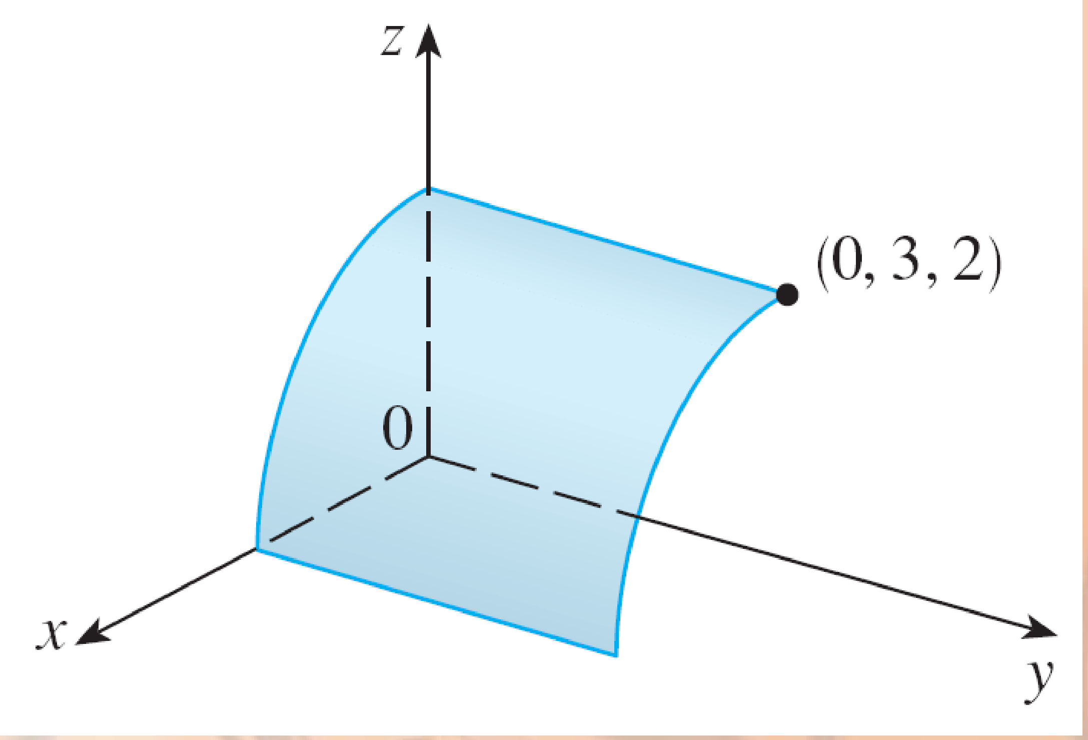
If **we restrict** *u* and *v* by writing:

$$0 \leq u \leq \pi/2 \qquad 0 \leq v \leq 3$$

then

$$x \geq 0 \qquad z \geq 0 \qquad 0 \leq y \leq 3$$

In that case, we get the quarter-cylinder with length 3.

