Dynamic Meteorology (METR 121A)
Lesson Plan

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Lesson: Vector Calculus for Atmospheric Dynamics

Timeframe:

Materials needed: Pencil, pens (standard and colored), paper, calculator, laptop computer (optional). Lecture notes, in-class activity assignments (handouts).

Objectives:

Part 1: Dot and Cross Products; Grad Operator; total vs. partial derivatives, directional derivative (grad of vector)

1. Demonstrate ability to apply and make calculations using dot and cross products of two vectors.
2. Demonstrate ability to apply and make calculations applying DIV and CURL of a vector.
3. Demonstrate ability to apply and make calculations using grad-operator.
4. Be able to physically explain what dot, cross and grad operators are, and whether a scalar or vector is produced when each operator is applied.
5. Explain verbally what the total and partial derivatives are, and the differences between them.

Part 2: Lagrangian vs. Eularian Derivatives; Advection

1. Be able to derive and/or write the mathematical definitions of Lagrangian and Eularian time derivatives.
2. Be able to identify the advection terms in the definition of Eularian time derivative. Be able to distinguish horizontal and vertical advection terms.
3. Be able to make calculations applying either Lagrangian or Eularian derivatives, as appropriate.
4. Explain physically what advection is.

Part 3: Application of Concepts on Weather Maps

1. Be able to diagram on a weather map the direction of grad-T, where T is temperature
2. Be able to diagram on a weather map the vectors that need to be dotted to calculate the components of horizontal advection.
3. Be able to identify on a weather map regions of positive and negative horizontal temperature advection.

Background: The course is among the first core upper division courses for Meteorology majors. It covers the fundamental theories governing wind flow, thermodynamics and weather systems as applied to the lower atmosphere (troposphere). Vector calculus is used throughout the course, requiring a lesson to review the basic vector calculus concepts and operators that will be used throughout the course and their uses in meteorology.

Introduction to Lesson: During the week before the class the students will watch lecture videos that cover the basic vector calculus concepts to be used in this lesson: dot products, cross products, the grad operator, and total vs. partial derivatives. An in-class quiz and follow-up discussion will then take place to verify understanding and address anything unclear to students from watching the videos. The in-class group activities will then have students carry out practice exercises where the operators are applied (Part 1), carry out an exercise that demonstrates horizontal advection (Part 2), and diagram on a weather map vectors following from vector-calculus operations to isolate areas of warm and cold air advection on the map (Part 3). Post-class homework exercises will be used to assess learning.

Procedure [Time needed, include additional steps if needed]:

Pre-Class Individual Space Activities and Resources:

Steps	**Purpose**	**Estimated Time**	**Learning Objective**
Step 1: Vector addition, Scalar Multiplication, Dot-Product (9 min): https://www.khanacademy.org/math/linear-algebra/vectors-and-spaces/dot-cross-products/v/vector-dot-product-and-vector-length Cross-Product (15 min): https://www.khanacademy.org/math/linear-algebra/vectors-and-spaces/dot-cross-products/v/linear-algebra-cross-product-introduction	Cover definition and obtain physical understanding of vector addition, scalar multiplication, and dot product. Cover definition and obtain physical understanding of cross product.	25 min	Part 1: #1 Part 1: #4
Step 2: - Gradient (10min): https://www.youtube.com/watch?v=U7HQ_G_N6vo - Divergence (10min): https://www.youtube.com/watch?v=JAXyLhvZ-Vg - Curl (3min): https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives/curl-grant-videos/v/2d-curl-intuition - Curl (7min): https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives/curl-grant-videos/v/3d-curl-formula-part-1	Cover definition and obtain physical understanding of vector addition, scalar multiplication, and dot product. Cover definition and obtain physical understanding of cross product.	30 min	Part 1: #2 Part 1: #3

Step 3: Partial derivatives (10min) https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives/partial-derivatives/v/partial-derivatives-introduction	Cover definition of partial derivative. Learn how to compute and understand verbally.	10min	Part 1: #5
Step 4: https://youtu.be/4HxvNCnlw9g (8min)	Cover Lagrangian vs. Eularian Time Derivatives & Advection	8 min	Part 2

In-Class Group Space Activities and Resources:

Steps	**Purpose**	**Estimated Time**	**Learning Objective**
Step 1: Lesson_2_InClass-Step1.docx (Selected problems) http://tutorial.math.lamar.edu/Classes/CalcIII/VectorFields.aspx (Selected problems)	Student practice in using vector calculus operators. Verify ability and identify trouble areas in learning.	20 minutes	Part 1: #1,2,3

