ME 130 Applied Engineering Analysis

Chapter 2

Mathematical Modeling

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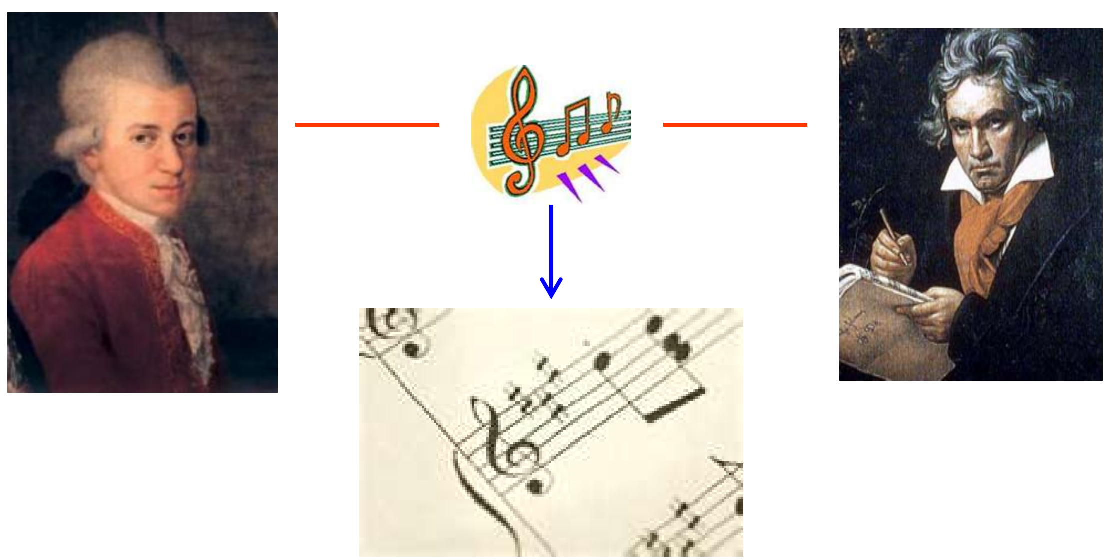
Chapter Learning Objectives

- Mathematical modeling in engineering analysis
- Elements of mathematical modeling
    - Functions and variables
    - Differentiation and derivatives
    - Integration and application of integration in engineering analysis
- Differential equations in mathematical modeling

Mathematical modeling involves:

Translating a physical situation into mathematical expressions

It is a similar action of writing MUSIC from the melodies in the minds of great composers, e.g., Beethoven, Mozart, etc.



Engineers' duties include:

CREATION,

DECISION MAKING, and

PROBLEM SOLVING

Performing each of these duties involves a process in reaching solutions

The subjects in these processes are translated into **FUNCTIONS**, and the factors that affecting the values of these subjects are **VARIABLES**.

VARIABLES include:

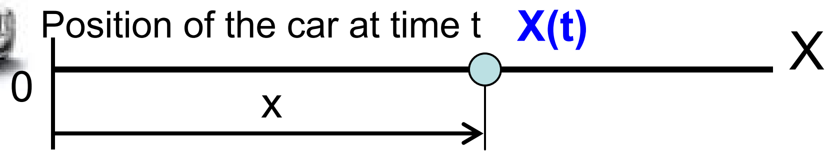
- Spatial variables: represented by coordinate systems with set reference points. Commonly used coordinate systems are:
  (x, y, z) in rectangular coordinates, or
  (r, θ, z) in cylindrical polar coordinates
- Temporal variable: time (t)
- x,y,z and t are INDEPENDENT variables

The process for solutions is to include the FUNCTIONS with VARIABLE in **APPROPRIATE MATH MODELS**, and reach math solutions

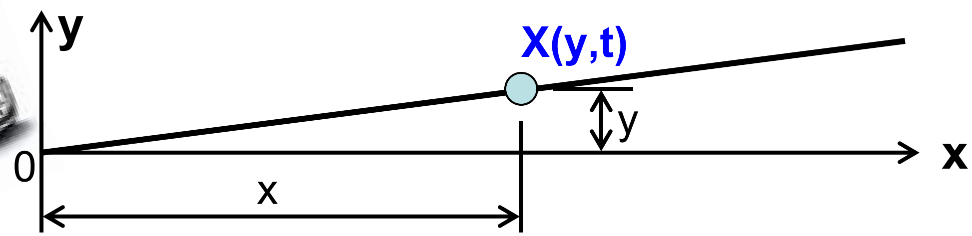
Example for FUNCTIONS and Variables: the function P

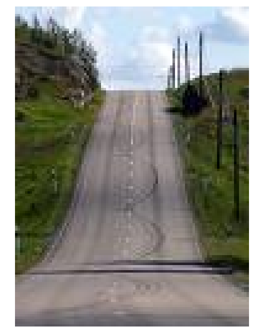
The position of a cruising vehicle P (the function) changes with time t (the variable):

Situation 1: Cruising at constant speed along a flat straight road:

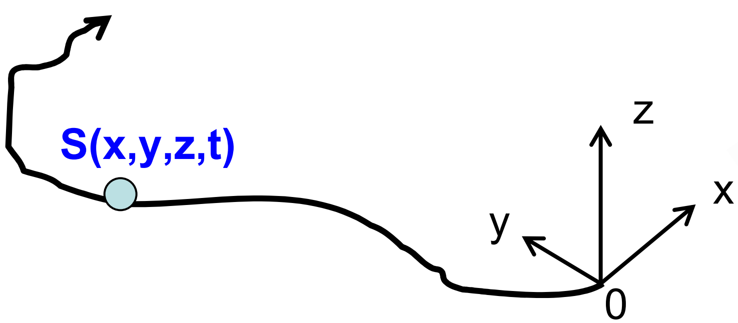


Situation 2: Cruising uphill at given time t:





Situation 3: Cruising along a winding rugged road:



Frequent Functions in ME Engineering Analyses

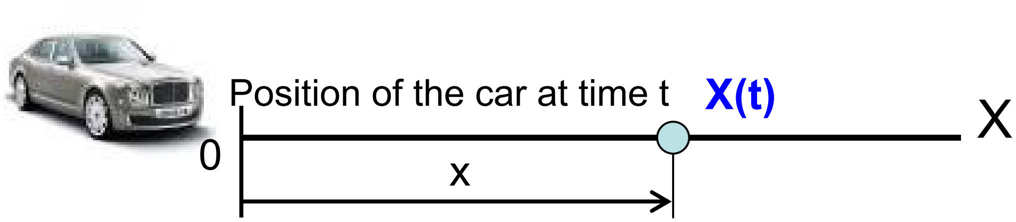
- Mass (**m**), weight (**W**), Length (**L**), Area (**A**), Volume (**V**) of solids
- Forces (**F**)
- Stress (**σ**), Strain (**ε**) in deformed solids
- Distance traveled by a moving rigid body (**S**)
- Temperature in solids and fluids (**T**)
- Velocity of a rigid body or fluid (**V**)

Properties of Functions

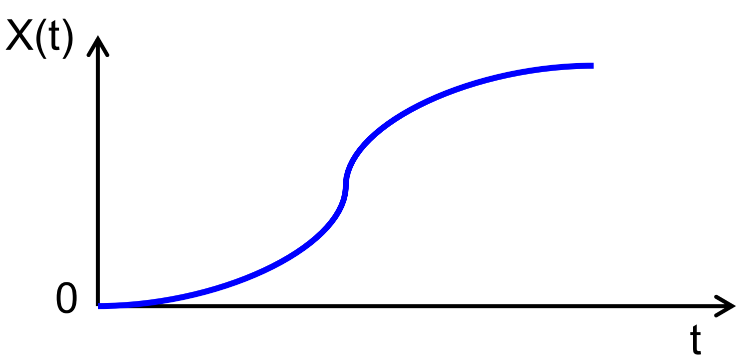
- Functions may change their values with the change of "independent variables" (spatial and temporal) - So, functions are "dependent variables"
- The value of a function is a CONSTANT-depending on the values of the associated independent variables.

