San Jose State University
Department of Mechanical Engineering

ME 130 Applied Engineering Analysis

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Chapter 7

Introduction to Partial Differential Equations

CONDENSED VERSION

2018 version

Chapter 7 Learning Objectives

- What is a partial differential equation (PDE)?
- What is “free vibration analysis of Cable structures and why it is an important vibration analysis?
- Partial differential equations for transverse vibration of strings (equivalent to long hanging cables in reality)
- Modes of vibration of strings (long hanging cables)

What is a Partial Differential Equation?

It is a differential equation that involves partial derivatives.

A partial derivative represents the rate of change of a function (a physical quantity) with respect to more than one independent variable.

Independent variables in partial derivatives can be:

(1) "*Spatial*" variables represented by (x,y,z) in a Cartesian coordinate system, or (r,θ,z) in a cylindrical coordinate system, and

(2) The "*Temporal*" variable represented by time, t.

Examples of partial derivatives of function F(x,t):

First order partial derivatives: $\frac{\partial F(x,t)}{\partial x}$ $\frac{\partial F(x,t)}{\partial t}$

Second order partial derivatives: $\frac{\partial^2 F(x,t)}{\partial x^2}$ $\frac{\partial^2 F(x,t)}{\partial t^2}$ $\frac{\partial^2 F(x,t)}{\partial x \partial t}$

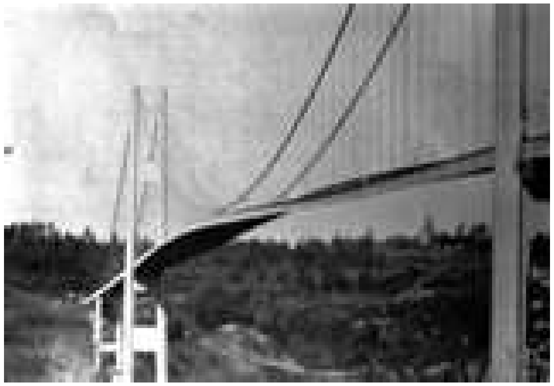
Partial Differential Equations for Transverse Vibration of Strings
- Why it is an important part of ME analysis

Transverse vibration of strings (or long flexible cable structures in reality) is used in common structures such as power transmission lines, guy wires, suspension bridges.

These structures, flexible in nature, are vulnerable to resonant vibrations, which may be devastating overall structural failure – resulting in colossal property losses.

Radio (TV) towers supported by guy wires

Long power transmission lines



A cable suspension bridge at the verge of collapsing

It is thus an important subject for mechanical and structural engineers in their safe design of this type of structures.

Derivation of Partial Differential Equation for Lateral Vibration of Strings

Idealizations and Assumptions:

Like all engineering analyses, there are a number of assumptions we need to make (idealizations, the Stage 2 in Engineering analysis as described in Section 1.4) that one needs to make for this particular case as follows.

(1) The string is flexible.
This assumption means that the string has no bending strength. Hence it cannot resist bending moment, and there is no shear force associated with its deflection.

(2) There exists a tension in the string (why?). It is designated as P, which is so large that the weight, but not the mass, of the string is neglected.

(3) Every small segment of the string along its length, i.e. the segment with a length Δx moves in the vertical direction only during vibration.

(4) The slope of the deflection curve α is small.

(5) The mass of the string along the length is constant, i.e. the string is made of same material along the length.

Vibration Analysis of Strings (or Long Cable Structures)

Mathematical modeling of vibrating:

Flexible structures with the derivation of appropriate differential equation may begin with the free-body diagram of forces applied to the vibrating "string" as illustrated in Figure 7.2 below.

