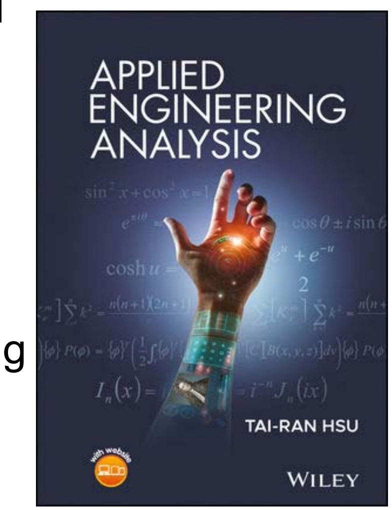
Applied Engineering Analysis
- slides for class teaching*

Chapter 3
Vectors and Vector Calculus

Chapter Learning Objectives

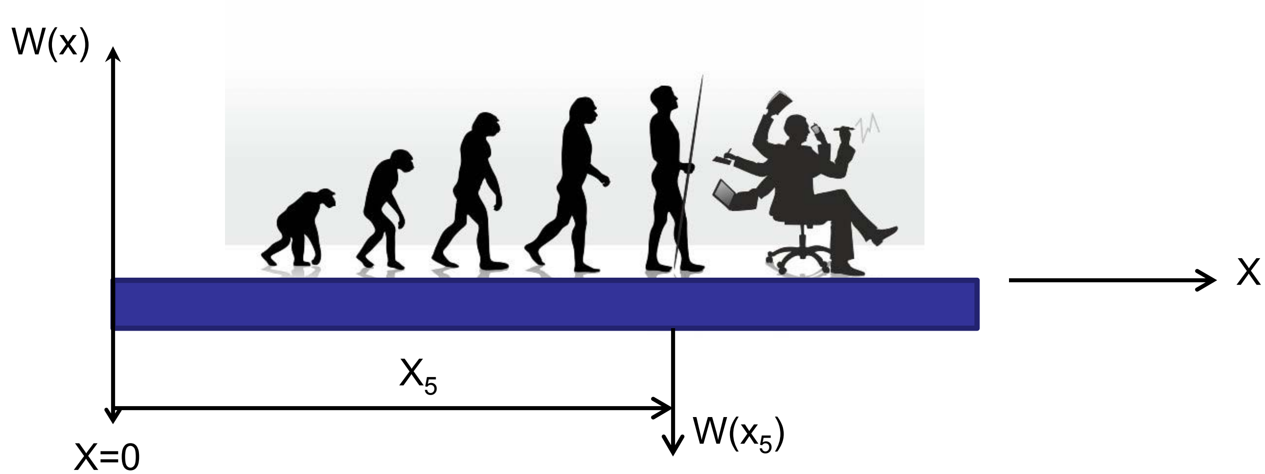
- To refresh the distinction between scalar and vector quantities in engineering analysis
- To learn the vector calculus and its applications in engineering analysis
- Expressions of vectors and vector functions
- Refresh vector algebra
- Dot and cross products of vectors and their physical meanings
- To learn vector calculus with derivatives, gradient, divergence and curl
- Application of vector calculus in engineering analysis
- Application of vector calculus in rigid body dynamics in rectilinear and plane curvilinear motion along paths and in both rectangular and cylindrical polar coordinate system



* Based on the book of "Applied Engineering Analysis", by Tai-Ran Hsu, Published by John Wiley & Sons, 2018

Scalar and Vector Quantities

Scalar Quantities: Physical quantities that have their values determined by the values of the variables that define these quantities. For example, in a beam that carries creatures of different weight with the forces exerted on the beam determined by the location x only, at which the particular creature stands.



Vector Quantities: There are physical quantities in engineering analysis, that has their values determined by NOT only the value of the variables that are associate with the quantities, but also by the directions that these quantities orient.

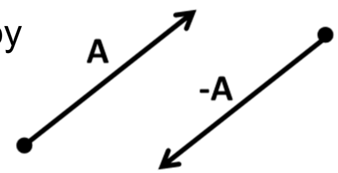
Example of vector quantifies include the velocities of automobile travelin in winding street called Lombard Drive in City of San Francisco the drivers adjusting the velocity of his(her) automobile according to the location of the street with its curvature, but also the direction of the automobile that it travels on that street.

Graphic and mathematical Representation of Vector Quantities

Vector are usually expressed in BOLDFACED letters, e.g. **A** for vector A

Graphic Representation of a Vector A:

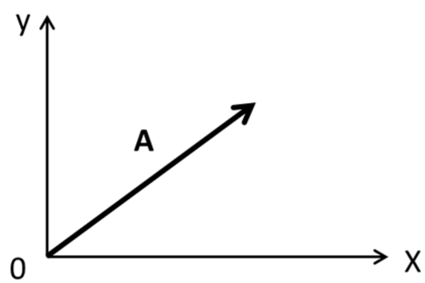
A vector **A** is represented by magnitude A in the direction shown by arrow head:



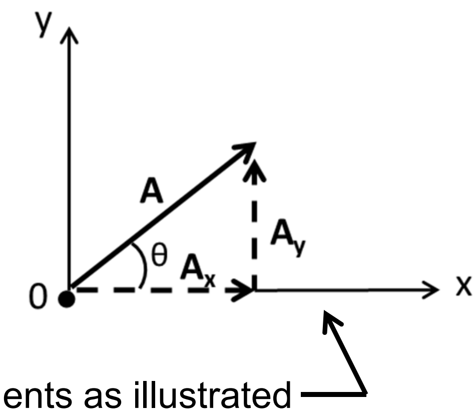
A –ve sign attached to vector A means the Vector orients in OPPOSITE direction

Mathematically it is expressed (in a rectangular coordinates (x,y) as:

With the magnitude expressed by the length of A:



With the magnitude expressed by the length of A: and the direction by θ:



Vector quantities can be DECOMPOSED into components as illustrated

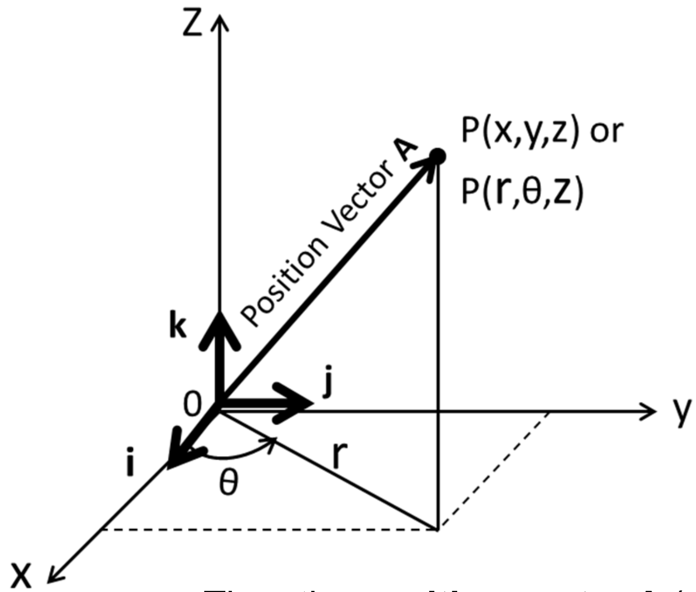
With MAGNITUDE:

$$|\mathbf{A}| = \sqrt{|\mathbf{A}_x|^2 + |\mathbf{A}_y|^2} = \sqrt{A_x^2 + A_y^2}$$

and DIRECTION:

$$\tan\theta = \frac{A_y}{A_x}$$

3.2 Vectors expressed in terms of Unit Vectors in Rectangular coordinate Systems
- A simple and convenient way to express vector quantities



Let: **i** = unit vector along the x-axis
j = unit vector along the y-axis
k = unit vector along the z-axis

in a rectangular coordinate system (x,y,z), or
a cylindrical polar coordinate system (r,θ,z).

All unit vectors **i**, **j** and **k** have a **magnitudes of 1.0 (i.e. unit)**

Then the **position vector A** (with it "root" coincides with th origin of the coordinate system) expressed in the following form:

$$\mathbf{A} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

where x = magnitude of the component of Vector A in the x-coordinate
y = magnitude of the component of Vector A in the y-coordinate
z = magnitude of the component of Vector A in the z-coordinate

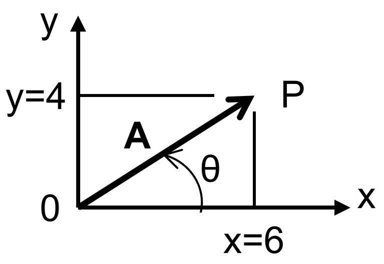
We may thus evaluate the magnitude of the vector **A** to be the sum of the magnitudes of all its components as:

$$|\mathbf{A}| = A = \sqrt{\left(\sqrt{x^2 + y^2}\right)^2 + z^2} = \sqrt{x^2 + y^2 + z^2}$$

Examples of using unit vectors in engineering analysis

Example 3.1: A vector **A** in Figure 3.2(b) has its two components along the x- and y-axis with respective magnitudes of 6 units and 4 units. Find the magnitude and direction of the vector **A**.