Step 2: Vector Calculus MATLAB.pdf	Help students visualize DIV, CURL, GRAD. Computer practice (MATLAB).	15 min	P1: #4
Step 3: Physical explanation. Develop in pairs complete sentences explaining what concepts mean for given examples. Go over in class	Help students connect math to physical understanding	5 min	P1: #4
Step 4: In-class review: Remind relationship between total and partial derivative. Identify advection. Identify Source and Sinks. Problem: pair off. Lesson_2_InClass-Step4.docx. One calculates rate of change at point from Lagrangian point of view. Other from Eularian. Verify same answer. Physical explanation: Develop sentence explaining "local" rate of change (at a point) as a result of air motion and advection and sources/sinks.	Reinforce understanding of Lagrangian vs. Eularian derivatives; advection	30 min	Part 2
Step 5: Weather map exercise. Diagraming direction of temperature gradient, areas of + and – temperature advection, and vectors that are involved in advection. Student prints out selected 850 mb weather map on computer.	Gain physical understanding within a meteorological weather prediction context for advection.	15 min	Part 3

Closure/Evaluation:

Analysis: *This in-class activity will provide the instructor direct indication as students are working on assignment problems of trouble areas that students have in in understanding concepts. This will improve greatly the traditional homework-only approach, whereby trouble areas are often only known about after homework is turned in.*

Post-Class Individual Space Activities: *Students will complete problems on worksheets that were no assigned in in-class activity. Students will complete additional problems on homework to verify understanding and apply to a meteorological problem of interest.*

Connections to Future Lesson Plan(s): *Vector calculus is used throughout the class. The Eularian time derivative equation with advection is the starting point from which equations of motion are derived. This will be covered in Lecture 3.*

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Lesson 2: In-class Activities Step 1

1. Complete the following formulas. Note that A, B, and C are arbitrary vectors, and *a* is scalar.

$\vec{A} \bullet (\vec{B} \times \vec{C}) =$

$\vec{A} \times (\vec{B} \times \vec{C}) =$

$\nabla \times \nabla a =$

$\nabla \bullet (\nabla \times \vec{A}) =$

$\nabla \bullet (a\vec{A}) =$

$\nabla(\vec{A} \bullet \vec{B}) =$

$\nabla \times (\vec{A} \times \vec{B}) =$

$(\vec{A} \bullet \nabla)\vec{A} =$

$\nabla \times (\nabla \times \vec{A}) =$

2. In Cartesian coordinate (x,y,z), the position vector is described as $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$, horizontal velocity is $\vec{V} = u\vec{\imath} + v\vec{\jmath}$. ϕ is a scalar (i.e., geopotential height). Complete the following. Indicate whether the results are a scalar or vector. If a vector, diagram the direction of the vector relative to the corresponding Φ, $\vec{V}$ or $\vec{k}$ fields (as appropriate).

$\nabla\phi =$

$\nabla \bullet \vec{V} =$

$k \bullet (\nabla \times \vec{V}) =$

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Lesson 2: In-class Activities Step 4

*** DRAFT: To be fine-tuned ***
Consider an air parcel traveling with the wind. At a certain time (t0) the position of the parcel and the temperature field is as shown below. Given this situation, we will calculate the temperature change at a later time at point ‘p’, shown also on the diagram. We will do this in both a Lagrangian and Eularian manner. There are no heat sources and sinks acting on the parcel.

Varies from west to east starting from 20C to 5C at 5C every 100 km. Wind is from west at 5m/s.

a) What is the value and direction of grad(T). Show work
b) Write expression for velocity vector.
c) Write values of wind components in meteorological notation.
d) Calculate how long it takes for the parcel at t0 to reach point ‘p’? Show all work and units. Call this deltat.
e) What is temperature of parcel when it reaches ‘p’? Remember, no sources or sinks!
f) Based on this, calculate the value of $\partial T/\partial t$ at point ‘p’ over the time period t0 to t0+deltat. Note that this is a partial derivative, since we are calculating rate of change over time at a fixed point ‘p’.
g) Calculate $\partial T/\partial t$ based on its Eularian expression, i.e.
h) How do values compare from part f and g?