The Derivatives

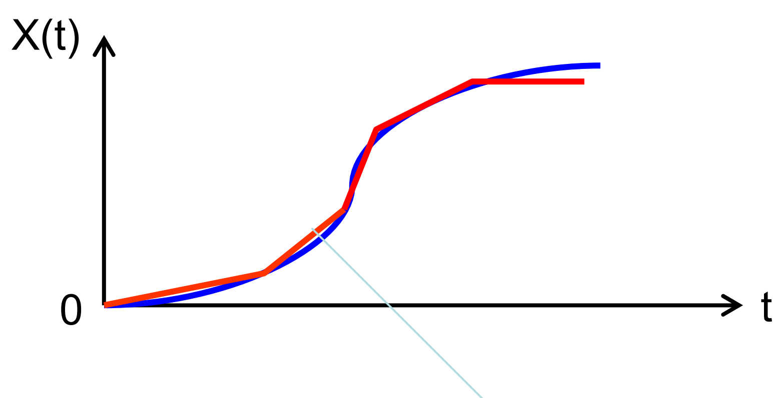
- Functions represent physical quantities in engineering analysis
- These physical quantities change their values with the change of associated independent variables, e.g., (x,y,z,t)
- Change of physical quantities (i.e., the functions) can be 'CONTINUOUS," or "INCREMENTAL":



Continuous variation: Real



Incremental variation: Unreal



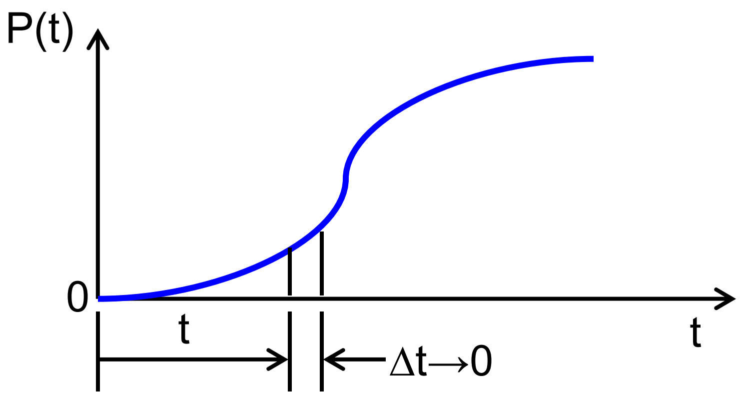
Definition of Derivatives

Mathematical expression representing the RATE of CONTINUOUS VARIATION of functions

Rate of a continuous variation can be viewed as variation of a function with INFINITESIMALLY SMALL increments of the associated independent variables:

$\Delta \mathbf{x} \to \mathbf{0}$ and/or $\Delta \mathbf{y} \to \mathbf{0}$ and/or $\Delta \mathbf{z} \to \mathbf{0}$ and/or $\Delta \mathbf{t} \to \mathbf{0}$

Continuous variation: Real

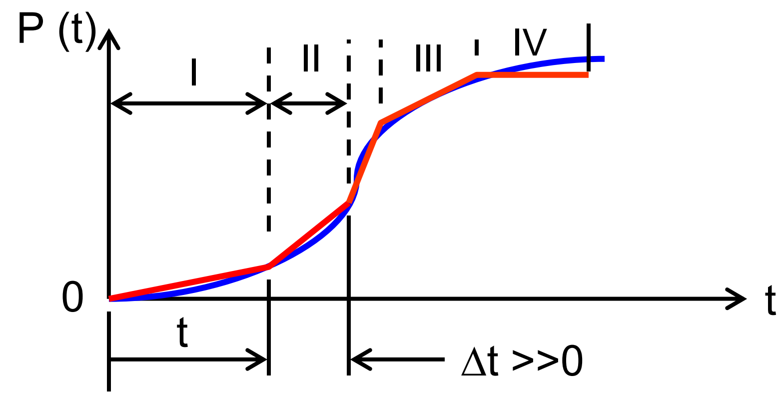


The rate of change of P(t):

$$\lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} = \lim_{\Delta t \to 0} \frac{P(t+\Delta t)}{\Delta t} = \frac{dP(t)}{dt} \qquad (2.1)$$

is the DERIVATIVE of function P(t)

Incremental variation: Unreal



The rates of change of P(t):

$$\left(\frac{\Delta P}{\Delta t}\right)_I \quad \left(\frac{\Delta P}{\Delta t}\right)_{II} \quad \left(\frac{\Delta P}{\Delta t}\right)_{III} \quad \left(\frac{\Delta P}{\Delta t}\right)_{IV}$$

No single derivative for all possible!

Orders of Derivatives

$$\frac{dy(x)}{dx}$$ = the first (1st) order derivative

$$\frac{d^2 y(x)}{d x^2} = \frac{d}{dx}\left(\frac{dy(x)}{dx}\right)$$ = the second (2nd) order derivative

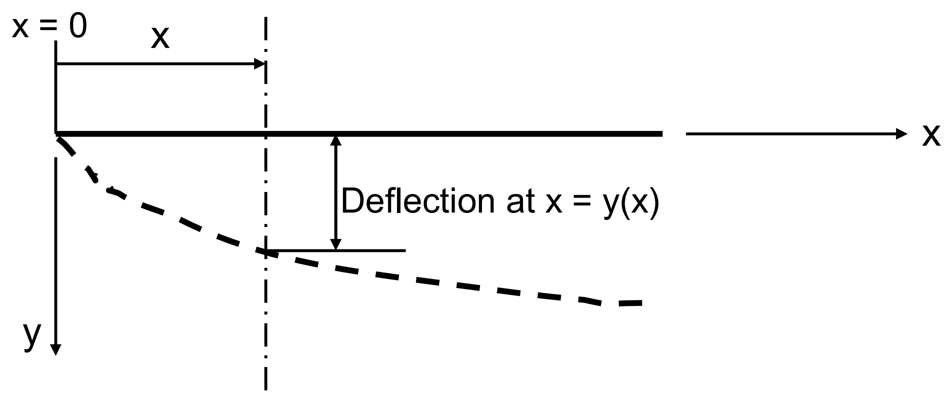
$$\frac{d^3 y(x)}{d x^3} = \frac{d}{dx}\left(\frac{d^2 y(x)}{d x^2}\right)$$ = the third (3rd) order derivative

$$\frac{d^4 y(x)}{d x^4} = \frac{d}{dx}\left(\frac{d^3 y(x)}{d x^3}\right)$$ = the fourth (4th) order derivative

ME analyses almost never involve derivatives with orders higher than 4

Physical Representations of Higher Order Derivatives

Deflection Curve of a Bent Beam:



$\dfrac{dy(x)}{dx}$ = The **slope** of the deflection curve of the bent beam evaluated at location x.