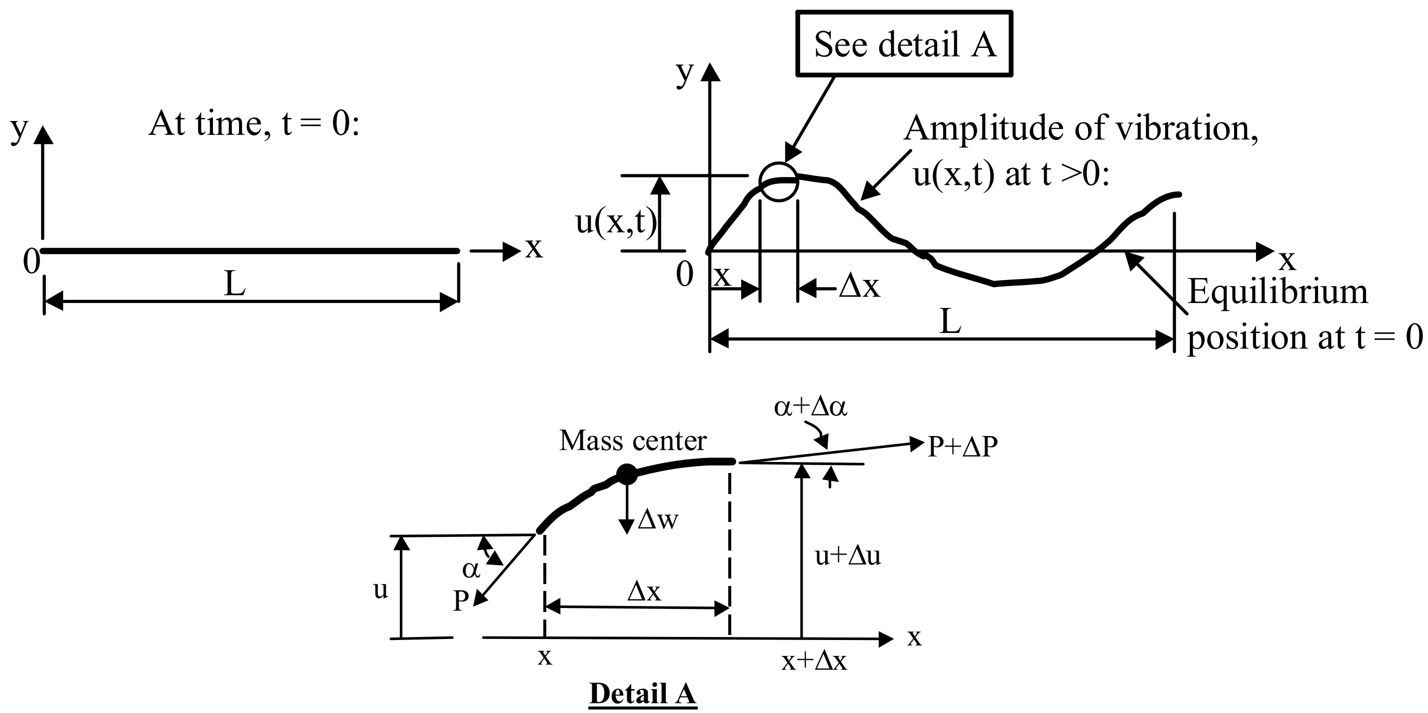


Figure 7.2 A Vibrating String

Let the mass per unit length of the string be (m).

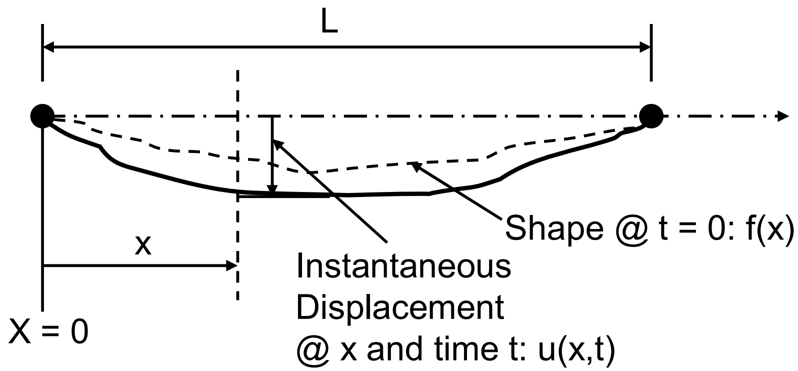
The total mass of string in an incremental length Δx will thus be ($m\Delta x$).

The condition for a dynamic equilibrium according to Newton's second law, or "*equation of motion*" can be expressed the relationship:

For a small segment of the cable, the following dynamic force is generated associate with the vibratory movement up and down from its original equilibrium position:

Applied forces: $\sum F$	=	Mass of Cable Segment M	X	Acceleration a

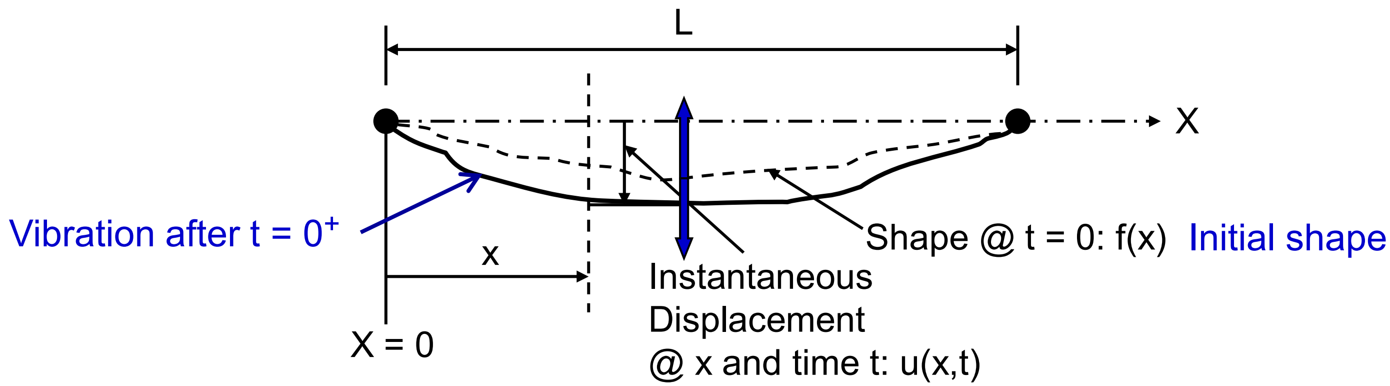
From which, we derive the partial differential equation to model the lateral vibration of an initially "sagged" string subject to an initial disturbance in the vertical direction. This situation is similar to a "mass" supported by a "spring" vibrate from its initial deflected position by a small instantaneous disturbance.



The initial shape of the string is described by function f(x) in the figure to the left.

The partial differential equation for lateral vibration of a string:

The length of the string = L, and it is fixed at both ends at x = 0 and x = L



The deflection of the string at x-distance from the end at x = 0 at time t into the vibration can be obtained by solving the following partial differential equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad \textbf{(7.1)}$$

in which $a = \sqrt{\frac{P}{m}}$ with P = tension in the string (or long cable) hung from both ends, and m = mass density of the string (or long cable) per unit length

A strong reminder: Make sure that you know how to determine the constant coefficient (a) in Equation (7.1) in the above expression with given cable diameter (d) and the mass density of the material (ρ) that makes the cable. Also make sure that you have the units of these quantities right so that you will get the unit for (a) to be the same as for a "velocity".

Equation (7.1) is often used for "wave propagations in solids," in which u(x,t) represents the amplitudes of the wave, x for the distance that wave travels and (a) to be the wave speed.