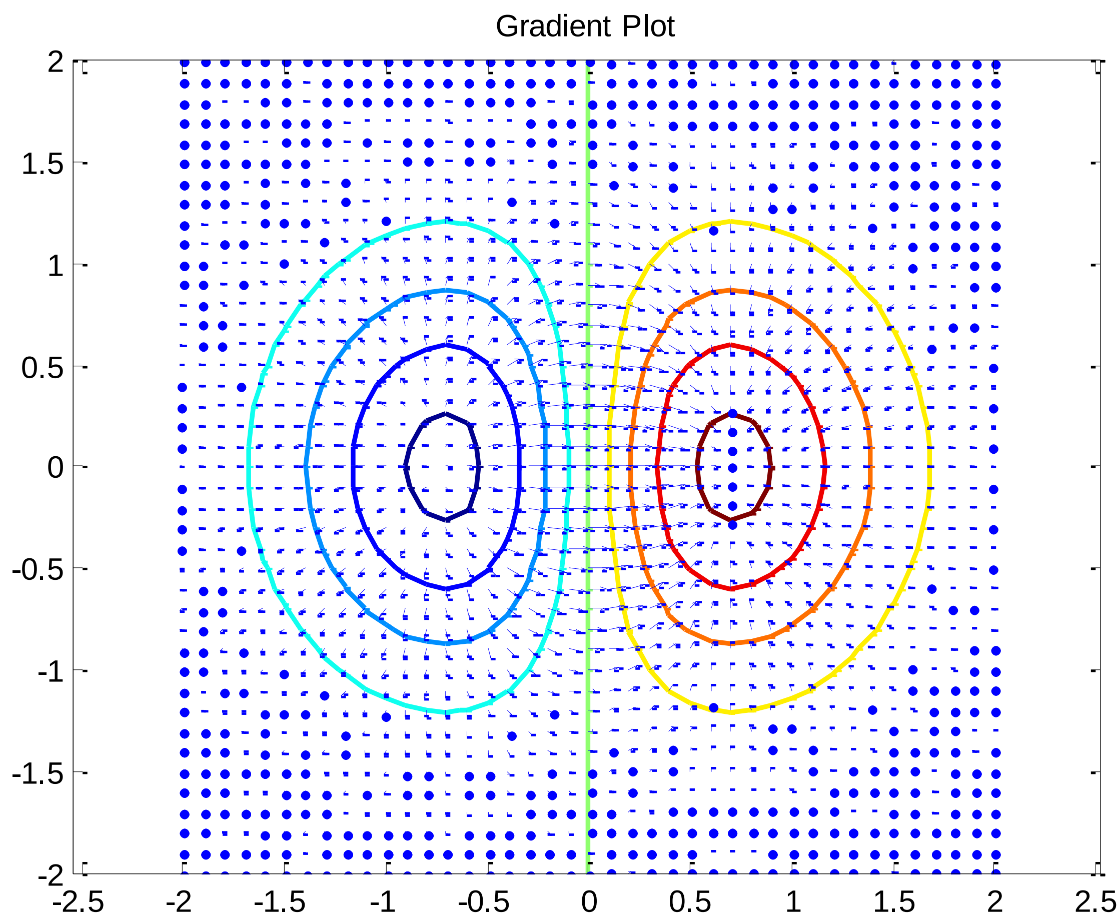
Vector Calculus Examples Using MATLAB

MATLAB can evaluate and plot most of the common vector calculus operations that we have previously discussed. Consider the following example problems:

Determine and Plot Contours of a Scalar Field and Plot a Vector Distribution of the Associated Gradient Field

Choosing the field $z = xe^{-(x^2+y^2)}$, over the domain $-2 < (x, y) < 2$ the MATLAB code is listed in the text box. Note the use of the "meshgrid" command to establish a grid of points in the domain of interest. The command "gradient" calculates ∇z, and "quiver" is a neat way to automatically plot the distribution of a vector field. The plot is shown below.