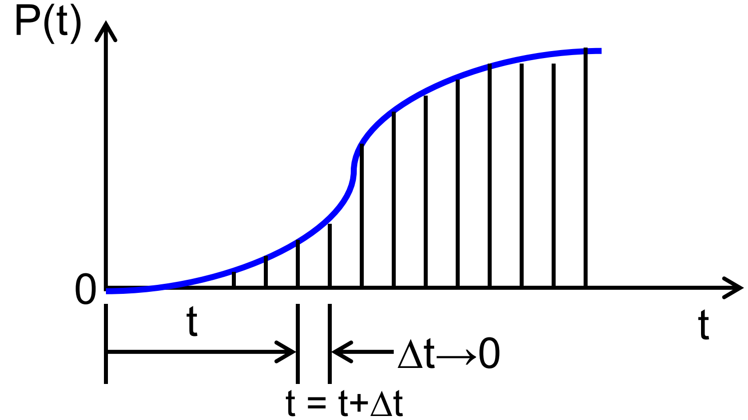
$C\dfrac{\partial^2 y(x)}{\partial x^2}$ = **Bending moment** at x with C being a constant.

$\alpha\dfrac{d^3 y(x)}{dx^3}$ = **Shear force** at x, with α being a constant.

The Integrals

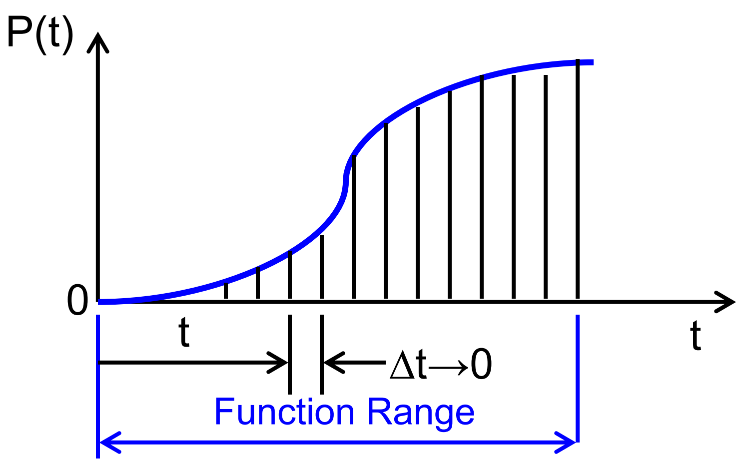
- Integration is a reverse process of differentiation
- Differentiations evaluate the **rate** of change of function values with infinitesimal increments of the associate variables, e.g., the rate of change function P(t) between t = t and t = t+Δt is:

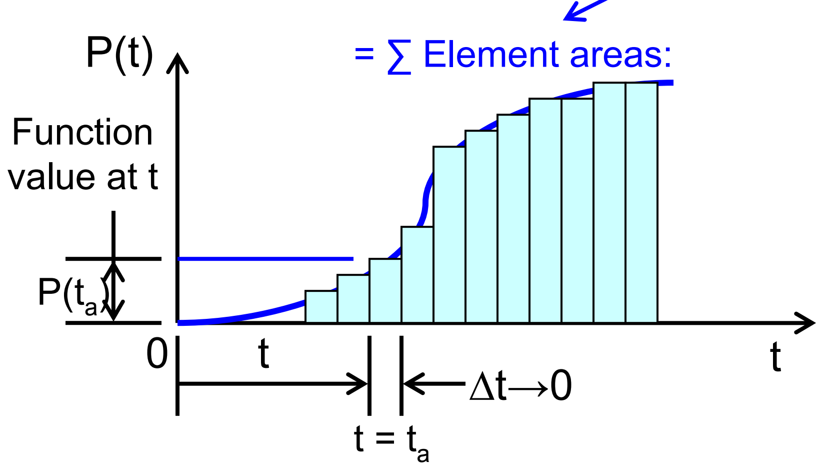
$$\frac{dP(t)}{dt}$$



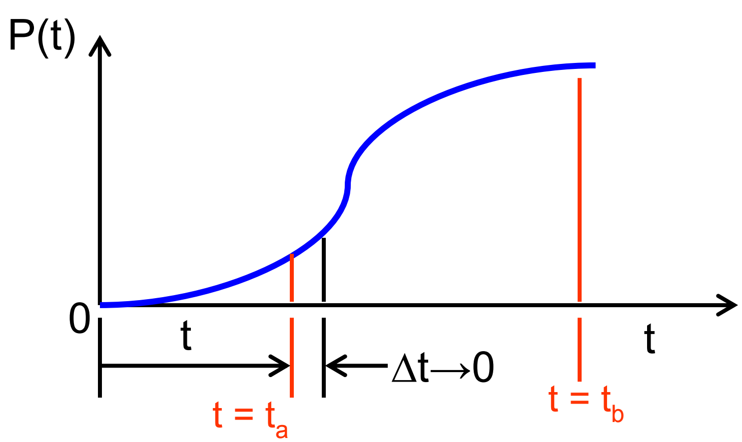
- Integration SUMS UP the function values obtained in all infinitesimal increments of variables associated with the function, e.g.,

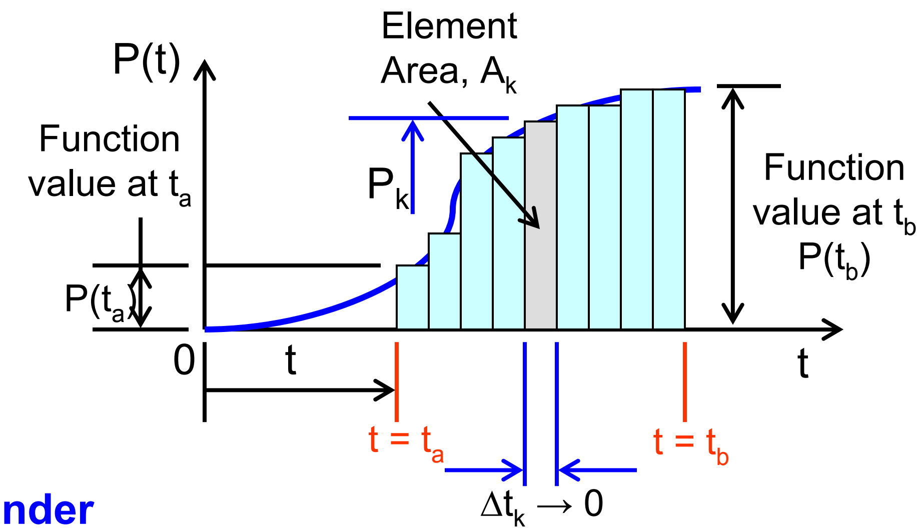
$$Integral, I = Sum\,of\ P(t)\ over\ a\ range\ of\ t = \int_{range} \frac{dP(t)}{dt} dt = \int_{range} dP(t)$$