The partial differential equation and the specific conditions:

The deflection of the string at x-distance from the end at x = 0 at time t into the vibration can be obtained by solving the following partial differential equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad (7.1)$$

where u(x,t) is the amplitude of the vibrating cable at position x and at time t

Required specific conditions are: (2 for the time variable t, and 2 for the space variable x):

Initial conditions = conditions of the string before vibration takes place:

u(x,0) = f(x) = the function describing the initial shape of the string before vibration, and

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0$$ = the initial velocity of the string before vibration (7.2a and b)

Boundary conditions = the end condition at two end supports at ALL times:

u(0,t) = 0, and (7.3 a and b)
u(L,t) = 0

Solution of Partial Differential Equation (7.1) by Separation of Variables Method

We realize a fact that there are two independent variables, i.e. x and t involved in the function u(x,t). We need to **separate these two variables** from the function u(x,t) by letting:

$$u(x,t) = X(x)\,T(t) \qquad (7.4)$$

in which the function X(x) involves the variable, x only,
and the other function T(t) involves the variable t only.

The relation in Eq. (7.4) leads to:

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial}{\partial x}\left[X(x)T(t)\right] = T(t)\frac{\partial X(x)}{\partial x} = T((t)X'(x)$$

$$\frac{\partial(x,t)}{\partial t} = \frac{\partial}{\partial t}\left[X(x)T(t)\right] = X(x)\frac{\partial T(t)}{\partial t} = X(x)T'(t)$$

Likewise, we may express the 2nd order partial derivatives as:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u(x,t)}{\partial x}\right] = T(t)X''(x) \qquad (7.5)$$

and

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial u(x,t)}{\partial t}\right] = T''(t)X(x)$$

By substituting Eq. (7.5) into Eq. (7.1), we get:

$$X(x)\frac{d^2T(t)}{dt^2} = a^2T(t)\frac{d^2X(x)}{dx^2}$$

Did you notice the partial derivative signs "∂" were replace by "d" in the above expression??

After re-arranging the terms, we have:

LHS = $\frac{1}{a^2T(t)}\frac{d^2T(t)}{dt^2} = \frac{1}{X(x)}\frac{d^2X(x)}{dx^2}$ = **RHS**

We realize the fact that: **LHS** = function of variable t ONLY,
and the **RHS** = function of variable x ONLY!

The **ONLY** possible way to have the above **LHS = RHS** is:

$$\frac{1}{a^2T(t)}\frac{d^2T(t)}{dt^2} = \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\beta^2 \qquad (7.6)$$

in which β = separation constant (+ve, or –ve). The value of which needs to be determined later in the mathematical manipulations. The negative sign for β^2 in Equation (7.6) is to ensure the negative quantity at the RHS of that expression for valid subsequent computations.

We will thus get **two ordinary differential equations** from (7.6):

$$\frac{1}{a^2T(t)}\frac{d^2T(t)}{dt^2} = \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\beta^2$$

LHS = -β²:

$$\frac{d^2T(t)}{dt^2} + a^2\beta^2\,T(t) = 0 \qquad (7.7)$$

RHS = -β²:

$$\frac{d^2X(x)}{dx^2} + \beta^2\,X(x) = 0 \qquad (7.8)$$

After applying same separation of variables as illustrated in Eq. (7.4) on the specified conditions in Eq. (7.2), we get the two sets of ordinary differential equations With specific conditions as:

$$\frac{d^2T(t)}{dt^2} + a^2\beta^2\,T(t) = 0 \qquad (7.7)$$

T(0) = f(x) (7.7a)

$$\left.\frac{dT(t)}{dt}\right|_{t=0} = 0 \qquad (7.7b)$$

$$\frac{d^2X(x)}{dx^2} + \beta^2\,X(x) = 0 \qquad (7.8)$$

X(0) = 0 (7.8a)

X(L) = 0 (7.8b)

Both Eqs. (7.7) and (7.8) are linear 2nd order DEs, we have learned how to solve them From Chapter 4:

T(t) = A Sin(βat) + B Cos(βat) (7.9a)

X(x) = C Sin(βx) + D Cos(βx) (7.9b)

Because we have assumed that u(x,t) = T(t)X(x) as in Eq. (7.4), upon substituting the Solutions of T(t) and X(x) in Eqs. (7.9a) and (7.9b), we have:

T(t) = A Sin(βat) + B Cos(βat)

X(x) = C Sin(βx) + D Cos(βx)

$$u(x,t) = [A \sin(\beta a t) + B \cos(\beta a t)][C \sin(\beta x) + D \cos(\beta x)] \quad (7.10)$$

where A, B, C, and D are arbitrary constants need to be determined from initial and boundary conditions given in Eqs. (7.7a,b) and (7.8a,b)

Determination of arbitrary constants:

Let us start with the solution: X(x) = C Sin(βx) + D Cos(βx) in Eq. (7.9b):

From Eq. (7.8a): **X(0) = 0**: ⟶

C Sin (β*0) + D Cos (β*0) = 0, which means that D = 0 ⟶ X(x) = C Sin(βx)

Now, from Eq. (7.8b): X(L) = 0: ⟶ X(L) = 0 = C Sin(βL)

At this point, we have the choices of letting C = 0, or Sin (βL) = 0 from the above relationship. A careful look at these choices will conclude that C ≠ 0 (why?), which leads to:

Sin (βL) = 0

The above expression is effectively a transcendental equation with infinite number of valid solutions with βL= 0, π, 2π, 3π, 4π, 5π…………nπ , in which n is an integer number. We may thus obtain the values of the "separation constant, β" to be:

$$\longrightarrow \quad \beta_n = \frac{n\pi}{L} \quad (n = 0, 1, 2, 3, 4...........) \quad (7.11)$$