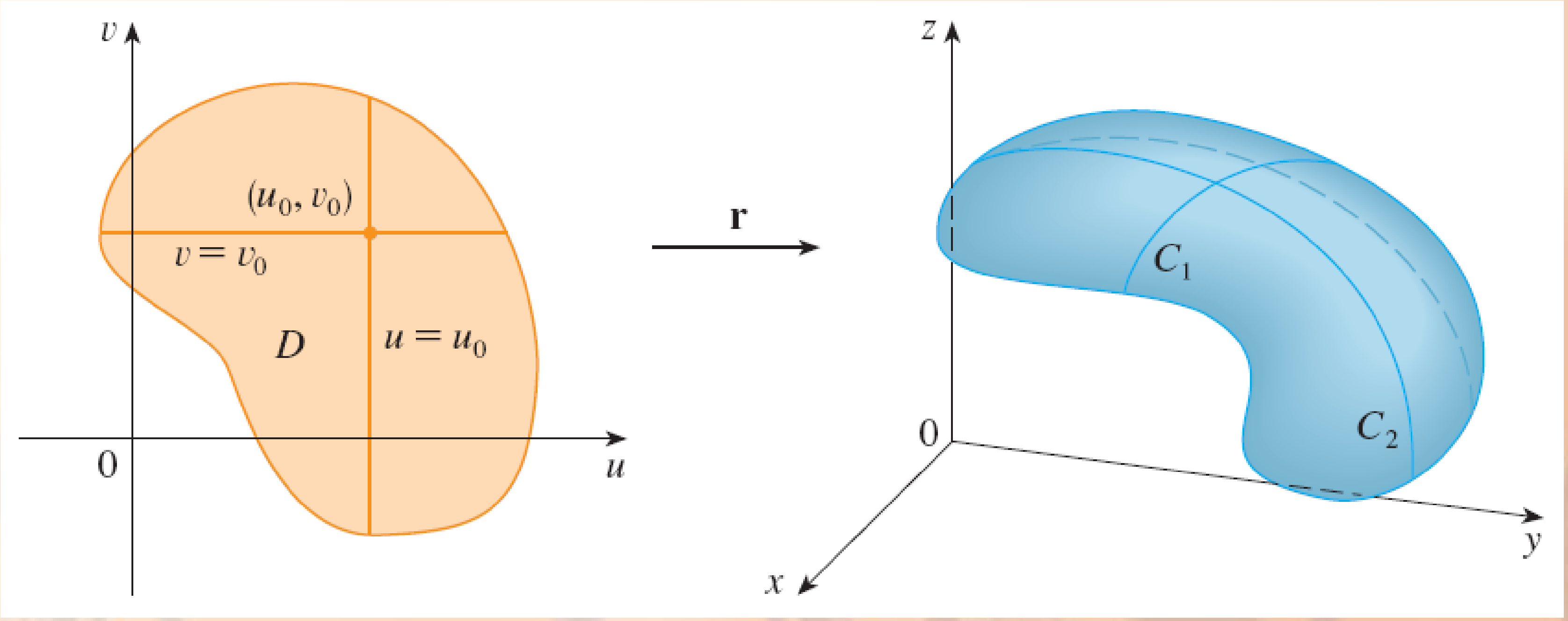


PARAMETRIC SURFACES

If a parametric surface *S* is given by a vector function **r**(*u*, *v*), then there are two useful families of curves that lie on *S*—one with *u* constant and the other with *v* constant.

- These correspond to vertical and horizontal lines in the *uv*-plane.

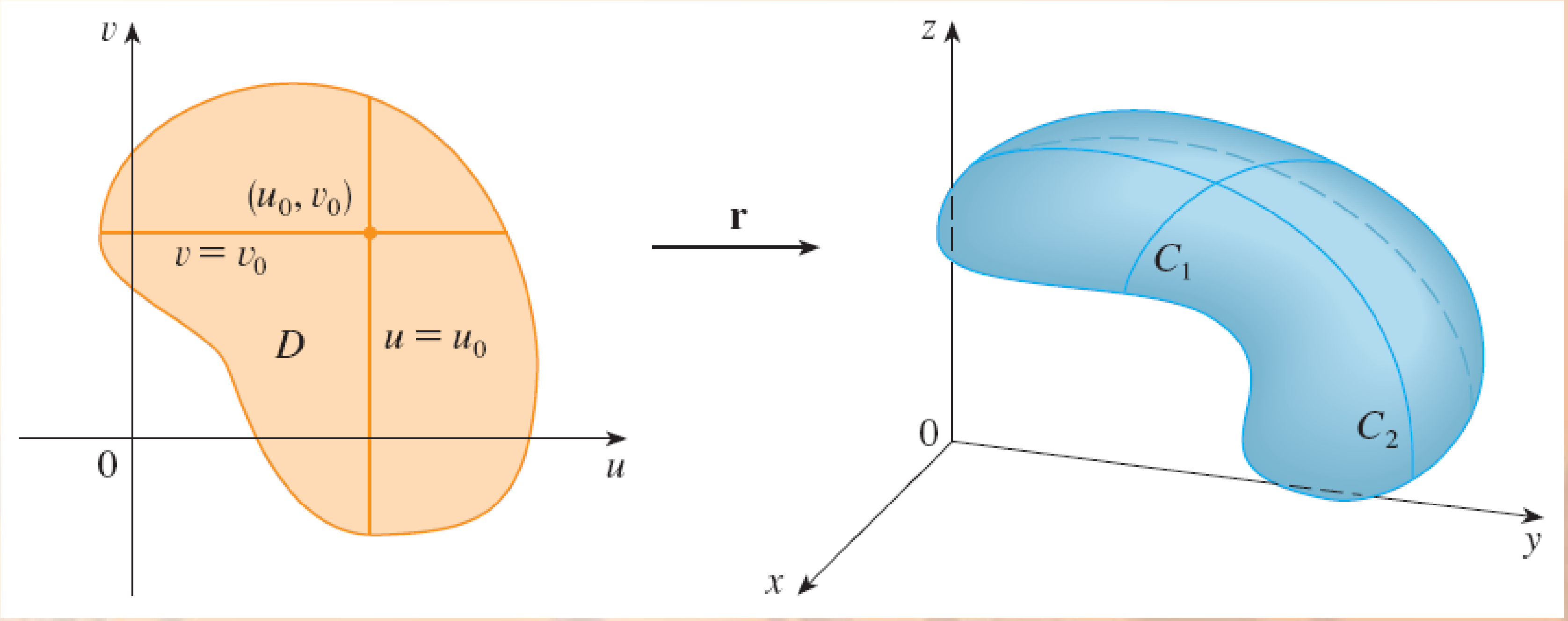
Keeping u constant by putting $u = u_0$, $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a curve C_1 lying on S.