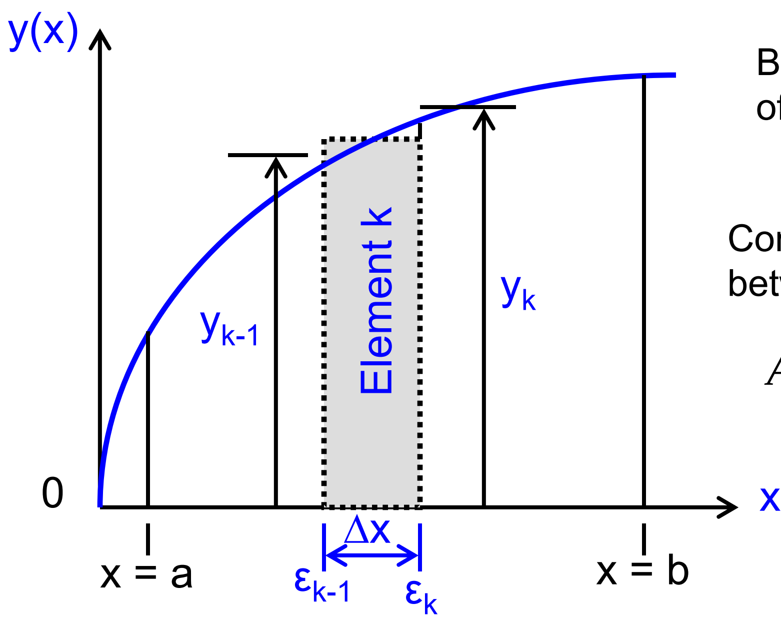


Integration of Function P(t) between t_a and t_b





- **Mathematical Formulation for Area Under the Curve Represented by Function y(x):**



Because the increment Δx is so small, the area of Element k under the curve can be made to equal:

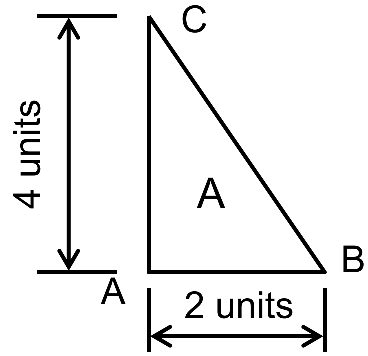
$$A_k \approx y_{k-1}\,\Delta x \approx y_k\,\Delta x$$

Consequently the total area under the curve between X = a and x = b is:

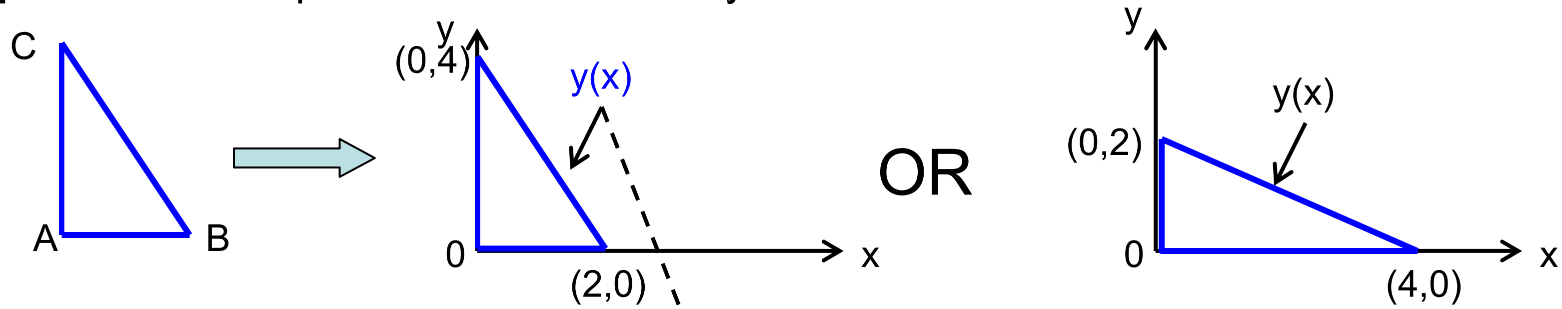
$$A = \sum_{k=1}^{n} y_k\,\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n} y_k\,\Delta x_k = \int_a^b y(x)dx \qquad (2.2)$$

NOTE: You need to formulate the function y(x) to obtain the area by Eq. (2.2)

Example 2.1: Determine the area of a triangle:



Step 1: To set the plane in a coordinate system with reference at O:



Step 2: Determine the function y(x):

$$y(x) = -2x + 4$$

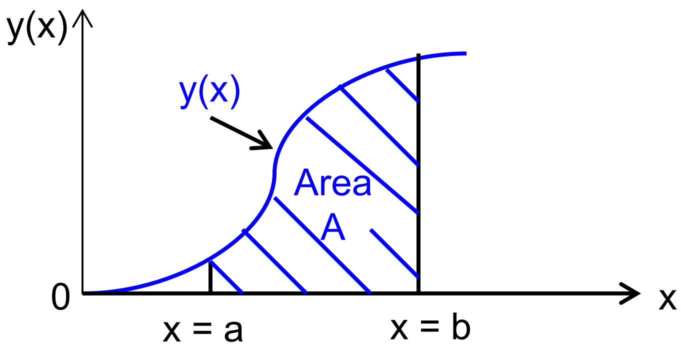
Step 3: Use Equation (2.2) to determine the area under function y(x):

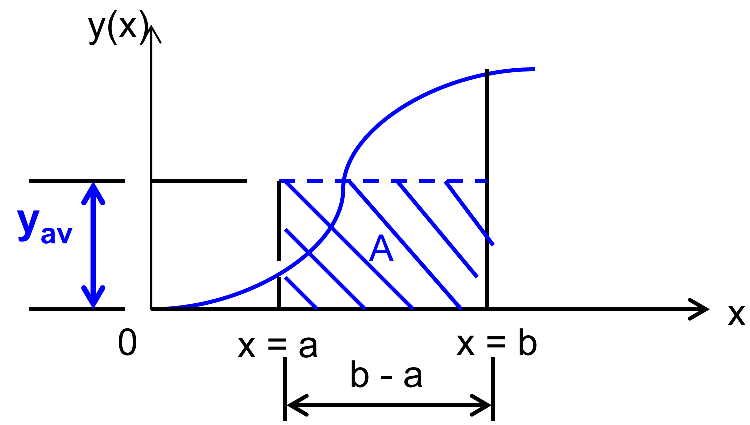
$$A = \int_0^2 y(x)\,dx = \int_0^2 (-2x + 4)\,dx = \left(-x^2 + 4x\right)\Big|_0^2 = 4 \text{ unit square}$$