Now, if we substitute the solution of X(x) in Eq. (7.9b) with D=0 and $\beta_n = n\pi/L$ with n = 1, 2, 3,..n into the solution of u(x,t) in Eq. (7.10) below:

$$u(x,t) = [A\,Sin(\beta at) + B\,Cos(\beta at)][\,C\,Sin(\beta x) + D\,Cos(\beta x)]$$

we get:

$$u(x,t) = \left(ASin\frac{n\pi}{L}at + BCos\frac{n\pi}{L}at\right)C\,Sin\frac{n\pi}{L}x \qquad (n = 1, 2, 3,,\ldots\ldots)$$

By combining constants A, B and C in the above expression, we have the interim solution of u(x,t) to be:

$$u(x,t) = \left(a_n\,Sin\frac{n\pi}{L}at + b_n\,Cos\frac{n\pi}{L}at\right)Sin\frac{n\pi}{L}x \qquad (n = 1, 2, 3,,\ldots\ldots)$$

We are now ready to use the two initial conditions in Eqs (7.2.a) and (7.2b) to determine constants a_n and b_n in the above expression:

Let us first look at the condition in Eq. (7.2b): $\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0 \longrightarrow$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0 = \frac{n\pi a}{L}\left.\left(a_n Cos\frac{n\pi at}{L} - b_n Sin\frac{n\pi at}{L}\right)\right|_{t=0} Sin\frac{n\pi}{L}x$$

But since $Sin\frac{n\pi}{L}x \neq 0$ (why?) $\longrightarrow$ $a_n = 0$ $\longrightarrow$

$$u(x,t) = \sum_{n=1}^{\infty} b_n\,Cos\frac{n\pi a}{L}t\;Sin\frac{n\pi a}{L}x \qquad (7.13)$$

The only remaining constants to be determined are: b_n as in Eq. (7.13)

Determination of constant coefficients b_n in Eq. (7.13):

$$u(x,t) = \sum_{n=1}^{\infty} b_n \, Cos \frac{n\pi a}{L} t \; Sin \frac{n\pi a}{L} x$$

The last remaining condition of **u(x,o) = f(x)** will be used for this purpose, in which f(x) is the initial shape of the string.

Thus, by letting u(x,0) = f(x), we will have:

$$u(x,0) = \sum_{n=1}^{\infty} b_n \, Sin \frac{n\pi x}{L} = f(x) \qquad \text{with } 0 \le x \le L \tag{7.13}$$

There are a number of ways to determine the coefficients b_n in Eq. (7.14). What we will do is to "borrow" the expressions that we derived from Fourier series in the form:

$$f(x) = \sum_{n=1}^{\infty} b_n \, Sin \frac{n\pi x}{L} \tag{7.14}$$

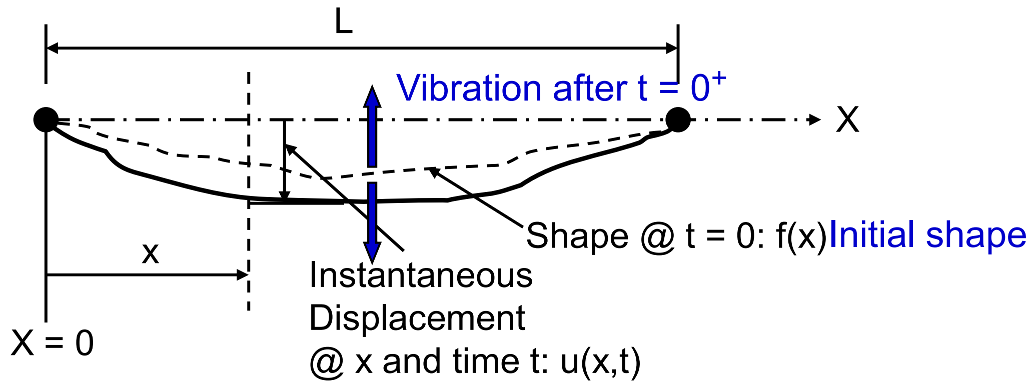
The coefficient b_n of the above Fourier series is:

$$b_n = \frac{2}{L} \int_0^L f(x) \, Sin \frac{n\pi x}{L} dx \tag{7.15}$$

The complete solution of the amplitude of vibrating string u(x,t) becomes:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{L} \left(\int_0^L f(x) \, Sin \frac{n\pi x}{L} dx \right) Cos \frac{n\pi at}{L} \, Sin \frac{n\pi x}{L} \tag{7.16}$$

Modes of Vibration of Strings



We have just derived the solution on the AMPLITUDES of a vibrating string, u(x,t) to be:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{L}\left(\int_0^L f(x)\, Sin\frac{n\pi x}{L}dx\right) Cos\frac{n\pi at}{L}\, Sin\frac{n\pi x}{L} \qquad (7.16)$$

We realize from the above form that the solution consists of INFINITE number of terms with n = 1, n = 2, n = 3,………… What it means is that each term alone is a VALID solution. Hence: u(x,t) with one term with n = 1 only is one possible solution, and u(x,t) with n = 2 only is another possible solution, and so on and so forth.

Consequently, because the solution u(x,t) also represents the INSTANTANEOUS SHAPE of the vibrating string, there could be many POSSIBLE instantaneous shape of the vibrating string depending on what the term in Eq. (7.16) is used.