GRID CURVES

Similarly, keeping v constant by putting $v = v_0$, we get a curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S.

- We call these curves grid curves.



Use a computer algebra system to graph the surface

$$\mathbf{r}(u, v) = \langle (2 + \sin v) \cos u, (2 + \sin v) \sin u, u + \cos v \rangle$$

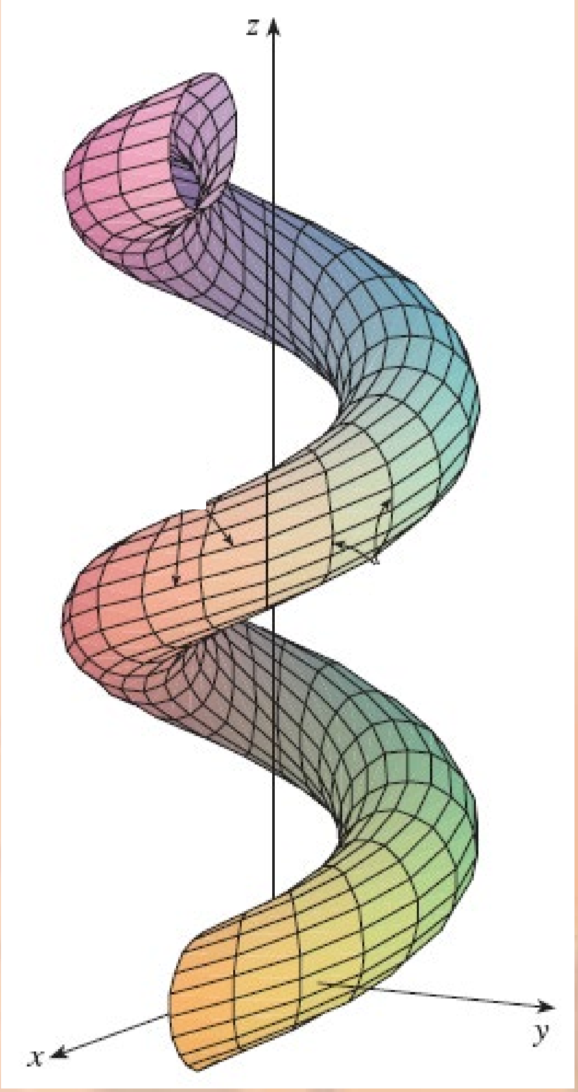
- Which grid curves have *u* constant?
- Which have *v* constant?

Example 2

We graph the portion of the surface with parameter domain

$$0 \le u \le 4\pi,\ 0 \le v \le 2\pi$$

- It has the appearance of a spiral tube.



PARAMETRIC REPRESENTATION

In Examples 1 and 2 we were given a vector equation and asked to graph the corresponding parametric surface.

- In the following examples, however, we are given the more challenging problem of finding a vector function to represent a given surface.
- In the rest of the chapter, we will often need to do exactly that.

PARAMETRIC REPRESENTATIONS **Example 4**

The vector **equation of the sphere**

$$x^2 + y^2 + z^2 = a^2$$

is given by:

$\mathbf{r}(\Phi, \theta) = a \sin \Phi \cos \theta \, \mathbf{i} + a \sin \Phi \sin \theta \, \mathbf{j} + a \cos \Phi \, \mathbf{k}$

- where $0 \leq \Phi \leq \pi$ and $0 \leq \theta \leq 2\pi$.