Solution: Let us first illustrate the vector **A** in the x-y plane:



The vector A may be expressed in terms of unit vectors **I** and **j** as:

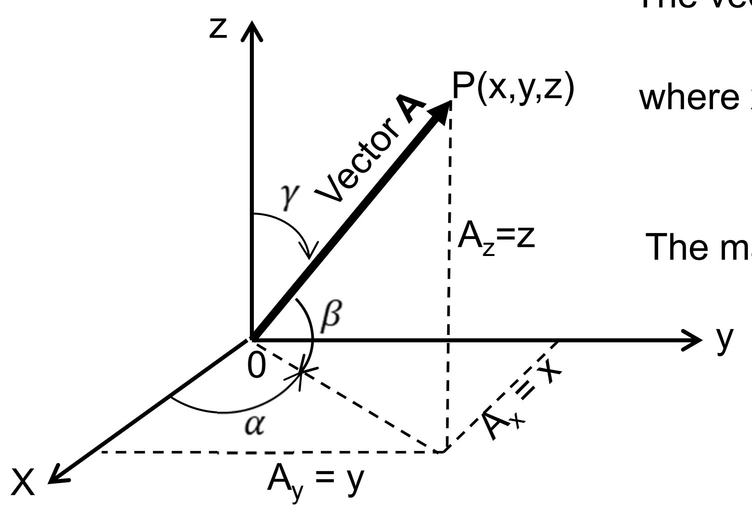
A = 6**i** + 4**J**

And the magnitude of vector **A** is:

$$\mathbf{A} = |\boldsymbol{A}| = \mathrm{A} = \sqrt{x^2 + y^2} = \sqrt{6^2 + 4^2} = \sqrt{52} = 7.21\ units$$

and the angle θ is obtained by: $tan\theta = \frac{y^{\alpha}}{x} = \frac{4}{6} = 0.67$

A Vector in 3-D Space in a Rectangular coordinate System:



The vector A may be expressed in terms of unit vectors **i, j and k** as:

A = x**i** + y**j** + z**k**

where x = magnitude of the component of Vector A in the x-coordinate
y = magnitude of the component of Vector A in the y-coordinate
z = magnitude of the component of Vector A in the z-coordinate

The magnitude of vector **A** is: $\mathbf{A} = |\boldsymbol{A}| = \mathrm{A} = \sqrt{x^2 + y^2 + z^2}$

The direction of the vector is determined by:

$$\cos\alpha = \frac{x}{|\boldsymbol{A}|}, \quad \cos\beta = \frac{y}{|\boldsymbol{A}|} \quad and \quad cos\gamma = \frac{z}{|\boldsymbol{A}|}$$

Addition and Subtraction of Two Vectors

Addition or subtraction of two vectors expressed in terms of **UNIT vectors** is easily done by the addition or subtraction of the corresponding coefficients of the respective unit vectors **i, j and k** as Illustrated below:

Given: The two vectors: Vector **A1**= x_1**i** + y_1**j** +z_1**k** and Vector $\mathbf{A_2}$ = x_2**i** +y_2**J** + z_2 **k**
We will have the addition and subtraction of these two vectors to be:

$$\mathbf{A_1} \pm A_2 = (x_1 \pm x_2)\mathbf{i} + (y_1 \pm y_2)\mathbf{j} + (z_1 \pm z_2)\boldsymbol{k}$$

Example 3.3 If vectors **A** = 2**i** +4**k** and **B** = 5**j** +6**k**, determine: (a) what planes do these two vectors exist, and (b) their respective magnitudes. (c) the summation of these two vectors

Solution:

(a) Vector **A** may be expressed as: **A** = 2**i** +0**j** + 4**k**, so it is positioned in the x-z plane in Figure 3.3. Vector **B** on the other hand may be expressed as: **B** = 0**i** + 5**j** +6**k** with no value along the x-coordinate. So, it is positioned in the y-z plane in a rectangular coordinate system.

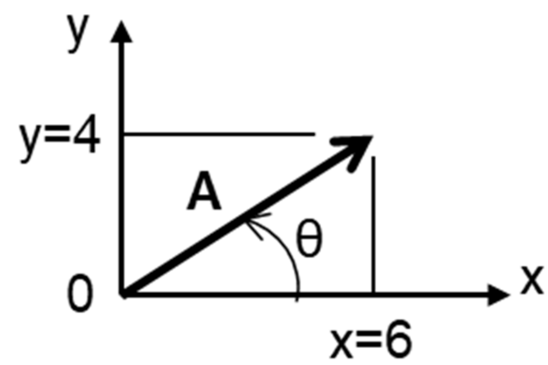
(b) The magnitude of vector **A** is: $|\mathbf{A}| = \mathrm{A} = \sqrt{2^2 + 4^2} = \sqrt{20} = 4.47$

and the magnitude of vector **B** is: $|\mathbf{B}| = \mathrm{B} = \sqrt{5^2 + 6^2} = \sqrt{61} = 7.81$

(c) The addition of these two vectors is:

$$|\boldsymbol{A}| + |B| = (2+0)\boldsymbol{i} + (0+5)\mathbf{j} + (4+6)\mathbf{k} = 2\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$$

Example 3.5



Determine the angle θ of a position vector **A = 6i + 4j** in an x-y plane.

Solution: We may express the vector A in the form of:

$$\mathbf{A} = 6\mathbf{i} + 4\mathbf{j}$$

with **i** and **j** to be the respective unit vectors along the x- and y-coordinates with the magnitudes: x = 6 units and y =4 units.

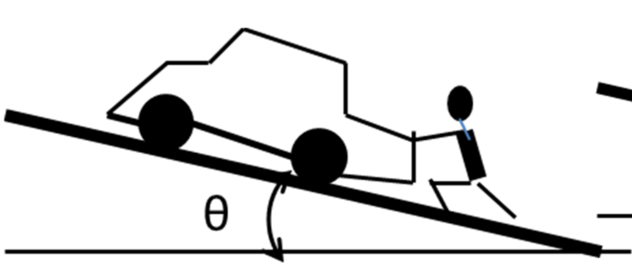
We may thus compute the magnitude of the vector A to be:

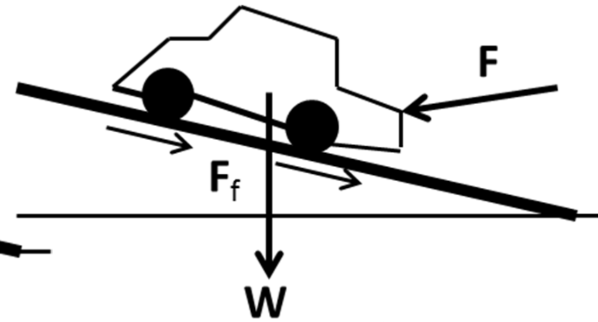
$$|A| = \sqrt{x^2 + y^2} = \sqrt{6^2 + 4^2} = \sqrt{52} = 7.21\ units$$

The angle θ may be calculated to be:

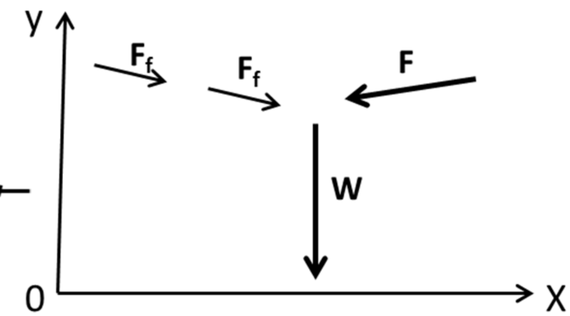
$$cos\theta = \frac{x}{|A|} = \frac{6}{7.21} = 0.832 \quad \text{or} \quad \theta = 33.68^o$$

3.5 **Example of Vector Quantity in 2-D Plane-Forces acting on a plane:**



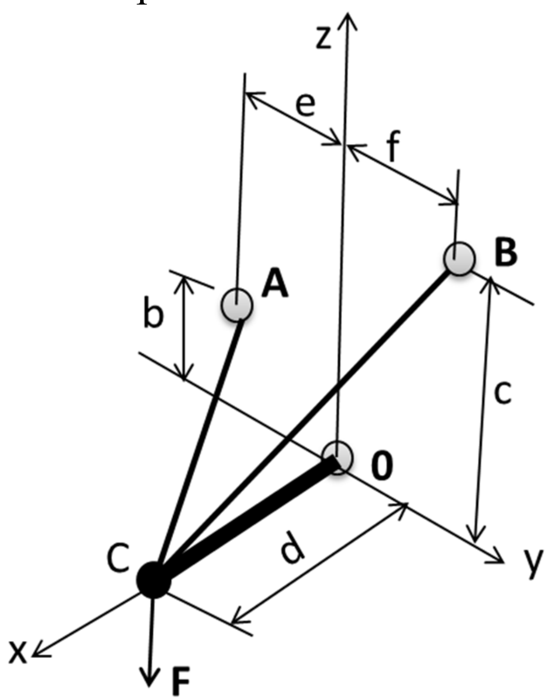


Force acting on a plane

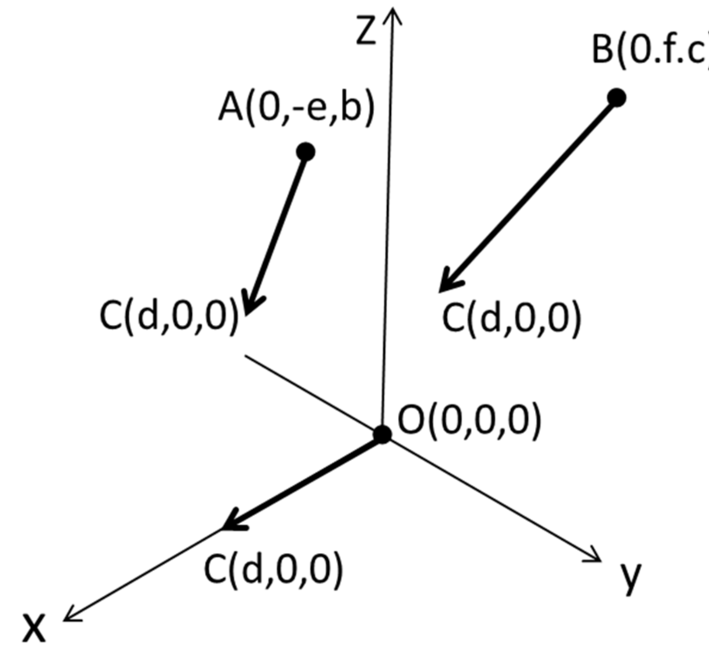


Example of Vector Quantity in 3-D Space - Forces acting in a space:

A space structure:



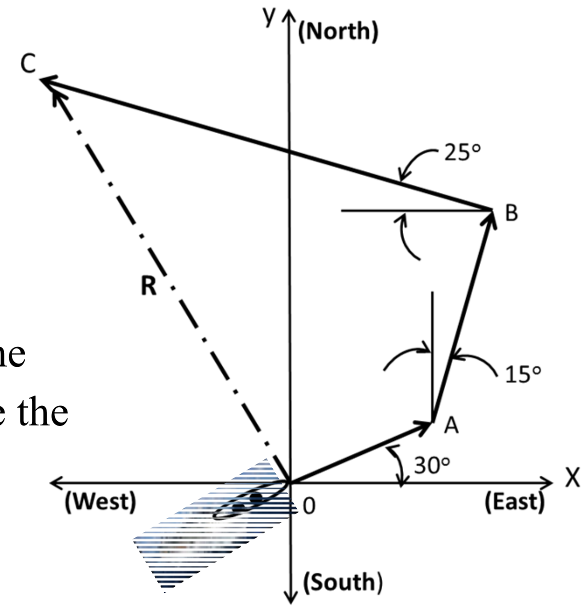
Force vectors in 3-D Space:



Example 3.6 **ON ADDITIONS AND SUBTRACTIONS OF VECOTORS**

A cruise ship begins its journey from Port O to its destination of Port C with intermediate stops over two ports at A and B as shown In the figure.