Predicting the possible forms (or INSTATANEOUS SHAPES) of a vibrating string is called **MODAL ANALYSIS**

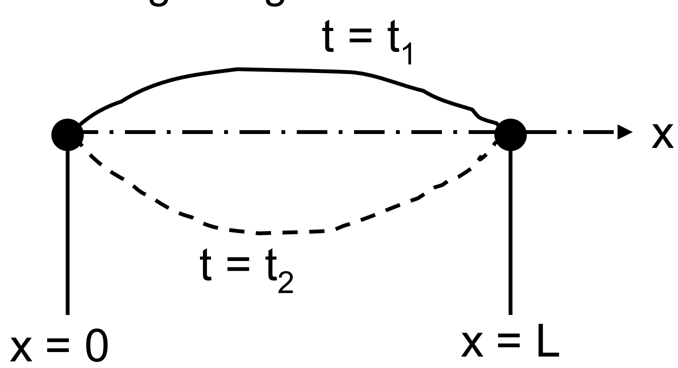
First Three Modes of Vibrating Strings:

We will use the solution in Eq, (7.16) to derive the first three modes of a vibrating string.

Mode 1 with n = 1 in Eq. (7.16):

$$u_1(x,t) = \left(b_1 Cos \frac{\pi a t}{L} \right) Sin \frac{n\pi}{L} x \qquad (7.17)$$

The SHAPE of the Mode 1 vibrating string can be illustrated according to Eq, (7.17) as:



We observe that the maximum amplitudes of vibration occur at the mid-span of the string

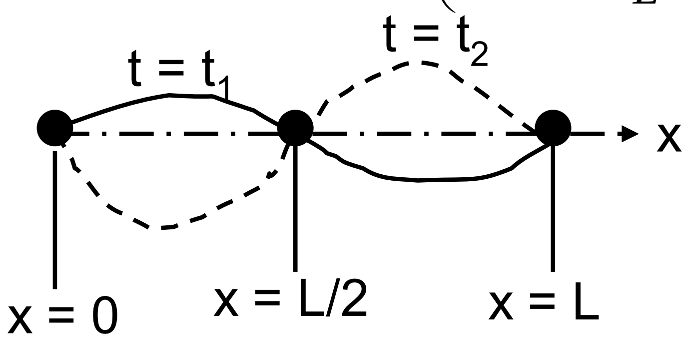
The corresponding frequency of vibration is obtained from the coefficient in the argument of the cosine function with time t, i.e.:

$$f_1 = \frac{\pi a / L}{2\pi} = \frac{a}{2L} = \frac{1}{2L}\sqrt{\frac{T}{m}} \qquad (7.18)$$

where T = tension in Newton or pounds, and M = mass density of string/unit length

Mode 2 with n = 2 in Eq. (7.16):

$$u_2(x,t) = \left(b_2 Cos\frac{2\pi a}{L}t\right) Sin\frac{2\pi}{L}x \qquad (7.19)$$

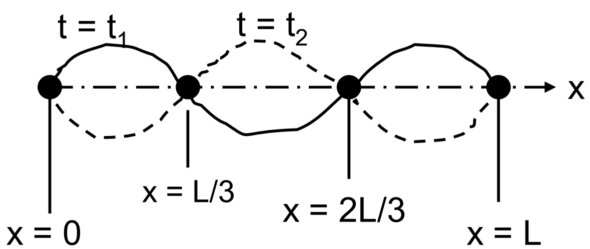


Maximum amplitudes occur at 2 positions, with zero amplitudes at two ends and mid-span

Frequency: $f_2 = \frac{2\pi a/L}{2\pi} = \frac{a}{L} = \frac{1}{L}\sqrt{\frac{T}{m}}$

Mode 3 with n = 3 in Eq. (7.16):

$$u_3(x,t) = \left(b_3 Cos\frac{3\pi a}{L}t\right) Sin\frac{3\pi}{L}x \qquad (7.21)$$



Maximum amplitudes occur at 3 positions, with Zero amplitudes at 4 locations.

Frequency: $f_3 = \frac{3\pi a/L}{2\pi} = \frac{3a}{2L} = \frac{3}{2L}\sqrt{\frac{T}{m}}$

Modal analysis provides engineers with critical information on where the possible maximum amplitudes may exist when the string vibrates, and the corresponding frequency of occurrence. Identification of locations of maximum amplitude allows engineers to predict possible locations of structural failure, and thus the vulnerable location of string (long cable) structures.

Chapter 8

Matrices and Solution to Simultaneous Equations Using Matrix Techniques

CONDENSED VERSION

for Spring 2018

Chapter Learning Objectives and Self Review and Re-learn

(NOTE: All self review and re-learned subjects will be in the final exam too. Students are encouraged to contact the Instructor for their difficulty in self-learning of the designated topics in this Chapter)

- **Matrices in engineering analysis**

- **Distinction between matrices and determinates (self review)**
- **Evaluation of determinants (self review)**
- **Different forms of matrices (Section 8.2 P. 198 to 201– self learning)**
- **Transposition of matrices (Section 8.3, p. 201)**
- **Addition, subtraction and multiplication of matrices (Section 8.4 201-204 self review & learning)**
- **Inversion of matrices (Section 8.3, p. 204-207, Self learning)**

- **Solution of simultaneous equations using matrix inversion method (Section 8.6)** **You are encouraged to invert matrices using available software. Most pocket electronic calculators have this built-in SW.**
- **Solution of large numbers of simultaneous equations using Gaussian elimination method (Section 8.7)**
  You MUST use the method and procedures included in the printed lecture notes of this course in your solution for solving simultaneous equations using Gaussian method in the final exam.

Matrices and Engineering Analysis

Matrices are the logical and convenient representations of LARGE NUMBERS OF REAL NUMBERS AND EQUATIONS that frequently occur in many engineering analyses,

Matrix algebra is for arithmetic manipulations of matrices. It is a vital tool to solve systems of linear equations

Matrix techniques are most efficient tools to solve very large number of simultaneous equations often occur in advance numerical analyses and engineering software codes, e.g., in finite element analysis and computer-aided design codes.