- **Read: Examples 2.4 (p. 22) and 2.5 (p. 23)**

Average of a CONTINUOUS varying physical quantity represented by a function)

Function y(x) represents the variation of a physical phenomenon:





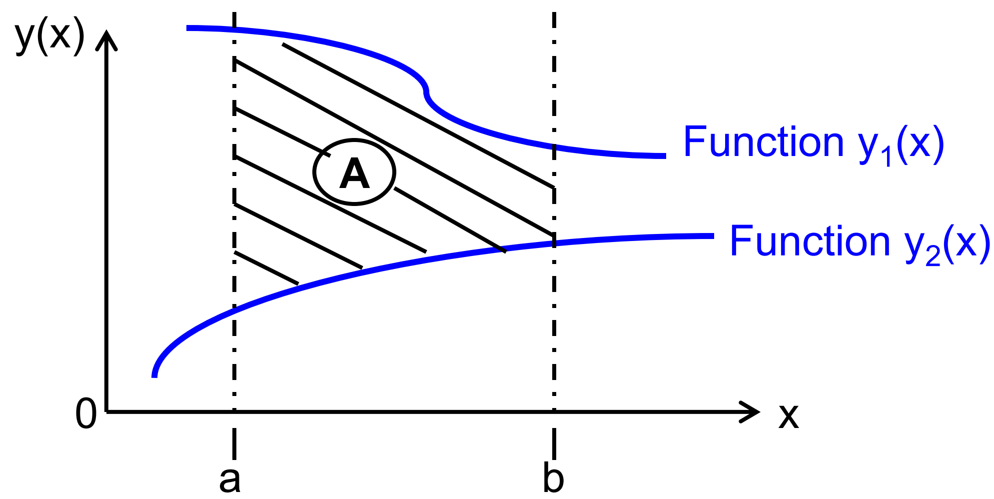
The average value of function y(x) between x=a and x=b is obtained by:

$$y_{av} = \frac{\int_a^b y(x)\,dx}{b-a}$$

$$y_{av} = \frac{Area\ A}{b-a}$$

Example 2.2 Determine the average temperature of a fabrication process (p. 20)

Plane Area Bonded by Two Curves

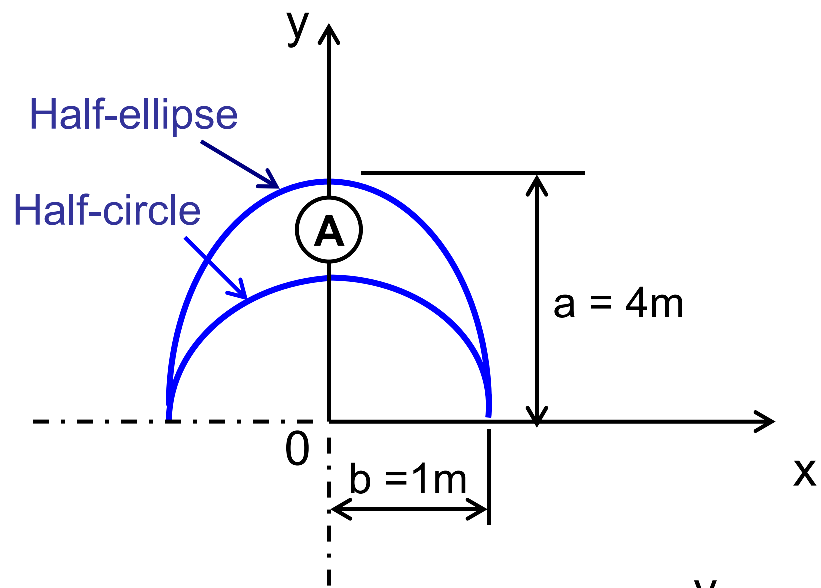


The area defined by the two functions between limits x = a and x = b is:

$$A = \int_a^b y_1(x)\,dx - \int_a^b y_2(x)\,dx = \int_a^b \left[y_1(x) - y_2(x)\right] dx \qquad (2.3)$$

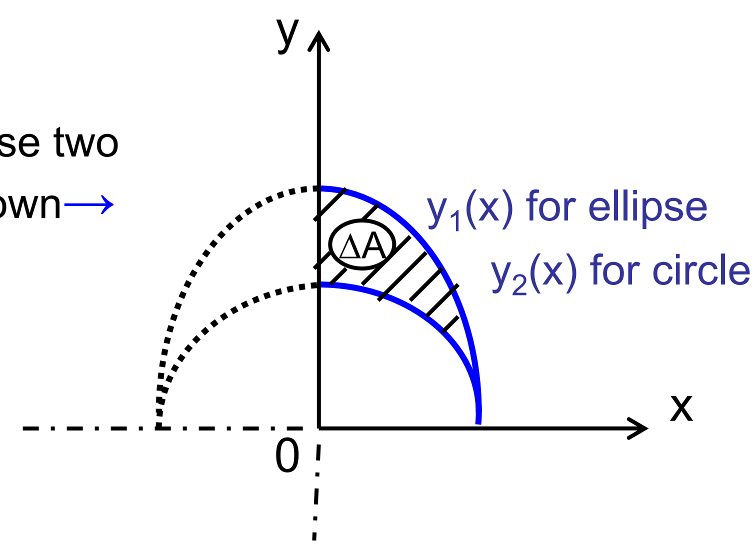
Example (Not in the printed lecture notes)

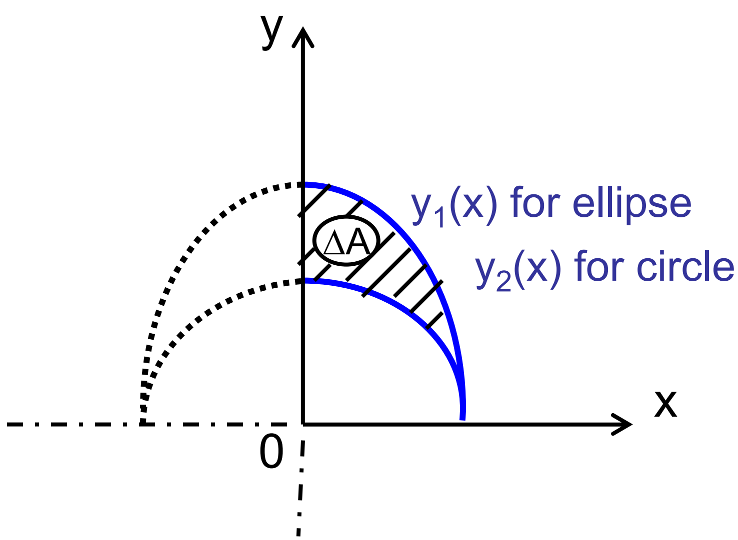
Determine the area of a plate with a geometry defined by two curves of ellipse and arc with dimensions shown below:



We may determine the area bonded by these two curves in an (x,y) coordinate system as shown→

(using the geometric symmetry about the y-coordinate)





Finding the functions for the elliptical part:

- The equation for an ellipse shown in the figure at left is:

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

from which, we get: $y(x)=\frac{a}{b}\sqrt{b^2-x^2}$

Thus, we have: $y_1(x)=\frac{4}{1}\sqrt{1^2-x^2}=4\sqrt{1-x^2}$

- The equation for a circle is: $x^2+y^2=r^2 \;\; with \;\; r=radius=1m$

Thus, the function $y_2(x)=\sqrt{1-x^2}$

- The area ΔA is by using Equation (2.3):

$$\Delta A=\int_0^1 y_1(x)\,dx-\int_0^1 y_2(x)\,dx=\int_0^1 4\sqrt{1-x^2}\,dx-\int_0^1\sqrt{1-x^2}\,dx=3\int_0^1\sqrt{1-x^2}\,dx$$

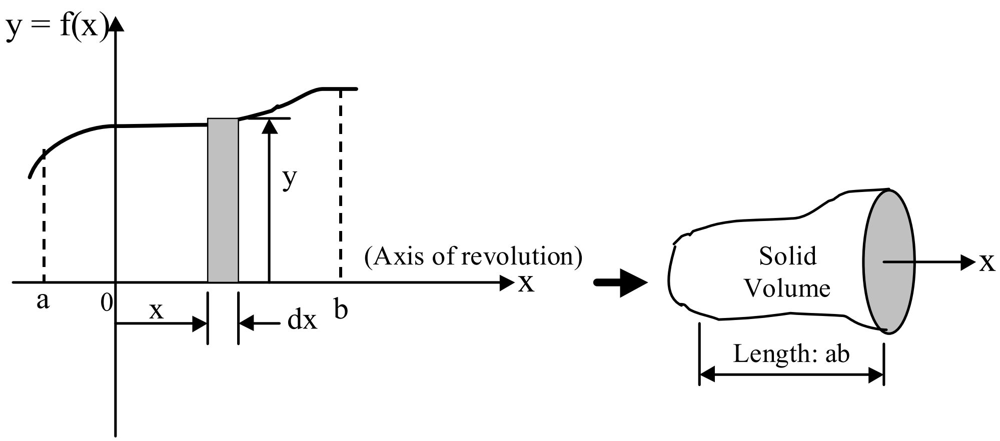
The above integral can be evaluated either using the CRC Math Tables, or by calculator to be: $\Delta A = 4.71\ m^2$. This leads to the total area defined by the half-ellipse and circle To be: **A** = **2ΔA** = **9.42 m^2**

Volume of Solids of Revolution

Solids of revolution: Solids with their geometry symmetrical to an axis of revolution. They are commonly used in machine component design.

Examples; Cylinders; cones, wine and coke bottles

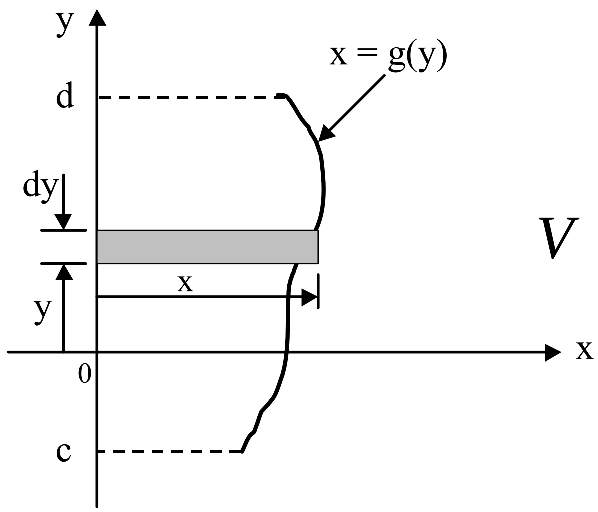
Mathematically, they are defined as:



The volume of revolution about the x-axis can be obtained by:

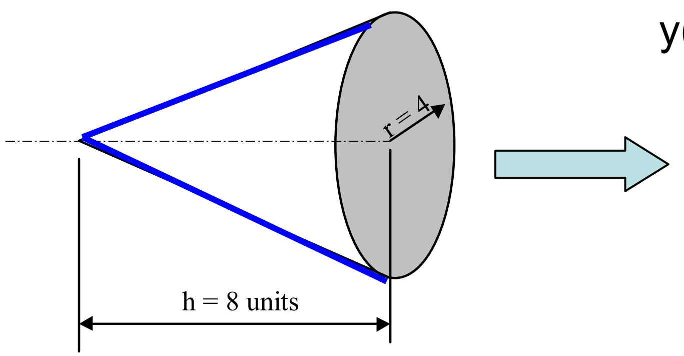
$$V = \int_a^b dv = \int_a^b \pi y^2 \, dx = \pi \int_a^b \left[f(x) \right]^2 dx \qquad (2.4)$$

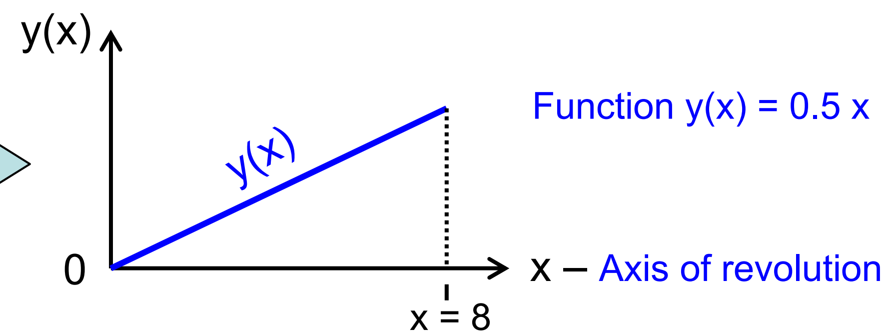
The volume of revolution about the y-axis can also be obtained by:



$$V = \pi \int_c^d x^2\, dy = \pi \int_c^d \left[g(y)\right]^2\, dy \qquad (2.5)$$

Example 2.6: Determine the volume of a right-cone by using the integration method.