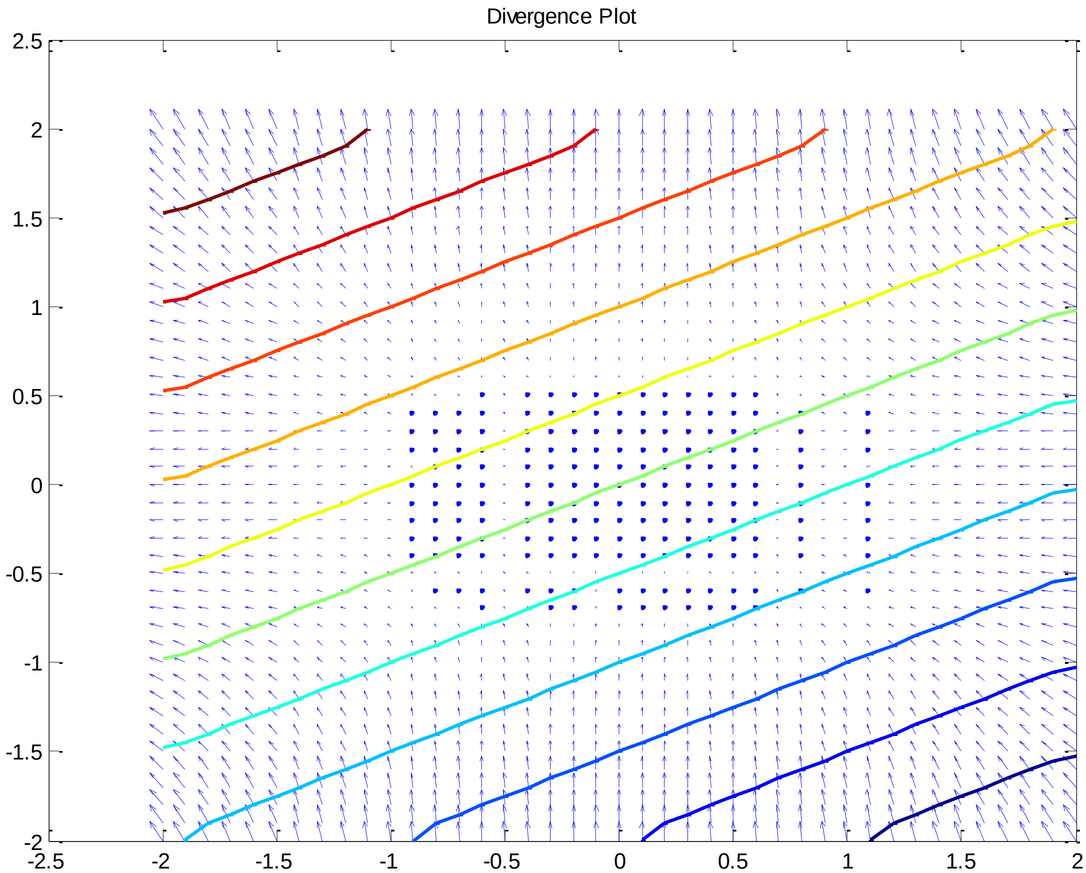
```
% Gradient Example
% Draws z-Contours and Then
Plots Gradient Vector Field
v = -2:0.1:2;
[x,y] = meshgrid(v);
z = x .* exp(-x.^2 - y.^2);
[px,py] = gradient(z,.2,.2);
figure(1)
contour(v,v,z,'linewidth',2)
hold on
quiver(v,v,px,py,'linewidth',1)
hold off
title('Gradient Plot');
axis equal
```



Calculate and Plot the Divergence of a Vector Field

Choosing the vector field $\mathbf{u} = -x^2\mathbf{i} + 2y^2\mathbf{j}$, we first plot the vector distribution using the "quiver" command. Next MATLAB computes the divergence of **u** using the "divergence" command and then plots contours on the same page. The code is listed in the text box. The plot is shown below.

