MATH 271A

Mathematical Logic: The Incompleteness Theorems

Time magazine includes only two mathematicians on its list of the 100 most influential people of the twentieth century, Alan Turing and Kurt Gödel. Both were mathematical logicians. (See `http://www.time.com/time/time100/scientist`.)

The main goal of Math 271A is to give a detailed proof of Gödel's most famous theorem, his first Incompleteness Theorem, widely considered (even outside the editorial board of Time) to be the most conceptually important theorem of the last century.

Modern mathematics is often understood to follow the *axiomatic method*: Given a structure to be studied—the natural numbers with addition and multiplication, all of the linear operators on $\mathbb{C}^n$, differential fields, Lie algebras, or whatever—one first adopts a set of statements that are evidently true as "axioms", then proves less obvious facts from these first principles. On this understanding, it is appealing to fix, once and for all, some good axioms regarding a structure of interest, thereby reducing mathematics to the project of deriving consequences of those fixed axioms.

The simplest statement of the first Incompleteness Theorem—it actually is better than this—is that in general this is impossible. Specifically, there is no "good" set of axioms describing the natural numbers with their usual addition and multiplication.

To be more precise, a *good* set of axioms T would be a collection of axioms (in a reasonable formal language) satisfying the following:

(1) Every axiom in T is a true proposition regarding the natural numbers with their usual addition and multiplication.

(2) Allowing that T is perhaps infinite, there exists an effective procedure by which members of T can be recognized. That is, it is possible to determine in some algorithmic manner which sentences of the formal language are axioms of T. (This stipulation precludes T from being simply the collection of all propositions true of the natural numbers.)

(3) Every proposition of the formal language is formally provable or disprovable from the axioms T. Thus, the propositions true of the natural numbers with their usual addition and multiplication are precisely those formally provable using the axioms T.

It is a straightforward matter to replace mention of the natural numbers with their usual addition and multiplication with mention of any other adequately complex structure.

On the other hand, there do exist "good" axioms for some structures that might initially seem as complex, for example, the real numbers with addition and multiplication. This make it worthwhile to understand not only the general ideas of the proof but also the nuts and bolts, which reveal what it is that makes some structure subject to the "incompleteness phenomenon" and others immune.

Before getting to the Incompleteness Theorems, it is necessary to have some background in mathematical logic. The first half of Math 271A will develop first-order logic from scratch, up through the proof of the Completeness and Compactness Theorems.