The ship sails 100 km in the direction 30° to northeast to Port A. From Port A, the ship sails 180 km in the direction 15° north east of Port A to Port B. The last leg of the cruise is from Port B to Port C in the direction of 25° northwest to the north of Port C. Find the total distance the ship traveled from Port O to Port C.



Solution:

We realize that the distances that the cruise ship sails are also specified by the specified direction, so the distances that the ship sail in each port are vector quantities. Consequently, we define the following position vectors, representing the change of the position while the ship sails:

Vector **A** = change position from O to Port A = 100 (cos30°) **i** + 100 (sin30°) **j** = 86.6 **i** + 50 **j**

Vector **B** = change position from Port A to Port B = 180 [cos(30+15)°] **i** + 180 [sin(30+15)°] **j** = 127.28 **i** + 127.28 **j**

Vector **C** = change position from Port B to Port C = 350[cos(90+25)°] **i** + 350[sin(90+25)°] **j** = -147.92 **i** + 317.21 **j**

The resultant vector **R** is the summation of the above 3 position vectors associated with unit vectors **i** and **j** is:

$$\mathbf{R} = \mathbf{A} + \mathbf{B} + \mathbf{C} = (86.6 + 127.28 +-147.92)\ \mathbf{i} + (50 + 127.28 + 317.21)\ \mathbf{j} = 65.96\ \mathbf{i} + 494.5\ \mathbf{j}$$

$$|\mathbf{R}| = \mathrm{R} = \sqrt{(65.96)^2 + (494.5)^2} = \sqrt{248881} = 498.88 \quad \mathrm{km}$$

3.4.4 Multiplication of Vectors

There are 3 types of multiplications of vectors: (1) Scalar product, (2) Dot product, and (3) Cross product

3.4.4.1 Scalar Multiplier

It involves the product of a scalar m to a vector **A.** Mathematically, it is expressed as:

$$\mathbf{R} = m\,(\mathbf{A}) = m\mathbf{A}$$

where m = a scaler quantity

Thus for vector $\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$, in which A_x, A_y and A_z are the magnitude of the components of vector **A** along the x-, y- and z-coordinate respectively.

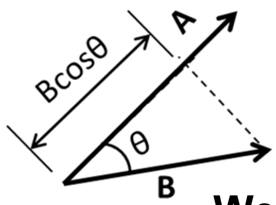
The resultant vector **R** is expressed as:

$$\mathbf{R} = mA_x\,\mathbf{i} + mA_y\,\mathbf{j} + mA_z\,\mathbf{k}$$

in which **i**, **j** and **k** are unit vectors along x-, y- and z-coordinates in a rectangular coordinate system respectively.

3.4.4.2 Dot Products

The DOT product of two vectors **A** and **B** is expressed with a “dot” between the two vectors as:



$$\mathbf{A} \bullet \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta = a\ scalar$$

where θ is the angle between these two vectors

We notice that the DOT product of two vectors results in a SCALAR

The algebraic definition of dot product of vectors can be shown as:

$$\mathbf{A} \bullet \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

where A_x, A_y and A_z = the magnitude of the components of vector **A** along the x-, y- and z-coordinate respectively,

and B_x, B_y and B_z = the magnitude of the components of vector **B** along the same rectangular coordinates.

Can you prove that $\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A}$??

Example 3.7 Determine (a) the result of dot product of the two vectors: **A** = 2**i** + 7**j** + 15**k** and **B** = 21**i** + 31**j** + 41**k**, and (b) the angle between these two vectors

Solution: (a) By using the above expression, we may get the result of the dot product of vectors **A** and **B** to be: **A●B** = 2x21 + 7x31+ 15x41 = 874

(b) We need to compute the magnitudes of both vectors **A** = 16.67 and **B** = 55.52 units, which lead to the angle θ between vectors **A** and **B** to be:

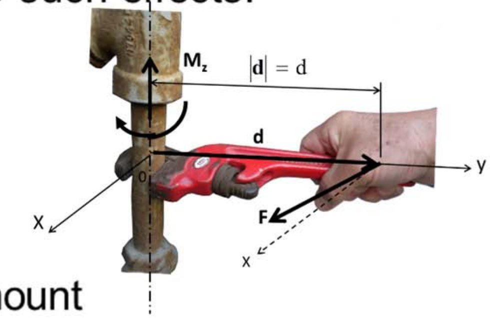
$$\cos\theta = \frac{A_x B_x + A_y B_y + A_z B_z}{AB} = \frac{2\times21+7\times31+15\times41}{16.67\times55.52} = \frac{874}{925.52} = 0.94433 \qquad \therefore \theta = 19.21^o$$

3.4.4.3 Cross Product

Physical examples for Cross product of vectors:

There are physical phenomena that we deal with as engineers that have a plane actions that result in some other physical quantity that occurs song the direction perpendicular to the plane of actions that produces this physical quantity. We offer two such effects:

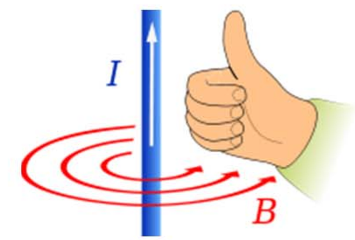
Case A: Force-application to a wrench that result a rotational effect of the pipe perpendicular to the plane on which force F and moment arm lie. i.e.:



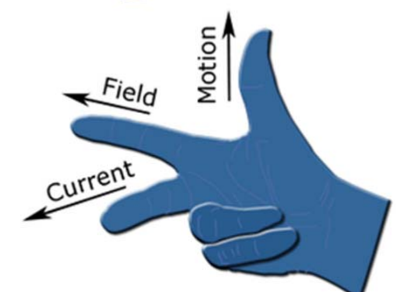
(**F**orce vector, **F**)$\times$ $(moment\ arm\ \boldsymbol{d}) \rightarrow Rotation\ of\ the\ pipe$ (amount and direction of rotation about z-axis)-a **vector**

Vectors on x-y plane

Case B: Produce a motion of an electric conductor by passing a current **i** in the conductor surrounded by a magnetic field **B**:



Here, we have the case in which the current passing the conductor in a magnetic field with a flux intensity B in the direction of the Middle and Index fingers of a right-hand respectively in the Fleming's right-hand rule, which lead to the prediction of the motion of the conductor represented by the thumb by the following expression:



Fleming's right-hand rule

$$Current, \boldsymbol{i} \times Intgensity\ of\ magnetic\ flux, \boldsymbol{B} = Velocity\ of\ motion\ of\ conductor, \boldsymbol{v}$$

(Vectors on a plane) (Vector in the direction perpendicular to the plane)

Mathematical expression of Cross product of vectors:

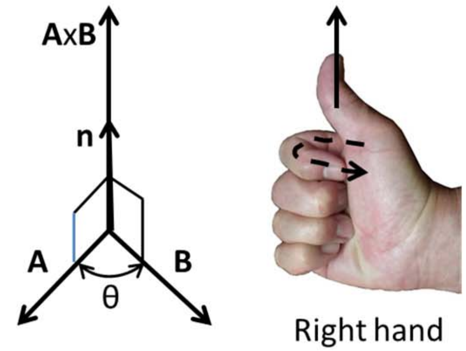
Cross product of two vectors applies to have both vectors lie on a plane but with the result of this Product in the direction perpendicular to the plane of these two vectors, as described in physical Situations illustrated in the preceding slide.

Cross products of two vectors A and B can be expressed as:

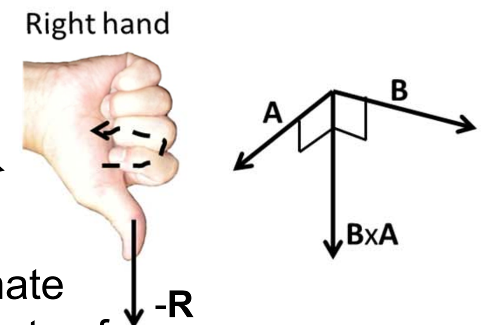
$$\boldsymbol{A} \times \boldsymbol{B} = \mathbf{R}$$

where the result of the cross product of vectors **A** and **B** is a Vector **R.**

The resultant vector ***R*** *is along the direction that is perpendicular* To the plane on which the vectors **A** and **B** lie.



One will realize that $\boldsymbol{A} \times \boldsymbol{B} \neq \boldsymbol{B} \times \boldsymbol{A}$. ***the case*** $\mathbf{B} \times \boldsymbol{A} = -\boldsymbol{R}$.



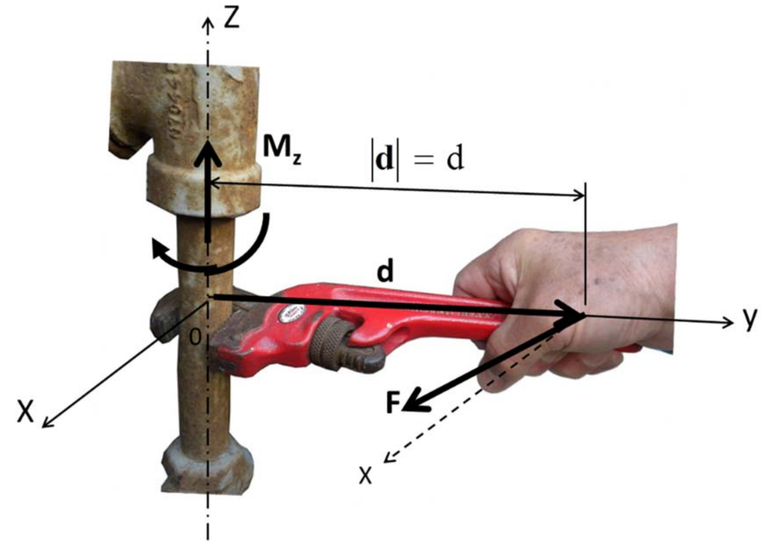
Cross product of vectors involving unit vectors:

A = A_x**i** + A_y**j** + A_z**k** and vector **B** = B_x**i** + B_y**j** + B_z**k** in a rectangular coordinate system with A_x, A_y and A_z, B_x, B_y and B_z being the magnitude of components of vector **A** and **B** along the x-, y- and z-coordinates respectively. We will have::

$$\mathbf{R} = \mathbf{A}x\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Example 3.8:

Determine the torque applied to the pipe in the Figure by a force F = 45 N with an angle θ = 60° to the y-axis at a distance d = 50 cm from the centerline of the pipe.