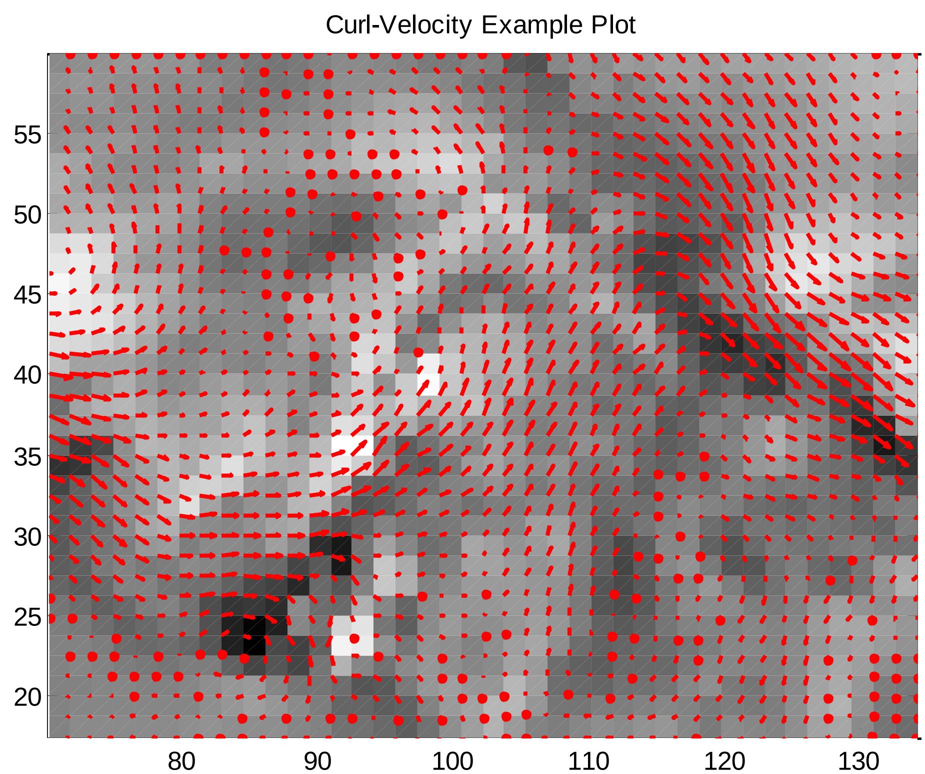
```
% Divergence Example
v = -2:0.1:2;
[x,y] = meshgrid(v);
u1=-x.^2;u2=2*y.^2;
figure(2)
quiver(x,y,u1,u2);hold on;
div=divergence(u1,u2);
contour(x,y,div,'linewidth',2)
hold off
title('Divergence Plot')
```



Determine the Curl of a Velocity Vector "wind" and Make Plots

We now wish to explore the behavior of a given set of velocity data from a file called "wind". This file represents air currents over North America and is built into the MATLAB package (see help for more details). The code starts by loading the "wind" file, makes particular choices for the spatial viewing and then determines the velocity components "u" and "v". The curl is evaluated using the "curl" command and then plotted using the "pcolor" command. The code is given in the text box, and the plot of the curl and velocity field distribution is shown below.

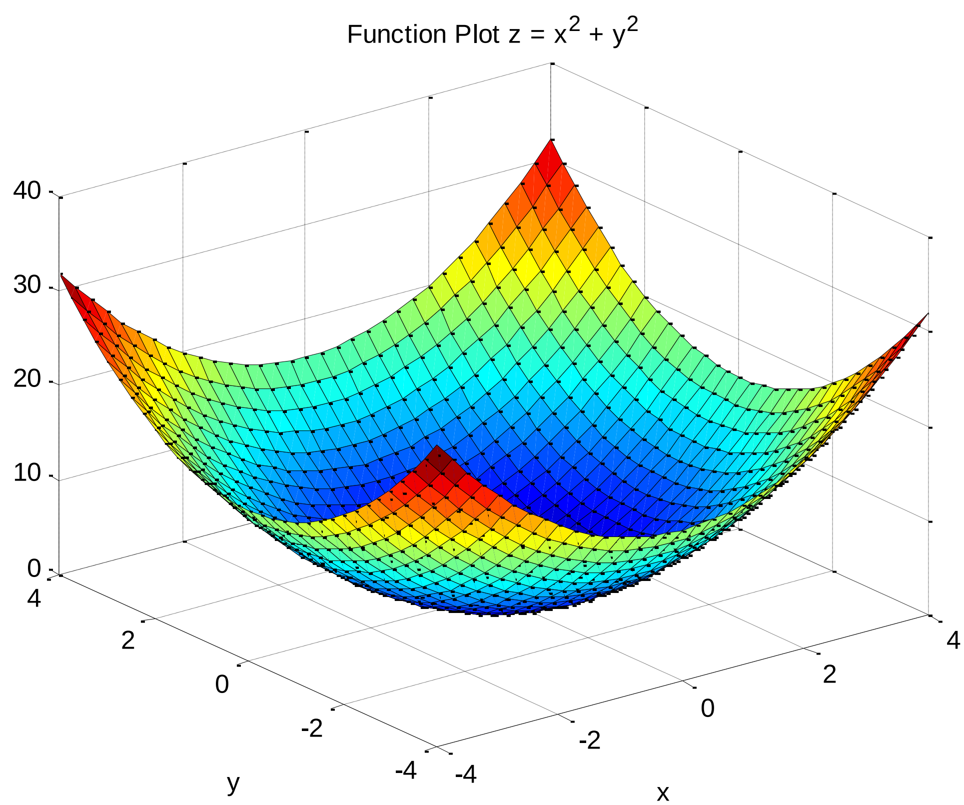
```
% Curl Example
% Views the Curl Angular Velocity in One Plane
% Plots the Velocity Vectors in Same Plane
load wind
% Choose Slice Plane and x,y Range
k = 3;
x = x(:,:,k); y = y(:,:,k);
u = u(:,:,k); v = v(:,:,k);
cav = curl(x,y,u,v);
figure(3)
pcolor(x,y,cav); shading interp
hold on;
quiver(x,y,u,v,'r','linewidth',1.5)
hold off
title('Curl-Velocity Example Plot')
colormap gray
```