What are Matrices?

- Matrices are used to express **arrays** of numbers, variables or data in a logical format that can be accepted by digital computers

- Matrices are made up with ROWS and COLUMNS:

Column (n): n=1 n=2 n=3 n = n

Row Number (m):

Abbreviation of Rectangular Matrices:

$$[A] = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \bullet & \bullet & \bullet & \bullet & a_{1n} \\ a_{21} & a_{22} & a_{23} & \bullet & \bullet & \bullet & \bullet & a_{2n} \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ a_{m1} & a_{m2} & a_{m3} & \bullet & \bullet & \bullet & \bullet & a_{mn} \end{bmatrix}$$

Matrix elements
i = row number
J = column number

Rows (m): m =1, m =2, •, •, m = m

- Matrices can represent vector quantities such as force vectors, stress vectors, velocity vectors, etc. All these vector quantities consist of several components

- Huge amount of numbers and data are common place in modern-day engineering analysis, especially in numerical analyses such as the finite element analysis (FEA) or finite deference analysis (FDA)

Different Forms of Matrices (section 8.2), Self learning:

1. Rectangular matrices
2. Square matrices
3. Row matrices
4. Column matrices
5. Upper triangular matrices
6. Lower triangular matrices
7. Diagonal matrices
8. Unity matrices

The transposition of a matrix [A] is designated by $[A]^T$:

- The transposition of matrix [A] is carried out by interchanging the elements in a square matrix across the diagonal line of that matrix:

Diagonal of a square matrix

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(a) Original matrix

$$[A]^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

(b) Transposed matrix

diagonal line

MATRIX ALGEBRA (Section 8.4, Self-learning)

1) Addition and subtraction of matrices

The involved matrices must have the SAME size (i.e., number of rows and columns):

$$[A] \pm [B] = [C] \ \ with \ \ elements \ c_{ij} = a_{ij} \pm b_{ij}$$

2) Multiplication with a scalar quantity (α):

$$\alpha\,[C] = [\alpha\, c_{ij}]$$

3) Multiplication of 2 matrices (self review and learn):

Multiplication of two matrices is possible only when:

The total number of columns in the 1st matrix
= the number of rows in the 2nd matrix:

$$\underset{(m \times p)}{[C]} = \underset{(m \times n)}{[A]} \times \underset{(n \times p)}{[B]}$$

The following recurrence relationship applies:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a_{in}b_{nj}$$

with i = 1, 2, ….,m and j = 1,2,….., n

Matrix Inversion (Section 8.5, Self-review and learn)

The inverse of a matrix [A], expressed as **[A]$^{-1}$**, is defined as:

$$[A][A]^{-1} = [A]^{-1}[A] = [I] \qquad (8.13)$$

(a **UNITY** matrix)

NOTE: the inverse of a matrix [A] exists ONLY if $|A| \neq 0$

where $|A|$ = the equivalent determinant of matrix [A]

Following are the general steps in inverting the matrix [A]: (self learning)

Step 1: Evaluate the equivalent determinant of the matrix. Make sure that $|A| \neq 0$

Step 2: If the elements of matrix [A] are a_{ij}, we may determine the elements of a co-factor matrix [C] to be:

$$c_{ij} = (-1)^{i+j}\,|A'| \qquad (8.14)$$

in which $|A'|$ is the equivalent determinant of a matrix [A'] that has all elements of [A] excluding those in ith row and jth column.

Step 3: Transpose the co-factor matrix, [C] to $[C]^T$.

Step 4: The inverse matrix $[A]^{-1}$ for matrix [A] may be established by the following expression:

$$[A]^{-1} = \frac{1}{|A|}[C]^T \qquad (8.15)$$

Solution of Simultaneous Equations Using Matrix Techniques

A vital tool for solving very large number of simultaneous equations

Solution of Simultaneous Equations Using Inverse Matrix Technique

Let us express the n-simultaneous equations to be solved in the following form:

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots + a_{1n}x_n = r_1 \\
&a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots + a_{2n}x_n = r_2 \\
&a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots + a_{3n}x_n = r_3 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots\ldots + a_{mn}x_n = r_n
\end{aligned}
\qquad (8.16)
$$

where a_{11}, a_{12}, ………, a_{mn} are constant coefficients
x_1, x_2, …………., x_n are the unknowns to be solved
r_1, r_2, …………., r_n are the "resultant" constants

The n-simultaneous equations in Equation (8.16) can be expressed in matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \bullet & \bullet & \bullet & \bullet & a_{1n} \\ a_{21} & a_{22} & a_{23} & \bullet & \bullet & \bullet & \bullet & a_{2n} \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ a_{m1} & a_{m2} & a_{m3} & \bullet & \bullet & \bullet & \bullet & a_{mn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \bullet \\ \bullet \\ x_n \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ \bullet \\ \bullet \\ r_n \end{Bmatrix} \quad (8.17)$$

or in an abbreviate form:

$$\mathbf{[A]\{x\} = \{r\}} \quad (8.18)$$

in which [A] = Coefficient matrix with m-rows and n-columns
{x} = Unknown matrix, a column matrix
{r} = Resultant matrix, a column matrix

Now, if we let $[A]^{-1}$ = the inverse matrix of [A], and multiply this $[A]^{-1}$ on both sides of Equation (8.18), we will get:

$$[A]^{-1}[A]\{x\} = [A]^{-1}\{r\}$$

Leading to: $[I]\{x\} = [A]^{-1}\{r\}$, in which [I] = a unity matrix

The unknown matrix, and thus the values of the unknown quantities $x_1, x_2, x_3, \ldots, x_n$ may be obtained by the following relation:

$$\mathbf{\{x\} = [A]^{-1}\{r\}} \quad (8.19)$$

(Self-read Example 8.6)