Solution:

We may express the force vector **F** = (Fsinθ) **i** + (Fcosθ) **j** = (45 sin 60°) **i** + (45 cos 60°) **j**, or **F** = 38.97**i** + 22.5**j.**

The moment arm vector **d** is and it may be expressed as: **d =** d**j =** 50 **j.**

The resultant vector $\mathbf{M_z} = \mathbf{F} \times \mathbf{d}$ can thus be computed using the above matrix form to be:

$$\boldsymbol{M_z} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 38.97 & 22.5 & 0 \\ 0 & 50 & 0 \end{vmatrix}$$

$$= (22.5 \times 0 - 0 \times 50)\boldsymbol{i} - (38.97 \times 0 - 0 \times 0)\mathrm{j} + (38.97 \times 50 - 22.5 \times 0)\boldsymbol{k} = 1948.5\mathbf{k}$$

The resultant torque on the pipe thus has a magnitude of $\mathbf{M_z}$ = 1948.5 N-cm in the direction along the z-axis

Example 3.10

If vectors **A** = **i** – **j** +2**k** and **B** = 2**i** + 3**j** - 4**k**, determine **A** x **B = C**.

Solution
We may use Equation (3.18) for the solution to be;

$$\mathbf{A}x\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 2 & 3 & 4 \end{vmatrix} = \mathbf{C}$$

in which the vector **C** = [(-1x4-(2x3]**i** + (1x4-2x2)**j** + [1x3-(-1x2)]**k** = -10**i** +5**k**

Useful Expressions of Multiplications of Vectors:

$$(\mathbf{A} \bullet \mathbf{B})\mathbf{C} \neq \mathbf{A}(\mathbf{B} \bullet \mathbf{C})$$

$$\mathbf{A} \bullet (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \bullet (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \bullet (\mathbf{A} \times \mathbf{B})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \bullet \mathbf{C})\mathbf{B} - (\mathbf{A} \bullet \mathbf{B})\mathbf{C}$$

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \bullet \mathbf{C})\mathbf{B} - (\mathbf{B} \bullet \mathbf{C})\mathbf{A}$$

Commutative law for dot product:

$$\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A}$$

$$\mathbf{A} \bullet (\mathbf{B} + \mathbf{C}) = \mathbf{A} \bullet \mathbf{B} + \mathbf{A} \bullet \mathbf{C}$$

$$m(\mathbf{A} \bullet \mathbf{B}) = (m\mathbf{A}) \bullet \mathbf{B} = \mathbf{A} \bullet (m\mathbf{B}) = (\mathbf{A} \bullet \mathbf{B})m$$

$$\mathbf{i} \bullet \mathbf{j} = \mathbf{j} \bullet \mathbf{k} = \mathbf{k} \bullet \mathbf{i} = 0$$

$$\mathbf{A} \bullet \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

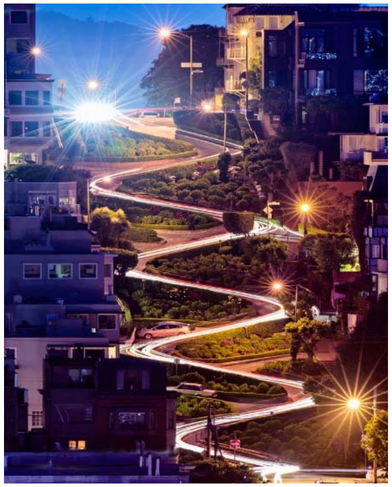
$$\mathbf{A} \bullet \mathbf{A} = A^2 = A_x^2 + A_y^2 + A_z^2$$

$$\mathbf{B} \bullet \mathbf{B} = B^2 = B_x^2 + B_y^2 + B_z^2$$

Vector Calculus

Vector calculus is used to solve engineering problems that involve vectors that not only need to be defined by both its magnitudes and directions, but also on their magnitudes and direction change CONTINUOUSLY with the time and positions.

There are many cases that this type of problems happen. We will illustrate the case by vehicles traveling on a steep and winding street by the name of Lombard Drive in the City of San Francisco (see pictures below). This 180 meters long paved crooked block involves eight sharp turns on a steep down slope at 27% which is much too steep by any standard for urban streets. Drivers driving their vehicles on that street need to constantly change the **velocity** (a vector quantity) of their cars in order to pass this steep and winding street. In other word, we have a situation in which the velocity v (a vector) with its values depending upon the locations on the street, and time, Or mathematically, we have a **vector function**: **v(x,y,z,t)** in which (x,y,z) is the position variables and t is the time variable. The same would happen to the vehicles cursing in racing tracks.



Definition of Vector Functions in Vector Calculus

We let **A**(u) = a Vector function, with u = variables that determine the value of the vector **A.**

Being a vector, A(u) may be expressed as:

$$\mathbf{A}(u) = \mathbf{A_x}(u) + \mathbf{A}_y(u) + \mathbf{A_z}(u) \quad \text{In general}$$

or with unit vectors in rectangular coordinate systems

$$\mathbf{A}(u) = A_x(u)\,\mathbf{i} + A_y(u)\,\mathbf{j} + A_z(u)\,\mathbf{k}$$

where $\mathbf{A_x}$, $\mathbf{A_y}$ and $\mathbf{A_z}$ denote the components of vector **A**(u) along the x-, y- and z-coordinate respectively, whereas A_x, A_y and A_z are the magnitudes of the components of vector **A**(u) along the same coordinates respectively.

The rate of change of the vector function (or DERIVATIVES) can be expressed the same way as other CONTINUOUS function to be:

$$\frac{dA(u)}{du} = \lim_{\Delta u \to 0} \frac{A(u+\Delta u) - A(u)}{\Delta u} \qquad \text{in general}$$

or with unit vectors in a rectangular coordinate system:

$$\frac{d\mathbf{A}(u)}{du} = \frac{dA_x(u)}{du}\mathbf{i} + \frac{dA_y(u)}{du}\mathbf{j} + \frac{dA_z(u)}{du}\mathbf{k}$$

and

$$d\mathbf{A} = \frac{\partial \mathbf{A}}{\partial x}dx + \frac{\partial \mathbf{A}}{\partial y}dy + \frac{\partial \mathbf{A}}{\partial z}dz$$

Example 3.12

If a position vector **r** in a rectangular coordinate system has both its magnitude and direction varying with time t, and its two components $\mathbf{r_x}$ and $\mathbf{r_y}$ vary with time according to functions:

$r_x = 1 – t^2$ and $r_y = 1+2t$ respectively.

Determine the rate of variation of the position vector with respect to time variable t.

Solution:

We may express the position vector **r** in the following form:

$$\mathbf{r}(t) = \mathbf{r_x}(t) + \mathbf{r_y}(t) = r_x(t)\,\mathbf{i} + r_y(t)\,\mathbf{j}$$

in which **i** and **j** are the unit vectors along the x- and y-coordinate respectively.

The rate of change of the position vector **r**(t) with respect to variable t may be obtained as::

$$\frac{d\mathbf{r}(t)}{dt} = \frac{dr_x(t)}{dt}\mathbf{i} + \frac{dr_y(t)}{dt}\mathbf{j} = \left[\frac{d}{dt}\left(1-t^2\right)\right]\mathbf{i} + \left[\frac{d}{dt}\left(1+2t\right)\right]\mathbf{j} = (-2t)\mathbf{i} + 2\mathbf{j}$$

Derivatives of the products of vectors:

$$\frac{\partial}{\partial x}(\mathbf{A}\bullet\mathbf{B}) = \mathbf{A}\bullet\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}\bullet\mathbf{B}$$

$$\frac{\partial}{\partial y}(\mathbf{A}\bullet\mathbf{B}) = \mathbf{A}\bullet\frac{\partial \mathbf{B}}{\partial y} + \frac{\partial \mathbf{A}}{\partial y}\bullet\mathbf{B}$$

$$\frac{\partial}{\partial z}(\mathbf{A}\bullet\mathbf{B}) = \mathbf{A}\bullet\frac{\partial \mathbf{B}}{\partial z} + \frac{\partial \mathbf{A}}{\partial z}\bullet\mathbf{B}$$

$$\frac{\partial}{\partial x}(\mathbf{A}\mathrm{x}\mathbf{B}) = \mathbf{A}\mathrm{x}\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}\mathrm{x}\mathbf{B}$$

$$\frac{\partial}{\partial y}(\mathbf{A}\mathrm{x}\mathbf{B}) = \mathbf{A}\mathrm{x}\frac{\partial \mathbf{B}}{\partial y} + \frac{\partial \mathbf{A}}{\partial y}\mathrm{x}\mathbf{B}$$

$$\frac{\partial}{\partial z}(\mathbf{A}\mathrm{x}\mathbf{B}) = \mathbf{A}\mathrm{x}\frac{\partial \mathbf{B}}{\partial z} + \frac{\partial \mathbf{A}}{\partial z}\mathrm{x}\mathbf{B}$$

Example 3.13

Determine d**A** if vector function **A(**x,y,z**) =** (x^2 siny**) i +** (z^2 cosy) **j** – (xy^2) **k**.