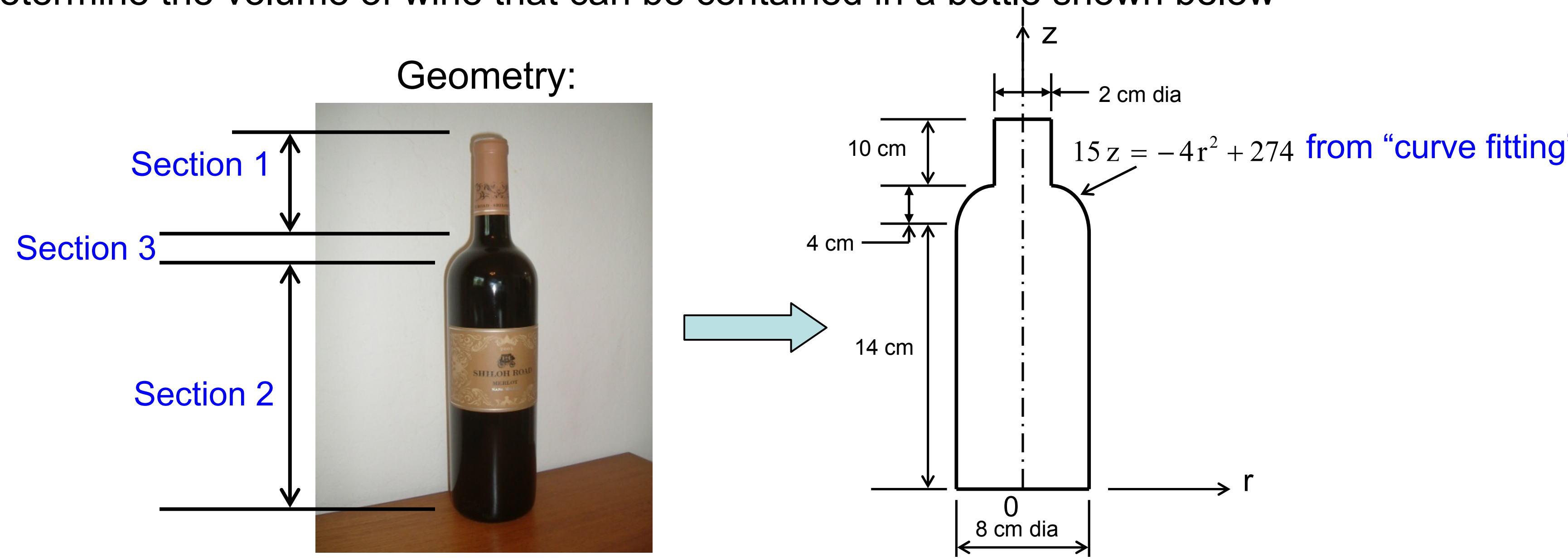


Use Equation (2.4) to determine the volume of revolution:

$$V = \pi \int_0^8 \left[y(x)\right]^2\, dx = \pi \int_0^8 (0.5x)^2\, dx = 42.67\pi$$

Example 2.8 (p. 27)

Determine the volume of wine that can be contained in a bottle shown below



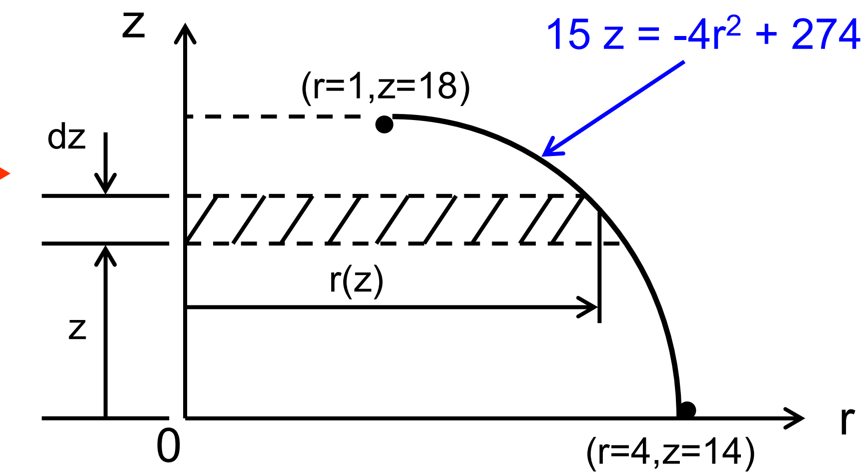
(b) Interior profile of a wine bottle

- Wine bottle is a "solid of revolution," with the coordinate z being the axis of revolution
- Because the axis of revolution coincides with a "vertical" coordinate, we will use Eq. (2.5) to get the volume of the 3 designated sections
- The volume of Section 1 and 2 are right cylinders. There is no need to use integration method

Volume of Section 1: $V_1 = \frac{\pi}{4} d_1^2 \,\ell_1 = 0.785\,(2)^2 \,x\,10 = 31.4 \;cm^3$

Volume of Section 2: $V_2 = \frac{\pi}{4} d_2^2 \,\ell_2 = 0.785\,(8)^2 \,x\,14 = 703.36 \;cm^3$

Volume of Section 3 – the curved section:



By using Equation (2.5):

$$V_3 = \int_{14}^{18} \pi [r(z)]^2 \, dz = \pi \int_{14}^{18} r^2 \, dz = \pi \int_{14}^{18} \frac{274 - 15z}{4} dz = 106.76 \ cm^3$$

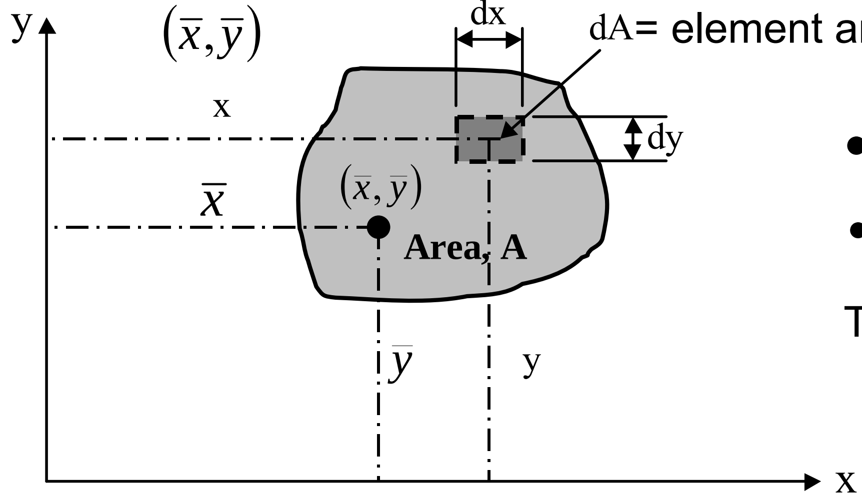
The total volume inside the wine bottle is:

$V = V_1 + V_2 + V_3 = 31.4 + 703.36 + 106.76 = 841.52$ cm^3

Centroid of Plane Areas

- "**Centroid**" is the location in a plane solid, e.g., plates, at which situates the "center of gravity"
- It is an important parameter in "rigid-body dynamic analysis" and "computer-aided-design"

Centroid location:



Define:

- Area moment about the x-axis: $M_x = \int y\,dA$
- Area moment about the y-axis: $M_y = \int x\,dA$

Then, the centroid location is:

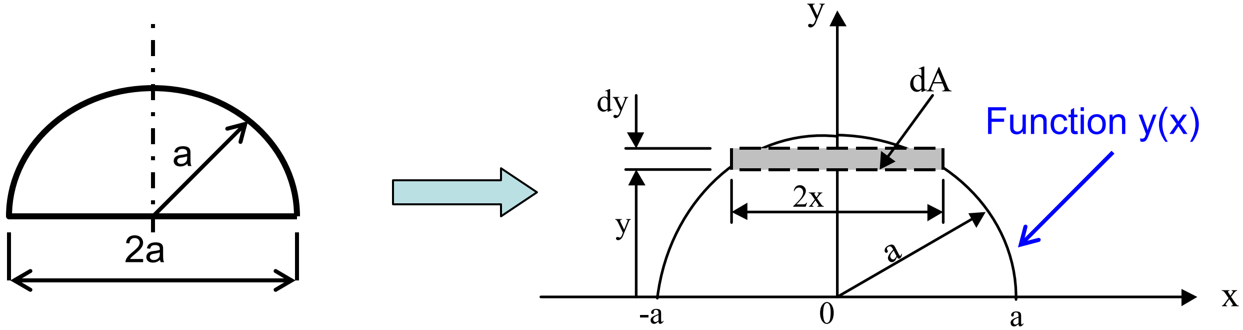
$$\bar{x} = \frac{M_y}{A} \quad \text{and} \quad \bar{y} = \frac{M_x}{A}$$