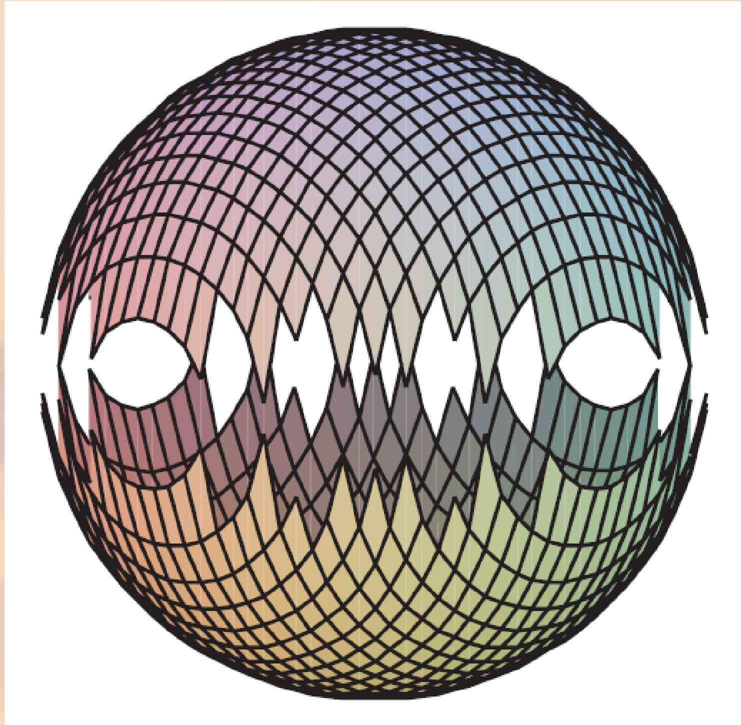
APPLICATIONS—COMPUTER GRAPHICS

One of the uses of parametric surfaces is in computer graphics.

The figure shows the result of trying to graph the sphere $x^2 + y^2 + z^2 = 1$ by:

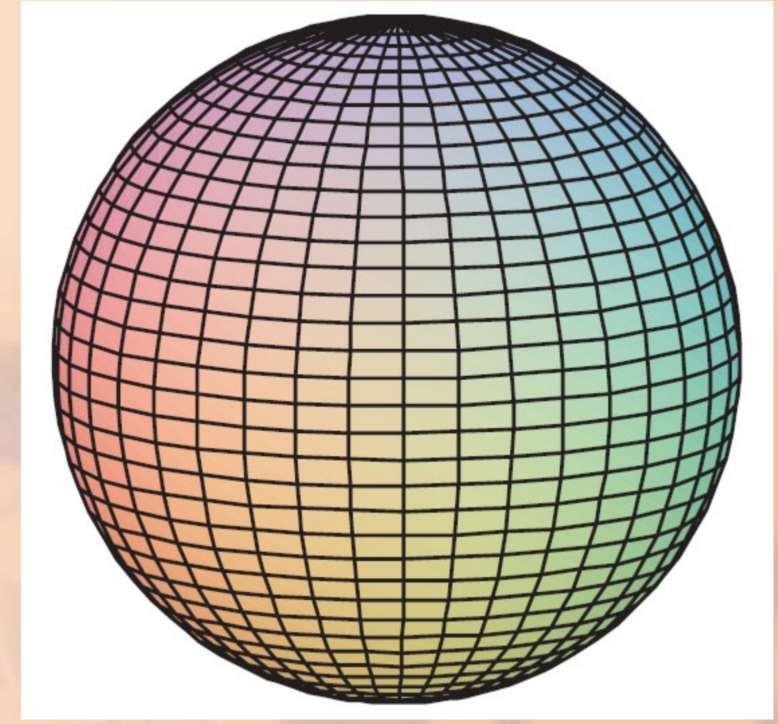
- Solving the equation for *z*.
- Graphing the top and bottom hemispheres separately.

Part of the sphere appears to be missing because of the rectangular grid system used by the computer.



COMPUTER GRAPHICS

The much better picture here was produced by a computer using the parametric equations found in Example 4.



PARAMETRIC REPRESENTATIONS **Example 5**

Find a parametric representation for the cylinder

$$x^2 + y^2 = 4 \qquad 0 \le z \le 1$$

Find a vector function that represents the elliptic paraboloid $z = x^2 + 2y^2$.