Solution of Simultaneous Equations Using Gaussian Elimination Method

The essence of Gaussian elimination method:

1) To convert the square coefficient matrix [A] of a set of simultaneous equations into the form of "Upper triangular" matrix in Equation (8.5) using an "*elimination procedure*"

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{\text{Via "elimination process}} [A]^{upper} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

2) The last unknown quantity in the converted upper triangular matrix in the simultaneous equations becomes immediately available.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2' \\ r_3'' \end{Bmatrix} \Longrightarrow x_3 = r_3''/a_{33}''$$

3) The second last unknown quantity may be obtained by substituting the newly found numerical value of the last unknown quantity into the second last equation:

$$a_{22}' x_2 + a_{23}' x_3 = r_2' \Longrightarrow x_2 = \frac{r_2' - a_{23}' x_3}{a_{22}'}$$

4) The remaining unknown quantities may be obtained by the similar procedure, which is termed as "*back substitution*"

The Gaussian Elimination Process:

We will demonstrate this process by the solution of 3-simultaneous equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = r_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = r_2 \qquad \text{(8.20 a,b,c)}$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = r_3$$

We will express Equation (8.20) in a matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} \qquad (8.21)$$

or in a simpler form: $[A]\{x\} = \{r\}$

We may express the unknown x_1 in Equation (8.20a) in terms of x_2 and x_3 as follows:

$$x_1 = \frac{r_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2 + \frac{a_{13}}{a_{11}}x_3$$

Now, if we substitute x_1 in Equation (8.20b and c) by $x_1 = \frac{r_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2 + \frac{a_{13}}{a_{11}} x_3$

we will turn Equation (8.20) from:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = r_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = r_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = r_3$$

→

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = r_1$$
$$0 + \left(a_{22} - a_{21}\frac{a_{12}}{a_{11}}\right)x_2 + \left(a_{23} - a_{21}\frac{a_{13}}{a_{11}}\right)x_3 = r_2 - \frac{a_{21}}{a_{11}}r_1 \qquad (8.22)$$
$$0 + \left(a_{32} - a_{31}\frac{a_{12}}{a_{11}}\right)x_2 + \left(a_{33} - a_{31}\frac{a_{13}}{a_{11}}\right)x_3 = r_3 - \frac{a_{31}}{a_{11}}r_1$$

You do not see x_1 in the new Equation (20b and c) anymore –
So, x_1 is "eliminated" in these equations after Step 1 elimination

The new matrix form of the simultaneous equations has the form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^1 & a_{23}^1 \\ 0 & a_{32}^1 & a_{33}^1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2^1 \\ r_3^1 \end{Bmatrix} \qquad (8.23)$$

$$a_{22}^1 = a_{22} - a_{21}\frac{a_{12}}{a_{11}} \qquad a_{23}^1 = a_{23} - a_{21}\frac{a_{13}}{a_{11}}$$
$$a_{32}^1 = a_{32} - a_{31}\frac{a_{12}}{a_{11}} \qquad a_{33}^1 = a_{33} - a_{31}\frac{a_{13}}{a_{11}}$$
$$r_2^1 = r_2 - \frac{a_{21}}{a_{11}}r_1 \qquad r_3^1 = r_3 - \frac{a_{31}}{a_{11}}r_1$$

The superscript index numbers ("1") indicates "elimination step 1" in the above expressions

Step 2 elimination involve the expression of x_2 in Equation (8.22b) in term of x_3:

from

$$0+\left(a_{22}-a_{21}\frac{a_{12}}{a_{11}}\right)x_2+\left(a_{23}-a_{21}\frac{a_{13}}{a_{11}}\right)x_3=r_2-\frac{a_{21}}{a_{11}}r_1 \qquad (8.22b)$$

to

$$x_2=\frac{r_2-\frac{a_{21}}{a_{11}}r_1-\left(a_{23}-a_{21}\frac{a_{13}}{a_{11}}\right)x_3}{\left(a_{22}-a_{21}\frac{a_{12}}{a_{11}}\right)}$$

and submitted it into Equation (8.22c), resulting in eliminate x_2 in that equation.

The matrix form of the original simultaneous equations now takes the form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{2} & a_{23}^{2} \\ 0 & 0 & a_{33}^{2} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}=\begin{Bmatrix} r_1^{2} \\ r_2^{2} \\ r_3^{2} \end{Bmatrix} \qquad (8.24)$$

We notice the coefficient matrix [A] now has become an "upper triangular matrix," from which we have the solution

$$x_3=\frac{r_3^{2}}{a_{33}^{2}}$$

The other two unknowns x2 and x1 may be obtained by the "back substitution process from Equation (8.24),such as:

$$x_2=\frac{r_2^{2}-a_{23}^{2}x_3}{a_{22}^{2}}=\frac{r_2^{2}-a_{23}^{2}\frac{r_3^{2}}{a_{33}^{2}}}{a_{22}^{2}}$$

Recurrence relations for Gaussian elimination process:

Given a general form of n-simultaneous equations:

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots\dots\dots + a_{1n}x_n = r_1 \\
&a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots\dots\dots\dots + a_{2n}x_n = r_2 \\
&a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots\dots\dots\dots + a_{3n}x_n = r_3 \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots\dots\dots\dots + a_{mn}x_n = r_n
\end{aligned} \quad (8.16)
$$

The following recurrence relations can be used in Gaussian elimination process:

For elimination:

i > n and j>n

$$a_{ij}^{n} = a_{ij}^{n-1} - a_{in}^{n-1} \frac{a_{nj}^{n-1}}{a_{nn}^{n-1}} \quad (8.25a)$$

$$r_{i}^{n} = r_{i}^{n-1} - a_{in}^{n-1} \frac{r_{n}^{n-1}}{a_{nn}^{n-1}} \quad (8.25b)$$

For back substitution

$$x_i = \frac{r_i - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}} \quad with \quad i = n-1, n-2, \dots\dots, 1 \quad (8.26)$$

Example

Solve the following simultaneous equations using Gaussian elimination method:

$$\begin{aligned} x + z &= 1 \\ 2x + y + z &= 0 \\ x + y + 2z &= 1 \end{aligned} \qquad \text{(a)}$$

Express the above equations in a matrix form:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} \qquad \text{(b)}$$

If we compare Equation (b) with the following typical matrix expression of 3-simultaneous equations:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix}$$

we will have the following:

$a_{11} = 1$ $a_{12} = 0$ $a_{13} = 1$ and $r_1 = 1$

$a_{21} = 2$ $a_{22} = 1$ $a_{23} = 1$ $r_2 = 0$

$a_{31} = 1$ $a_{32} – 1$ $a_{33} = 2$ $r_3 = 1$

Let us use the recurrence relationships for the elimination process in Equation (8.25):

$$a_{ij}^{n} = a_{ij}^{n-1} - a_{in}^{n-1}\frac{a_{nj}^{n-1}}{a_{nn}^{n-1}} \qquad r_{i}^{n} = r_{i}^{n-1} - a_{in}^{n-1}\frac{r_{n}^{n-1}}{a_{nn}^{n-1}} \qquad \text{with } i > n \text{ and } j > n$$

Step 1 n = 1, so i = 2,3 and j = 2,3

For i = 2, j = 2 and 3:

i = 2, j = 2: $a_{22}^{1} = a_{22}^{0} - a_{21}^{0}\frac{a_{12}^{0}}{a_{11}^{0}} = a_{12} - a_{21}\frac{a_{12}}{a_{11}} = 1 - 2\frac{0}{1} = 1$

i = 2, j = 3: $a_{23}^{1} = a_{23}^{0} - a_{21}^{0}\frac{a_{13}^{0}}{a_{11}^{0}} = a_{23} - a_{21}\frac{a_{13}}{a_{11}} = 1 - 2\frac{1}{1} = -1$

i = 2: $r_{2}^{1} = r_{2}^{o} - a_{21}^{0}\frac{r_{1}^{0}}{a_{11}^{0}} = r_{2} - a_{21}\frac{r_{1}}{a_{11}} = 0 - 2\frac{1}{1} = -2$

For i = 3, j = 2 and 3:

i = 3, j = 2: $a_{32}^{1} = a_{32}^{0} - a_{31}^{0}\frac{a_{12}^{0}}{a_{11}^{0}} = a_{32} - a_{31}\frac{a_{12}}{a_{11}} = 1 - 1\frac{0}{1} = 1$

i = 3, j = 3: $a_{33}^{1} = a_{33}^{0} - a_{31}^{0}\frac{a_{13}^{0}}{a_{11}^{0}} = a_{33} - a_{31}\frac{a_{13}}{a_{11}} = 2 - 1\frac{1}{1} = 1$

i = 3: $r_{3}^{1} = r_{3}^{0} - a_{31}^{0}\frac{r_{1}^{0}}{a_{11}^{0}} = r_{3} - a_{31}\frac{r_{1}}{a_{11}} = 1 - 1\frac{1}{1} = 0$

So, the original simultaneous equations after Step 1 elimination have the form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^1 & a_{23}^1 \\ 0 & a_{32}^1 & a_{33}^1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2^1 \\ r_3^1 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -2 \\ 0 \end{Bmatrix}$$

We now have:

$$a_{21}^1 = 0 \quad a_{22}^1 = 1 \quad a_{23}^1 = -1$$

$$a_{31}^1 = 0 \quad a_{32}^1 = 1 \quad a_{33}^1 = 1$$

$$r_2^1 = -2 \quad r_3^1 = 0$$

Step 2 n = 2, so i = 3 and j = 3 (i > n, j > n)

i = 3 and j = 3: $$a_{33}^2 = a_{33}^1 - a_{32}^1 \frac{a_{23}^1}{a_{22}^1} = 1 - 1\frac{(-1)}{1} = 2$$

$$r_3^2 = r_3^1 - a_{32}^1 \frac{r_2^1}{a_{22}^1} = 0 - 1\frac{(-2)}{1} = 2$$

The coefficient matrix [A] has now been triangularized, and the original simultaneous equations has been transformed into the form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^1 & a_{23}^1 \\ 0 & 0 & a_{33}^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2^1 \\ r_3^2 \end{Bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -2 \\ 2 \end{Bmatrix}$$

We get from the last equation with (0) x_1 + (0) x_2 + 2 x_3 = 2, from which we solve for x_3 = 1. The other two unknowns x_2 and x1 can be obtained by back substitution of x_3 using Equation (8.26):

$$x_2 = \left(r_2 - \sum_{j=3}^{3} a_{2j} x_j \right) / a_{22} = (r_2 - a_{23} x_2) / a_{22} = [-2 - (-1)(1)]/1 = -1$$

and $$x_1 = \left(r_1 - \sum_{j=2}^{3} a_{1j} x_j \right) / a_{11} = [r_1 - (a_{12} x_2 + a_{13} x_3)] / a_{11}$$

$$= \{1 - [0(-1) + 1(1)]\}/1 = 0$$

We thus have the solution: x = x_1 = 0; y = x_2 = -1 and z = x_3 = 1