Expressions for area moments and centroid location:

$$M_x = \int_{x_1}^{x_2} \left(\frac{1}{2} y\right) dA = \frac{1}{2}\int_{x_1}^{x_2} y^2 dx \quad \Rightarrow \quad \bar{y} = \frac{M_x}{A} = \frac{1}{2}\frac{\int_{x_1}^{x_2} [y(x)]^2\, dx}{\int_{x_1}^{x_2} y(x) dx} \qquad (2.7a)$$

$$M_y = \int_{x_1}^{x_2} x dA = \int_{x_1}^{x_2} xy\, dx \quad \Rightarrow \quad \bar{x} = \frac{M_y}{A} = \frac{\int_{x_1}^{x_2} x[y(x)] dx}{\int_{x_1}^{x_2} y(x) dx} \qquad (2.7b)$$

Example 2.9 Determine the location of the centroid in a plate of semi-circular geometry



The function y(x) can be derived from the equation of circle: $x^2 + y^2 = a^2$ in the forms:

$$y(x) = \sqrt{a^2 - x^2} \qquad \text{or} \qquad x(y) = \sqrt{a^2 - y^2}$$

By using Equation (2.7), we have:

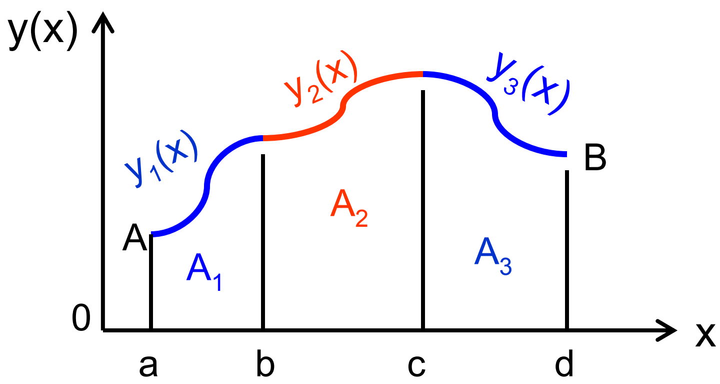
$$\bar{y} = \frac{\int y dA}{A} = \frac{2\int xy\,dy}{2\int x\,dy} = \frac{\int_0^a y\sqrt{a^2 - y^2}\,dy}{\int_0^a \sqrt{a^2 - y^2}\,dy}$$

leading to:

$$\bar{y} = \left. \frac{-\frac{1}{3}\sqrt{\left(a^2 - y^2\right)^3}}{\frac{1}{2}\left[y\sqrt{a^2 - y^2} + a^2 \operatorname{Sin}^{-1}\frac{y}{|a|}\right]} \right|_0^a = \frac{\frac{1}{3}a^3}{\frac{1}{2}a^2\left(\frac{\pi}{2}\right)} = \frac{4a}{3\pi}$$

The location $\bar{X}$ is at x = 0 because of the symmetry of geometry about y-axis

Centroid of Plane Areas Enclosed by Multiple Functions – Not available in the printed notes



Areas of individual elements:

$$A_1 = \int_a^b y_1(x)\,dx$$

$$A_2 = \int_b^c y_2(x)\,dx$$

$$A_3 = \int_c^d y_3(x)\,dx$$

Calculate centroids of individual elements using Equations (2.7a) and (2.7b):

$(\bar{x}_1, \bar{y}_1)$ for Element 1

$(\bar{x}_2, \bar{y}_2)$ for Element 2

$(\bar{x}_3, \bar{y}_3)$ for Element 3

The centroid for the plane defined by Abad $(\bar{x}, \bar{y})$ can be obtained by the following expressions:

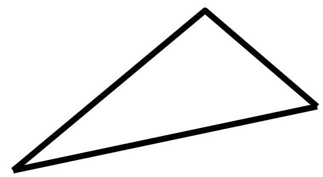
$$\bar{x} = \frac{\bar{x}_1 A_1 + \bar{x}_2 A_2 + \bar{x}_3 A_3}{A_1 + A_2 + A_3}$$

and

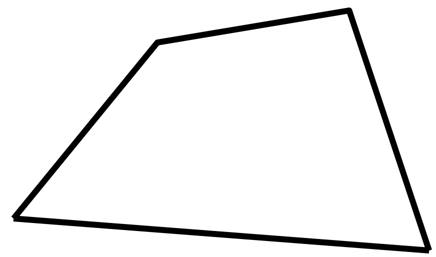
$$\bar{y} = \frac{\bar{y}_1 A_1 + \bar{y}_2 A_2 + \bar{y}_3 A_3}{A_1 + A_2 + A_3}$$

Centroid of Plane Areas Enclosed by Multiple Functions – Not available in the printed notes

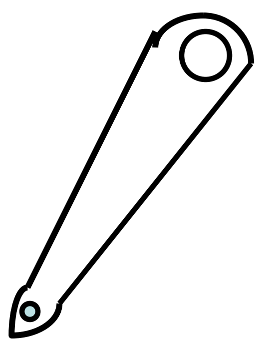
Examples: Determine the locations of the centroid in the following plates:



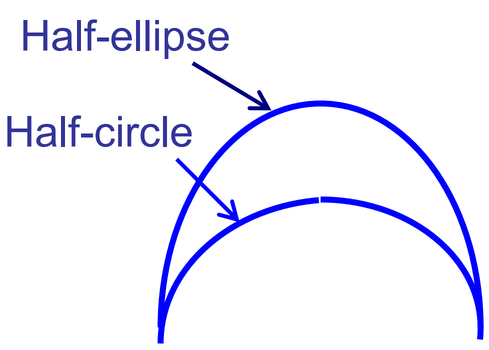
A triangular plate



A joint in a large mechanism



A robotic arm



A special plate

Differential Equations in Mathematical Modeling

What are differential equations?

Equations involving derivatives (of different orders)

How differential equations are derived?

They are derived from the laws of physics

What are the laws of physics relevant to engineering applications?

- **Fundamental laws of Physics:**
  - Conservation of energy – The first law of thermodynamics
  - Conservation of momentum
  - Conservation of mass
- **Application laws of physics in ME:**
  - Newton's laws for solid mechanics (static and dynamic)
  - Fourier law for heat conduction in solids
  - Newton's cooling law for convective heat transfer in fluids
  - Bernoullis law for fluid dynamics