PARAMETRIC REPRESENTATIONS

In general, a surface given as $z = f(x, y)$ can be parameterized as:

$$x = x \qquad y = y \qquad z = f(x, y)$$

Parameterizations of surfaces

are not unique.

- The next example shows two ways to parametrize a cone.

Find a parametric representation for the surface

$$z = 2\sqrt{x^2 + y^2}$$

One possible representation is:

$x = x$ $y = y$ $z = 2\sqrt{x^2 + y^2}$

- So, the vector equation is:

$$\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + 2\sqrt{x^2 + y^2}\,\mathbf{k}$$

If we use the polar coordinates, then:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = 2\sqrt{x^2 + y^2} = 2r$$

Therefore:

$$\mathbf{r}(r, \theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + 2r\,\mathbf{k}$$

$$r \geq 0 \quad \text{and} \quad 0 \leq \theta \leq 2\pi$$

In certain situations, Solution 2 might be preferable.

For instance, if we are interested only in the part of the cone that lies **below the plane *z* = 1**, all we have to do in Solution 2 is change the parameter domain to:

$$0 \le r \le \tfrac{1}{2} \qquad 0 \le \theta \le 2\pi$$

TANGENT PLANE

If a surface S is given by

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

Then the a **normal** vector for the **tangent plane** at a point P_0 is given by $\mathbf{r}_u \times \mathbf{r}_v$

Find the tangent plane to the surface with parametric equations

$$x = e^{2u} \qquad y = \cos(v^2) \qquad z = 1 + u + v$$

at the point (1, 1, 1).

Example 9

Note that the point (1, 1, 1) corresponds to the parameter values $u = 0$ and $v = 0$.

SURFACE AREAS

Suppose a surface *S* is given by **r**(*u*, *v*) , where (*u*, *v*) $\in$ *D*.

Then, the **surface area** of *S* is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

A parameterization for a sphere of radius a is

$\mathbf{r}(\Phi, \theta) = \langle a \sin \Phi \cos \theta, a \sin \Phi \sin \theta, a \cos \Phi \rangle$

Then:

$$\mathbf{r}_\varphi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= a^2 \sin^2\phi\cos\theta\,\mathbf{i} + a^2\sin^2\phi\sin\theta\,\mathbf{j}$$

$$+ a^2\sin\phi\cos\phi\,\mathbf{k}$$

Thus,

$$|r_\phi \times r_\theta|$$

$$=\sqrt{a^4 \sin^4 \phi \cos^2 \theta + a^4 \sin^4 \phi \sin^2 \theta + a^4 \sin^2 \phi \cos^2 \phi}$$

$$=\sqrt{a^4 \sin^4 \phi + a^4 \sin^2 \phi \cos^2 \phi}$$

$$= a^2 \sqrt{\sin^2 \phi} = a^2 \sin \phi$$

Therefore for the sphere, if

r(Φ, θ)= < $a \sin \Phi \cos \theta$, $a \sin \Phi \sin \theta$, $a \cos \Phi$ **>**

Then:

$$\left| \mathrm{r}_{\varphi} \times \mathrm{r}_{\theta} \right| = a^2 \sin(\phi)$$

Find the surface area of a sphere of radius *a* by using the following parameterization of the sphere:

$$r(\Phi, \theta) = \langle a \sin \Phi \cos \theta, a \sin \Phi \sin \theta, a \cos \Phi \rangle$$

The area of the sphere is:

$$A = \iint_D |\mathbf{r}_\phi \times \mathbf{r}_\theta|\, dA = \int_0^{2\pi} \int_0^{\pi} a^2 \sin\phi \, d\phi \, d\theta$$

$$= a^2 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi$$

$$= a^2 (2\pi) 2 = 4\pi a^2$$

SURFACE AREAS **Example**

Find the area of the part of the sphere

$x^2 + y^2 + z^2 = 16$ that lies above the cone

$$z = \sqrt{x^2 + y^2}$$

- Note that: $r(\Phi, \theta) = < 4 \sin \Phi \cos \theta, 4 \sin \Phi \sin \theta, 4 \cos \Phi >$ where $0 \leq \Phi \leq \pi/4$, $0 \leq \theta \leq 2\pi$