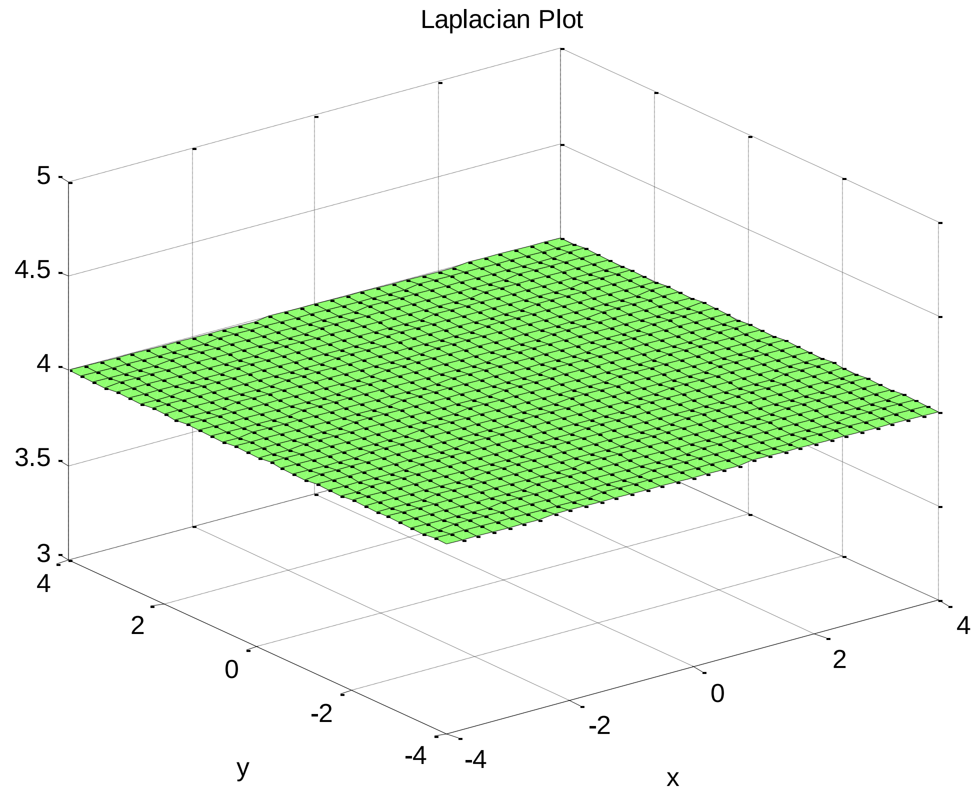


Calculate and Plot Function and its Laplacian

Next we wish to calculate the Laplacian of a given scalar field function ϕ, and make appropriate three-dimensional plots of each distribution over the domain of interest. The scalar function is specified by $\phi = x^2 + y^2$ over the domain $-4 < (x, y) < 4$. The MATLAB code is given in the box. The Laplacian is calculated using the "del2" command, but notice that $\nabla^2\phi = 4 * \mathrm{del2}$ and that the command requires input of the spacing distance of the meshgrid. Also notice the use of the "surf" command to do the 3-D plotting. Results are shown below. Note the analytical value $\nabla^2\phi = 4$ for this case.

```
% Laplacian Example
% Calculate & Plot Function and
Laplacian
[x,y] = meshgrid(-4:0.5:4,-4:0.5:4);
U = x.*x+y.*y;
V = 4*del2(U);
figure (4)
surf(x,y,U)
xlabel('x');ylabel('y');
title('Function Plot z = x^2 + y^2')
figure(5)
surf(x,y,V)
xlabel('x');ylabel('y');
title('Laplacian Plot')
```




Laplacian Plot
5
4.5
4
3.5
3
4
2
0
-2
-4
y
-4
-2
0
2
4
x