Solution:

$$d\mathbf{A} = \frac{\partial \mathbf{A}}{\partial \mathbf{x}}dx + \frac{\partial \mathbf{A}}{\partial y}dy + \frac{\partial \mathbf{A}}{\partial z}dz = \left[(\sin y)\mathbf{i}\frac{d}{dx}(x^2) - (y^2)\mathbf{k}\frac{dx}{dx}\right]dx + \left[x^2\mathbf{i}\frac{d}{dy}(\sin y) + z^2\mathbf{j}\frac{d}{dy}(\cos y)\right]dy$$

$$+\left[(\cos y)\mathbf{j}\frac{d}{dz}(z^2)\right]dz$$

$$= \left[(2x\sin y)\mathbf{i} - y^2\mathbf{k}\right]dx + \left[(x^2\cos y)\mathbf{i} - (z^2\sin y)\mathbf{j} - 2xy\mathbf{k}\right]dy + \left[(2z\cos y)\mathbf{j}\right]dz$$

$$= (2x\sin y\,dx + x^2\cos y\,dy)\mathbf{i} + (2z\cos dz - z^2\sin y\,dy)\mathbf{j} - (y^2\,dx + 2x\,y\,dy)\mathbf{k}$$

3.5.3 Gradient, Divergence and Curl

Gradient, divergence and curl are frequently used when dealing with variations of vectors using a vector operator designated by ∇(Pronounced del) defined as follows:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$ in a rectangular coordinate system

3.5.3.1 Gradient

Gradient relates to the variation of the **magnitudes of vector quantities** with a scalar quantity ϕ, defined by:

$$grad\phi = \nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

Example 3.14-A**:**

Determine the gradient of a scalar quantity $\phi = xy^2z^3$ which is the magnitude of a vector $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$:

$$grad\phi = \nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

$$= \frac{\partial}{\partial x}\left(xy^2z^3\right) + \frac{\partial}{\partial y}\left(xy^2z^3\right) + \frac{\partial}{\partial z}\left(xy^2z^3\right) = y^2z^3 + 2xyz^3 + 3xy^2z^2$$

3.5.3.2 **Divergence:**

Divergence of vector function **A**(x,y,z) implies the RATE of "growth" or "contraction" of this vector function in its components along the coordinates. The divergence of the vector function **A(**x,y,z**)** is defined as:

$$div\,\mathbf{A} = \nabla \bullet \mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \bullet \left(A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}\right) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Example 3.14-B**:**

Determine the div (φ**A**) if the gradient of a scalar quantity $\phi = xy^2z^3$ which is the magnitude of a vector $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$:

$$div\,\mathbf{A} = \nabla \bullet \mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \bullet \left(A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}\right) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

where A_x, A_y and A_z are the magnitude of the components of vector **A** along the x-, y- and z-coordinate respectively.

3.5.3.3 **Curl**:

The curl of a vector function **A** is related to the "rotation" of this vector. It is defined as:

$$curl\,\mathbf{A} = \nabla\times\mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times\left(A_x\,\mathbf{i}+A_y\,\mathbf{j}+A_z\,\mathbf{k}\right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \mathbf{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_y & A_z \end{vmatrix} - \mathbf{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ A_x & A_z \end{vmatrix} + \mathbf{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ A_x & A_y \end{vmatrix}$$

$$= \left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\mathbf{i} - \left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right)\mathbf{j} + \left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\mathbf{k}$$

Example 3.14-B:

Determine the curl (φ**A**) if the gradient of a scalar quantity $\phi = xy^2z^3$ which is the magnitude of a vector $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$:

$$curl\,(\phi\mathbf{A}) = \nabla\times(\phi\mathbf{A}) = \left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times\left(\phi A_x\,\mathbf{i}+\phi A_y\,\mathbf{j}+\phi A_z\,\mathbf{k}\right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi A_x & \phi A_y & \phi A_z \end{vmatrix} = \mathbf{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi A_y & \phi A_z \end{vmatrix} - \mathbf{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \phi A_x & \phi A_z \end{vmatrix} + \mathbf{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \phi A_x & \phi A_y \end{vmatrix}$$

$$= \left(\frac{\partial \phi A_z}{\partial y}-\frac{\partial \phi A_y}{\partial z}\right)\mathbf{i} - \left(\frac{\partial \phi A_z}{\partial x}-\frac{\partial \phi A_x}{\partial z}\right)\mathbf{j} + \left(\frac{\partial \phi A_y}{\partial x}-\frac{\partial \phi A_x}{\partial y}\right)\mathbf{k}$$

Three (3) Useful Expressions for Differentiating Vector Functions

$$\nabla \bullet (\phi \mathbf{A}) = (\nabla \phi) \bullet \mathbf{A} + \phi (\nabla \bullet \mathbf{A})$$

$$\nabla^2 \phi = Laplacian\ operator\ of\ \phi = \nabla \bullet \nabla \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$div\ curl\ \mathbf{A} = 0$$

where the scalar quantity ϕ is the magnitude of the Vector **A**

3.6 Applications of Vector Calculus in Engineering Analysis

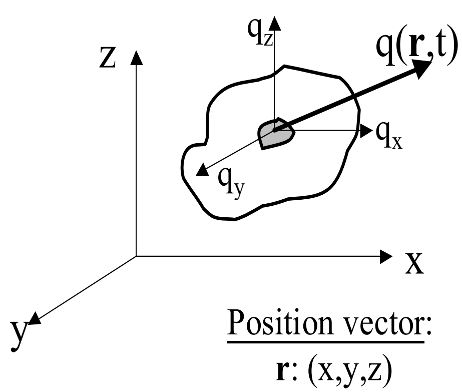
3.6.1 In Heat Conduction:

The vector quantity heat flux transmitting in solids **q** as we will derive the Fourier law in Section 7.5.2 has the form:

$$q = \mathbf{q}(x) = -k\frac{dT(x)}{dx}$$

for heat transmits in the direction of x in a rectangular coordinate system in which k is the thermal conductivity of the solid. The scalar quantity T(x) is the temperature variation along the x-coordinate.

The Fourier law of heat conduction as presented in the above expression may lead to the Derivation of the following heat conduction equation to solve for the temperature distribution T(**r**,t) in which r represents the position variable defining the solid as illustrated in Figure 7.18:



$$\frac{\partial T(\mathbf{r}.t)}{\partial t} = \propto \nabla^2 T(\mathbf{r},t)$$

where α is the thermal diffusivity of the solid = k/ρc, with ρ, c = Mass density and specific heats of the solid

For the case in which no thermal condition varies with time t, the heat conduction equation is simply reduced to a Laplace equation:

$$\nabla^2 \mathrm{T}(\mathbf{r},t) = 0$$

3.6 **Applications of Vector Calculus in Engineering Analysis**-Cont'd

3.6.2 **In Fluid Mechanics:**

The Law of Continuity governs the flow of fluids in space. Mathermatica; expression of this law is:

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \bullet \nabla \rho + \rho \nabla \bullet \mathbf{v} = 0$$

where **v** is the velocity of flowing fluid (a vector), which varies with the position (x,y,z in a rectangular coordinate system, i.e. v=v(x,y,z,t) , and ρ is the mass density of the fluid (a scalar, treated as a constant for non-compressible fluids).

Again, if no flow condition taken place during the motion of the fluid, we will have the following math condition:

$$\nabla \cdot v(\mathbf{r}, t) = 0$$

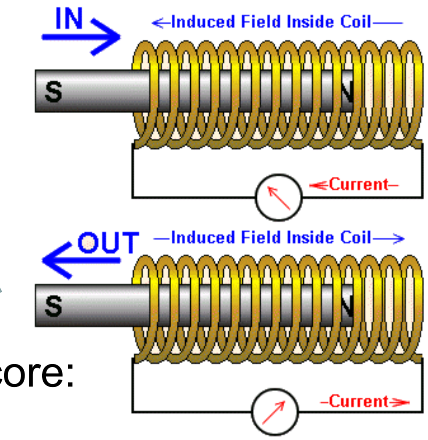
For fluid flowing in a conduit or open channels, the following Bernoulli equation is applied:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left|\mathbf{v}\right|^2 + \frac{p}{\rho} + \mathbf{U} = 0$$

where φ =potential energy, e.g. provided by gravitation of the conduit
p = pressure that "drives" the fluid to flow
U = body force vector of the fluid

3.6 Applications of Vector Calculus in Engineering Analysis-Cont'd

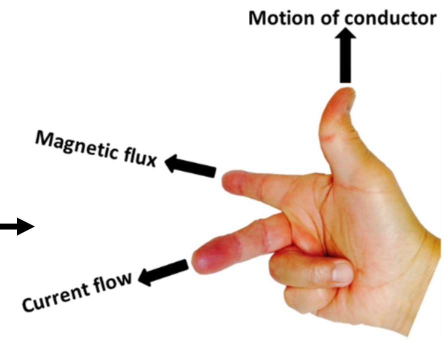
3.6.3 In Electromagnetism with Maxwell Equations:



Maxwell equations are widely used to model the movement of a magnetized soft iron core that can slide in both direction In a coil of electric conduct field as illustrated in the figure:

We realize that all the 3 quantities governing the motion of the iron core:

The velocity of the motion, **v**
the electric field, **E**, and
the magnetic flux density, **B** are vector quantities (from S to N),
and they may vary with the positions (**r**) in the space r and time (t).
Orientations of these 3 quantities follow the right-hand rule as:



Following are five Maxwell equations that enable engineers to model the motion of assess the motion of the magnetic soft ion core:

$$\nabla \bullet (\varepsilon \mathbf{E}) = \rho \qquad \nabla \bullet \mathbf{B} = 0 \qquad \nabla\, x\, \mathbf{H} = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla\, x\, \mathbf{E} = -\frac{\partial B}{\partial t} \qquad \nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Where ε =permittivity or dielectric constant of the medium between the conductor and the magnetic field,
ρ = charge density in the conductor, and
J = electric current flow

3.7 **Application of Vector Calculus in Rigid Body Dynamics**

Dynamics analysis is an important part of the design of any moving machine or structure, regardless of their sizes from giant space stations and a jumble jet airplane to small components of sensors and actuators in the minute scales in micrometers.

Dynamics analysis involves both kinematics and kinetics of moving solids;

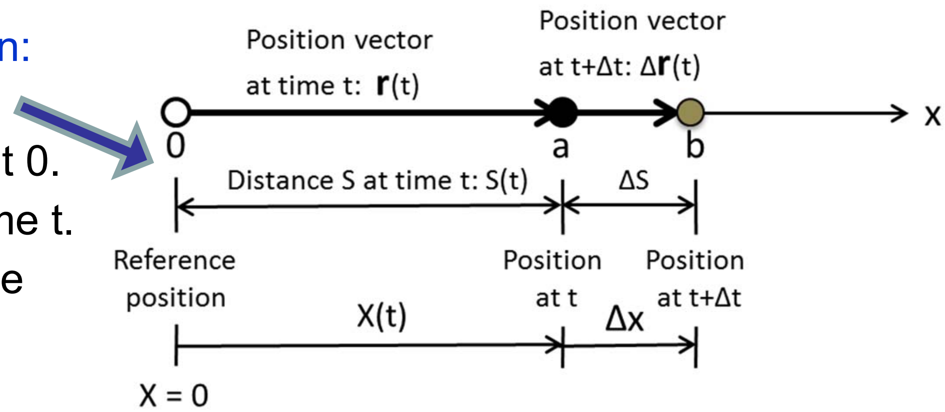
"**Kinematics**" is the study of the geometry of motion. It relates displacement, velocity and acceleration of moving solids at given times.