SURFACE AREA OF THE GRAPH OF A FUNCTION

If a surface S is given by $z = f(x, y)$, then

- The parametric equations are:

$$x = x \qquad y = y \qquad z = f(x, y)$$

Thus,

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}\right)\mathbf{k}$$

and

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \dfrac{\partial f}{\partial x} \\ 0 & 1 & \dfrac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$$

Thus, we have:

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}$$

$$= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

And so we have the next formula:

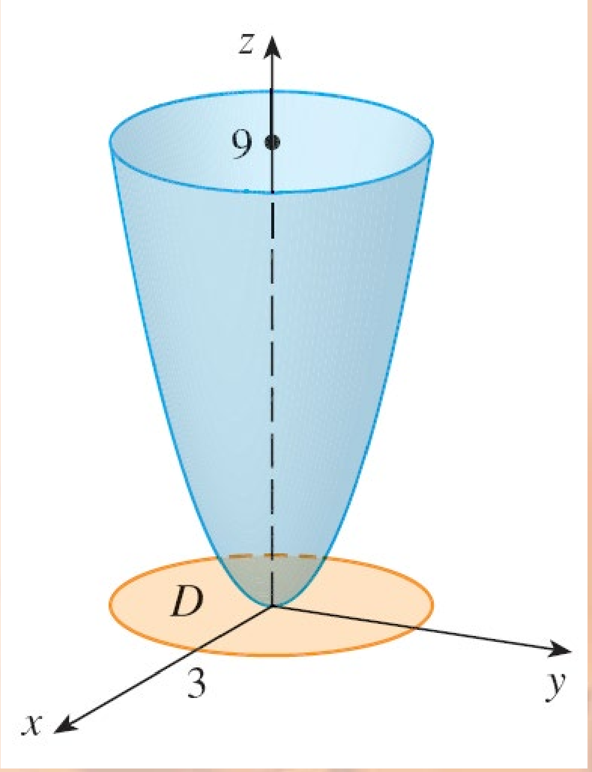
 Formula 9

If S is given by $z = f(x, y)$,

Then, the **surface area** is given by:

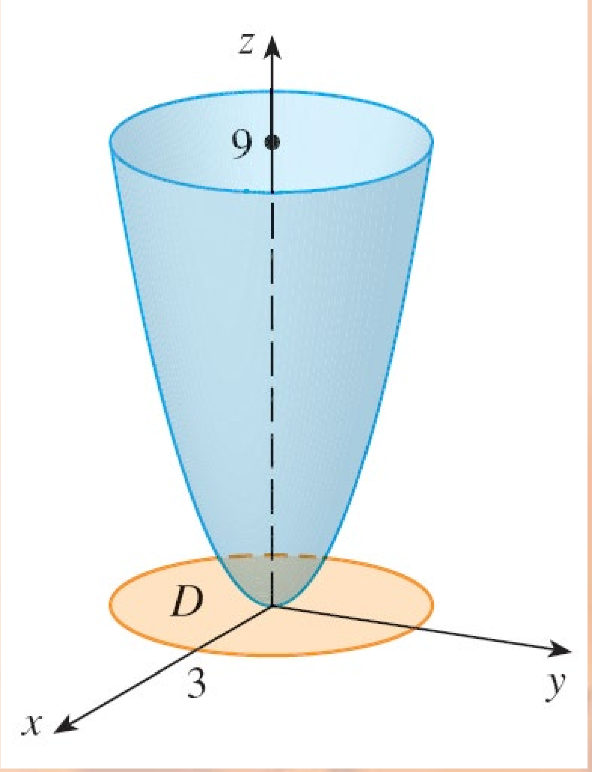
$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.



The plane intersects the paraboloid in the circle $x^2 + y^2 = 9, z = 9$

Therefore, the surface lies above the disk *D* with center the origin and radius 3.



So we have:

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA$$

$$= \iint_D \sqrt{1 + 4(x^2 + y^2)}\, dA$$

Converting to polar coordinates,

we obtain:

$$A = \int_0^{2\pi} \int_0^3 \sqrt{1+4r^2}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^3 r\sqrt{1+4r^2}\, dr$$

$$= 2\pi \left(\tfrac{1}{8}\right) \tfrac{2}{3} (1+4r^2)^{3/2} \Big]_0^3$$

$$= \frac{\pi}{6}(37\sqrt{37} - 1)$$

Surface Area Formulas

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA$$