"**Kinetics**" relates the forces acting on moving rigid body, the mass of the body, and the motion of the body. It is also used to predict the motion caused by given forces or to determine the forces required to produce a given motion.

Since "Dynamic" is a stand-alone course in almost all mechanical engineering schools in the world, we will limit our learning in this course in the application of vector calculus in "kinematics" of rigid bodies in motion and the coverage will be confined in planar motions. We will leave other topics for in-depth studies in distinct courses in Dynamics.

3.7 Application of Vector Calculus in Rigid Body Dynamics – Cont'd

3.7.1 Rigid Body Motion in Rectilinear Motion:



The rigid body is originally located at Point 0. It travels to a new position at Point a in time t. We may thus define its new position by the position vector function **r**(t) which can be expressed by:

$$\mathbf{r}(t) = S(t)\mathbf{i} \quad \text{or} \quad \mathbf{r}(t) = x(t)\mathbf{i}$$

where **i** = unit vector along the x-coordinate

The velocity vector function of the rigid body **v**(t) can be expressed as:

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \frac{d}{dt}[S(t)\mathbf{i}] = \left(\frac{d}{dt}[S(t)]\right)\mathbf{i} = v(t)\,\mathbf{i}$$

The corresponding acceleration vector function **a**(t) for the moving body may be:

$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \frac{d}{dt}\left(\frac{d}{dt}\mathbf{r}(t)\right) = \frac{d}{dt}\left[\frac{d}{dt}S(t)\mathbf{i}\right] = \frac{d^2}{dt^2}[S(t)\mathbf{i}] = \left[\frac{d^2S(t)}{dt^2}\right]\mathbf{i} = a(t)\mathbf{i}$$

3.7.1 Rigid Body Motion in Rectilinear Motion-Cont'd:

Example 3.15

A rigid body is traveling along the x-axis in a rectangular coordinate system. Assume that the instantaneous position of the body may be represented by a function of $x(t) = 11t^2 - 2t^3$ in meter, in which t is time in second. We further Assume the mass of the body is negligible.

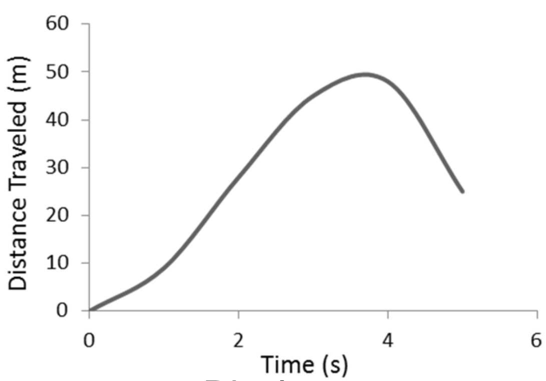
Determine the vector functions of the velocity and acceleration of the moving rigid body.

Solution:

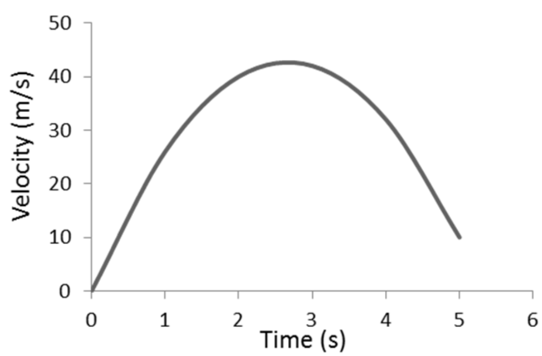
We may determine the magnitude of the velocity vector function **v**(t) and the magnitude of the acceleration vector function **a**(t) as shown below:

$$\mathbf{v}(t) = \frac{d\mathbf{x}(t)}{dt} = \frac{d}{dt}\left(11t^2 - 2t^3\right) = 22t - 6t^2 \qquad \mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \frac{d}{dt}\left(22t - 6t^2\right) = 22 - 12t$$

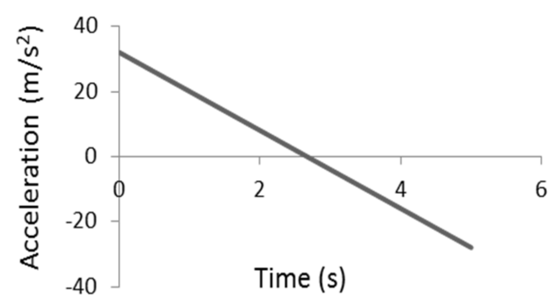
The vector functions of the velocity and acceleration of the moving rigid body thus can be expressed as:: **v**(t) = $(22t - 6t^2)$ **i** and **a**(t) = (22-12t) **i** . Graphic solutions are:



Displacement



Velocity

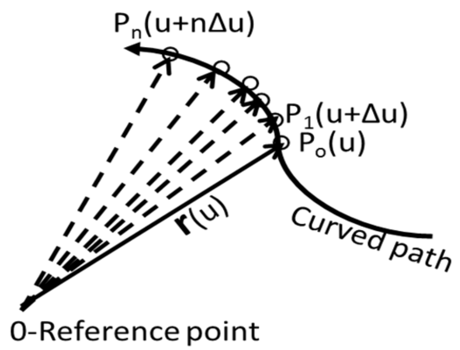


Acceleration

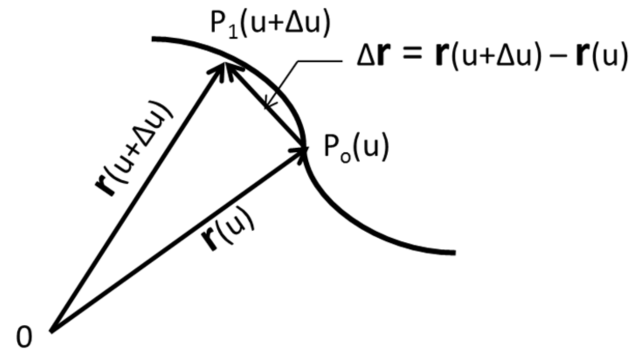
3.7 Application of Vector Calculus in Rigid Body Dynamics – Cont'd

3.7.2 Plane Curvilinear Motion in Rectangular Coordinates

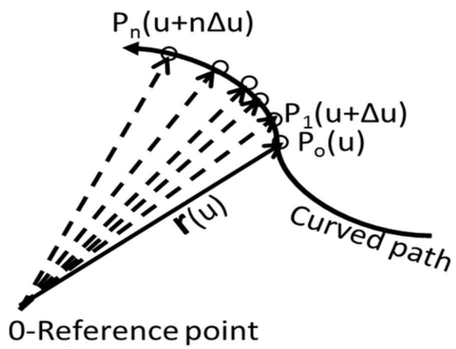
The case of rigid body moving on a curvilinear plane is a lot more complicated than that of linear movement in the proceeding case, simply because the position (and thus the direction) of the rigid body motion changes at all times in addition to the change of time in the fixed direction, as in the liner motion. This situation is illustrated below:



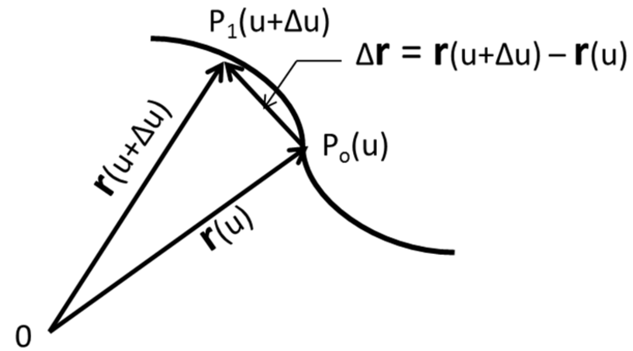
A rigid body traveling on a curved path



Shift of position vectors on a curved path



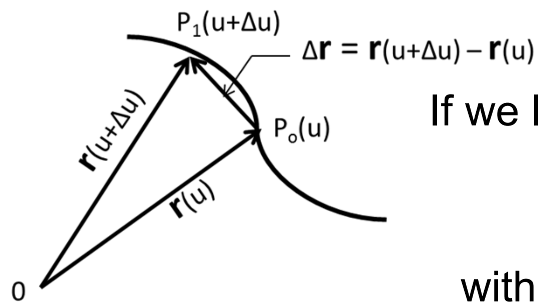
A rigid body traveling on a curved path in x-y plane



Shift of position vectors on a curved path in x-y plane

3.7 Application of Vector Calculus in Rigid Body Dynamics – Cont'd

3.7.2 Plane Curvilinear Motion in Rectangular Coordinates-Cont'd



If we let the RATE of change of the Position vector **r**(t) to be defined as:

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

with **i**, **j** = unit vector along the x- and y- coordinates respectively

We will have both the velocity acceleration vectors expressed to be:

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \left[\frac{dx(t)}{dt}\right]\mathbf{i} + \left[\frac{dy(t)}{dt}\right]\mathbf{j} \quad \text{and} \quad \mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \left[\frac{d^2x(t)}{dt^2}\right]\mathbf{i} + \left[\frac{d^2y(t)}{dt^2}\right]\mathbf{j}$$

Example 3.16 If the position of a rigid body moving on a curved path at time t is: **r**(t) = t**i** +2t^3**j.** Determine the velocity v(t) and acceleration vector functions at that instant.

Solution: We will first express the components of the position vector from the given expression of r(t) to be: x(t) = t and y(t) = 2t^3, from which, we will get the velocity and acceleration vector functions and their respective magnitudes as:

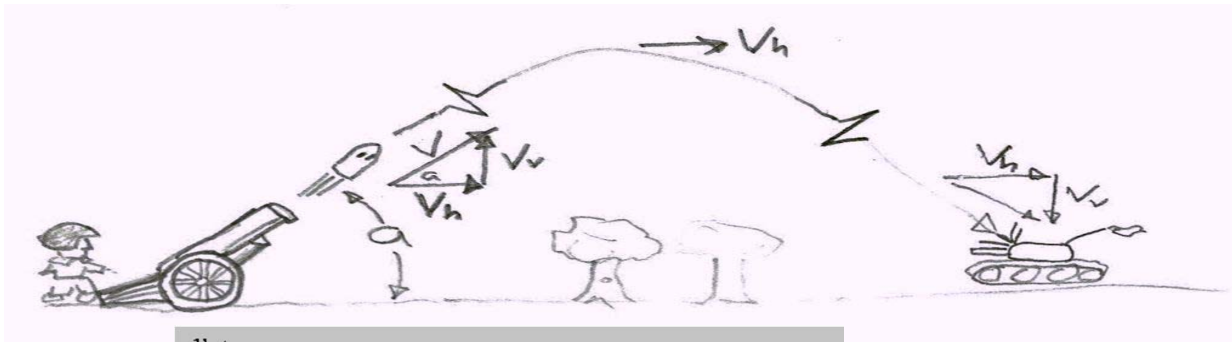
$$\mathbf{v}(t) = \left[\frac{d(t)}{dt}\right]\mathbf{i} + \left[\frac{d(2t^3)}{dt}\right]\mathbf{j} = \mathbf{i} + 6t^2\mathbf{j} \quad \text{and} \quad \mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \left[\frac{d(1)}{dt}\right]\mathbf{i} + \left[\frac{d(6t^2)}{dt}\right]\mathbf{j} = 0\,\mathbf{i} + 12t\,\mathbf{j} = 12t\,\mathbf{j}$$

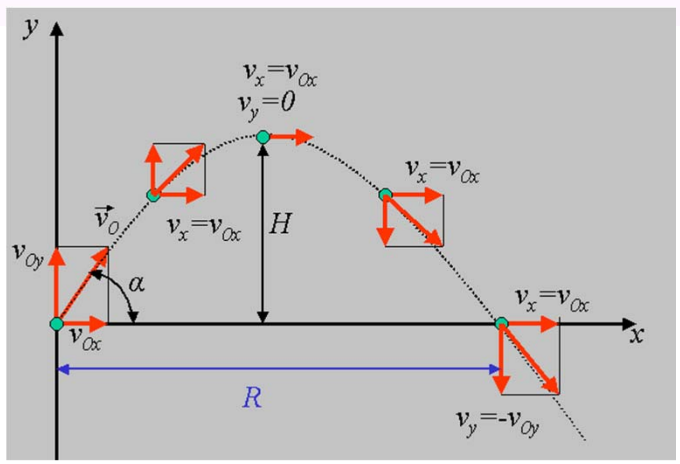
and the magnitudes:

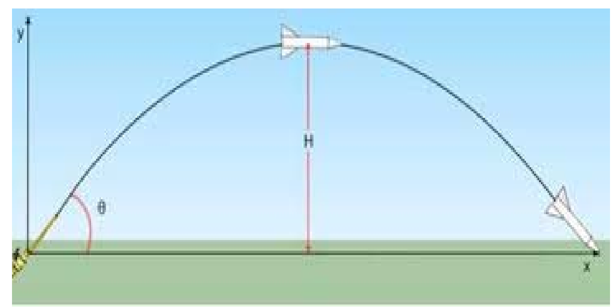
$$v(t) = |\mathbf{v}(t)| = \sqrt{1^2 + (6t)^2} = \sqrt{1 + 36t^2} \text{ m/s} \quad \text{and} \quad a(t) = |\mathbf{a}(t)| = \sqrt{(12t)^2} = 12t \text{ m/s}^2$$

3.7.3 Application in the Kinematics of Projectiles

The case of analysis that we will be dealing with here is of great interest to military personnel; the situation is for operators in either artillery or missile launching to determine what the "jump slope angle" (angles α or σ or θ) should be chosen to hit the enemy targets as shown in the pictures below:





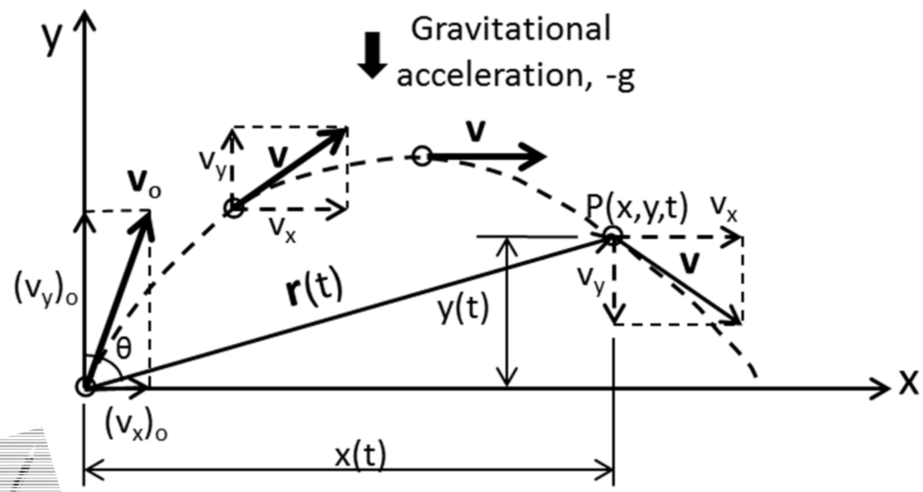


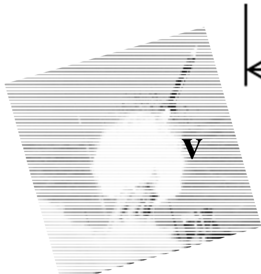
By neglecting the weight of the projectile (the cannon shells or the missile), and with the given "jump slope angle" and initial velocity V0, and assume there is no head-on wind to the projectile, one may compute the maximum height H and the range R that the projectile will fly.

Readers will find that vector calculus will offer a much speedier solution to this type of analysis.

3.7.3 Application in the Kinematics of Projectiles-Cont'd

We will begin our analysis using the diagram as shown in the diagram shown below:





Following formula will be used in this type of analysis:

Decomposition of the initial velocity: $\mathbf{v_o} = (v_x)_o\,\mathbf{i} + (v_y)_o\,\mathbf{j}$

Magnitudes of the components of initial velocity: $(v_x)_o = v_o \cos\theta$ and $(v_y)_o = v_o \sin\theta$

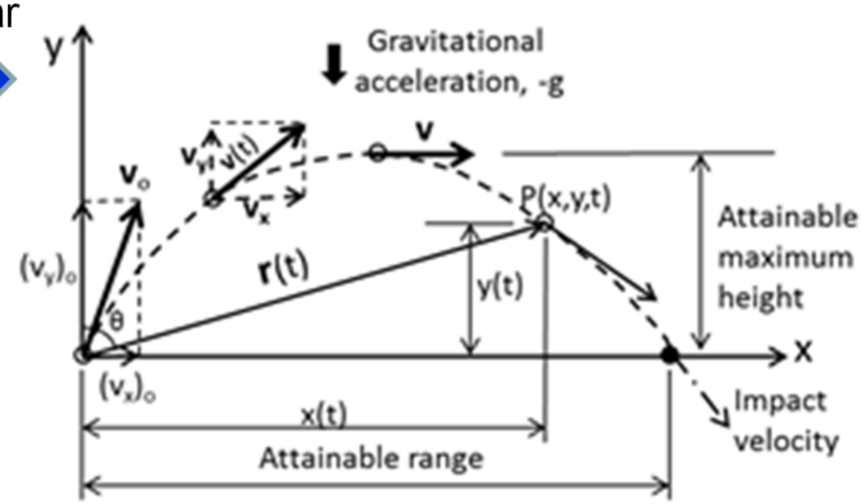
Position vector: $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$

Velocity vector function in terms of acceleration vector function: $\mathbf{v}(t) = \int \mathbf{a}(t)dt + c_1 = \int (-g\mathbf{j})dt + c_1$

Instantaneous position vector function: $\mathbf{r}(t) = \int \mathbf{v}(t)dt + c_2$ where c_1 and c_2 are constants

Example 3.17

A projectile is launched at the origin of a rectangular coordinate system (x-y) as shown in Figure with an initial velocity $\mathbf{V_o}$ = 200 m/s at a jump slope angle $\theta = 30^o$. Determine the following:



(a) The instantaneous position vector function of the projectile **r**(t)
(b) The maximum height the projectile attains y_m
(c) The range R, and
(d) The impact velocity v_e

Solution:

We will make the following necessary idealizations:
(1) The projectile will fly on a path on the x-y plane
(2) The projectile has negligible weight (Note; the mass is, however, accounted for)
(3) The projectile will not face with head-on wind
(4) The "jump slope angle" of the projector is fixed at θ
(5) The projectile leaves the projector at an initial velocity v_o
(6) The projectile is only subjected to gravitational force at all times while flying in its path.

The last assumption allows us to have the acceleration vector:

$$\mathbf{a}(t) = -g\mathbf{j}, \quad \text{or} \quad \mathbf{a}(t) = -9.81\mathbf{j} \text{ m/s}^2$$

of the projectile. The above expression of the acceleration of the projectile will lead to the velocity vector function of the projectile as shown in the next slide:

$$\mathbf{v}(t) = \int \mathbf{a}(t)dt + c_1 = \int (-9.81\mathbf{j})dt + c_1 = (-9.81\mathbf{j})t + c_1$$

in which the integration constant c_1 may be determined from the initial velocity of the projectile with the following relation using the above expression but with t = 0:

$$\mathbf{v_0} = \mathbf{v}(t)\big|_{t=0} = (\mathbf{v_x})_o + (\mathbf{v_y})_o = (200\cos 30^o)\mathbf{i} + (200\sin 30^o)\mathbf{j} = 173.21\mathbf{i} + 100\mathbf{j}$$

yielding: c_1 = **v**(0)=173.21**i** + 100 **j.** We may thus establish the velocity vector function to be:

$$\mathbf{v}(t) = -9.81t\mathbf{j} + c_1 = -9.81t\mathbf{j} + (173.21\mathbf{i} + 100\mathbf{j})$$
$$= 173.21\mathbf{i} + (100 - 9.81t)\mathbf{j}$$

Now with the velocity vector function of the projectile v(t) identified as shown above, we may obtain the instantaneous position of the projectile by using the expression:

(a) Determine the instantaneous position vector **r**(t) of the projectile:

$$\mathbf{r}(t) = \int \mathbf{v}(t)dt + c_2$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)dt + c_2 = \int [173.21\mathbf{i} + (100 - 9.81t)\mathbf{j}]dt + c_2$$

$$= (173.21t)\mathbf{i} + (100t)\mathbf{j} - \left(\frac{9.81}{2}t^2\right)\mathbf{j} + c_2 = (173.21t)\mathbf{i} + (100t - 4.905t^2)\mathbf{j} + c_2$$

The initial condition of **r**(0) = 0 allows us to determine that c_2 =0, we thus have the instantaneous position vector function of the projectile **r**(t) to be:

$$\mathbf{r}(t) = (173.21t)\mathbf{i} + (100t - 4.905t^2)\mathbf{j}$$

This expression leads to the components of the position vector **r**(t) = x(t)**i** +y(t)**j** to be:

$x(t) = 173.21t$ and $y(t) = 100t - 4.905t^2$

(b) Determine the maximum height of the flying projectile (y_m):

We have just derived the instantaneous position vector function to be:

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

with the two components along the horizontal direction, i.e. x(t) = 173.21 t
and the other component along the y-direction to be $y(t) = 100t-4.95t^2$.

The time at which the projectile would reach its maximum height in the flight y_m could be obtained by finding the maximum value of the vector function y(t) using the principle of calculus by the following procedure::

Step 1: $\left.\frac{dy(t)}{dt}\right|_{t=t_m} = \left.\frac{d}{dt}\left(100t - 4.905t^2\right)\right|_{t=t_m} = 0$ leads to the following equation:

$100 - 9.81\ t_m = 0$, from which we solve for $t_m = 5.0968$ s.

Step 2: To ensure that this t_m would result in $y(t_m)$ to be the maximum of the function y(t), we will need to show that $\left.\frac{d^2y(t)}{dt^2}\right|_{t=t_m} = -9.81 < 0$

So, the t_m = 5.0968 s indeed is the time for the projectile to reach the maximum height, y_m.

Step 3: The attainable max. height of the projectile is thus equal to:

$$y_m = y(t_m) = 100(5.0968) - 4.95\ (5.0968)^2 = 382.26\ \text{m}$$

(c) The attainable Range (R):

The maximum attainable range of the projectile R can be determined by a physical condition that the flying height at time t_e is zero. Mathematically, we will have: $y(t_e) = 0$. Since we have already derived: $y(t) = 100t - 4.905t^2$ We may solve the time t_e from the following equation:

$$y(t_e) = 100\, t_e - 4.902\, t_e^2 = 0$$

from which, we get $t_e = 20.3874$ s meaning it would take this long to have the projectile to reach the maximum value in the horizontal distance, i.e. $R = x(t_e)$. We thus have:

$$R = x(t_e) = x(20.3874) = 173.21\ (20.3874) = 3531.3 \text{ m}$$

(d) The impact velocity at the target

We have already derived the instantaneous velocity vector function to be:

$$\mathbf{v}(t) = 173.21\mathbf{i} + (100 - 9.81t)\mathbf{j}$$

We have computed the time required to hit the target, $t_e = 20.3874$ s. This lead to the determination of the final, or impact velocity of the projectile to be:

$$\mathbf{v}_e = \mathbf{v}(t_e) = \mathbf{v}(20.3874) = 173.21\mathbf{i} + (100 - 9.81x20.3874)\mathbf{j} = 173.21\mathbf{i} - 100\,\mathbf{j}$$

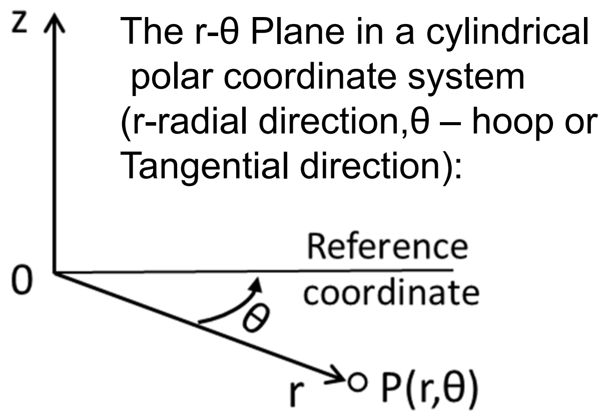
and the magnitude of the vector $\mathbf{v}_e$ is:

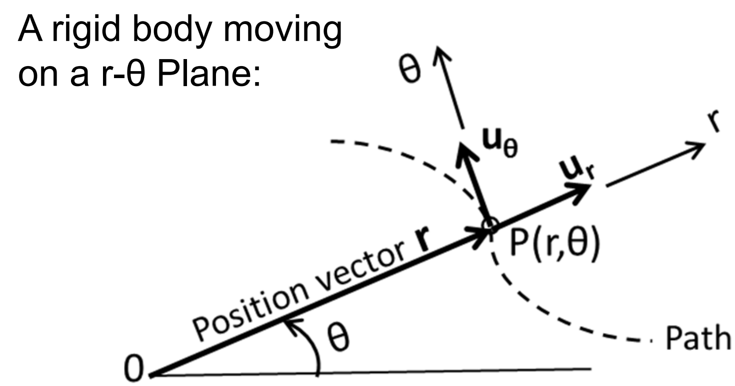
$$|\mathbf{v}_e| = \sqrt{(173.21)^2 + (-100)^2} = 200\ m/s$$

The direction of the impact velocity is: $\theta = \tan^{-1}(100/173.21) = -18.43^{o}$ from the x coordinate.

3.7.4 Plane Curvilinear Motion in Cylindrical Coordinates

In this Section, we will deal with motion of a rigid body on the plane defined by r-θ in a cylindrical polar coordinate system as illustrated below:





The position vector P(r,θ) of the moving rigid body, shown in the figure is expressed as:

$$\mathbf{r}(r,\theta) = r\,\mathbf{u}_r + \theta\mathbf{u}_\theta$$

where $\mathbf{u}_r$ = the unit vector along the r-coordinate, and

$\mathbf{u}_\theta$ = the unit vector along the coordinate that follows the trend of positive θ-coordinate (i.e. in the counter-clockwise direction) in the direction that is perpendicular to the r-coordinate.

The instantaneous position vector function **r**(t) may take the form of the following expression:

$$\mathbf{r}(t) = r(t)\mathbf{u}_r(t) + \theta\mathbf{u}_\theta(t)$$

in which both the unit vectors are function of time t.

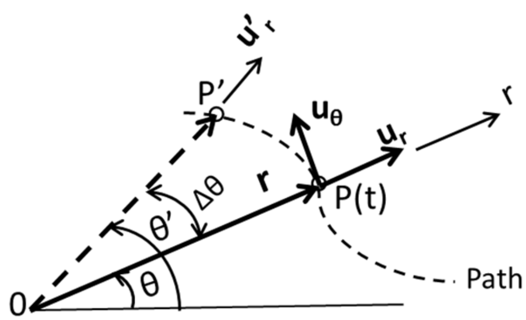
NOTE: r(t) in the above expression is the magnitude of vector **r**(t) at time t.

3.7.4 **Plane Curvilinear Motion in Cylindrical Coordinates –** Cont'd

The instantaneous velocity vector of the rigid body **v**(t) is expressed in the following way:

$$\mathbf{v}(t) = \frac{d\boldsymbol{r}(t)}{dt} = \frac{d}{dt}[\boldsymbol{r}(t)\boldsymbol{u_r}(t)]$$
$$= r(t)\frac{d[\mathbf{u_r}(t)]}{dt} + \boldsymbol{u_r}(t)\frac{d[r(t)]}{dt}$$
$$= r(t)\dot{\mathbf{u}}_{\mathbf{r}}(t) + \dot{r}(t)\mathbf{u_r}(t)$$

We need to realize a fact that the motion of a rigid body long a curved path as shown below will make the unit vectors change its magnitude due to a simultaneous change of the angular position Δθ, as illustrated in the figure below. Consequently, we will have the following Expressions to account such effects:

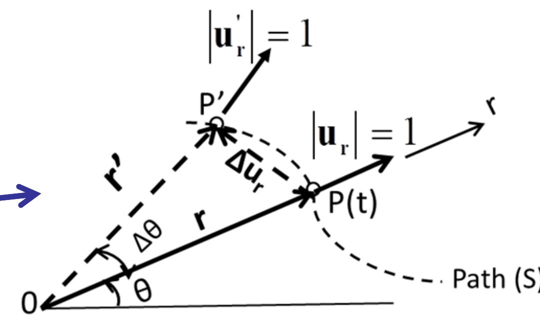


$$\mathbf{u}_{\mathbf{r}}'(t) = \mathbf{u}_{\mathbf{r}}(t) + \Delta\mathbf{u}_{\mathbf{r}}(t)$$

$$\Delta\mathbf{u}_{\mathbf{r}}(t) = (\Delta\theta)\ \mathbf{u}_{\boldsymbol{\theta}}(t)$$

where Δθ is the corres$\Delta \mathbf{u}_r$ponding variation of the θ-coordinate associated with the shift of the position vector from position P to P' as shown in the figure:

$\Delta \mathbf{u}_r(t) = (\Delta\theta)\ \mathbf{u}_\theta\ (t)$ for very small $\Delta \mathbf{u}_r$



3.7.4 **Plane Curvilinear Motion in Cylindrical Coordinates – Cont'd**

The rate of change of the instantaneous unit vector can thus be expressed as:

$$\dot{\boldsymbol{u}} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{u_r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\dot{}(\Delta\theta)\boldsymbol{u_\theta}}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{\Delta\theta}{\Delta t}\right) \mathbf{u_\theta}$$

from which we may derive the following useful relations:

$\dot{\mathbf{u}}_r(t) = \dot{\theta}(t)\mathbf{u_\theta}$ leads to the instantaneous velocity vector function in the form:

$$\mathbf{v}(t) = r(t)\dot{\theta}(t)\mathbf{u_\theta} + \dot{r}(t)\mathbf{u_r}$$

in which $\mathbf{u_\theta}$ *and* $\mathbf{u_r}$ *are the u nit vectors in the* θ *and* $r -$ coordinate respectively.

Consequently, we will have the components of the velocity vector in both the r- and θ-coordinates expressed to be:

$v_r(t) = \dot{r}(t)$ the component of instantaneous velocity along the r-coordinate

and $v_\theta(t) = r(t)\,\dot{\theta}(t)$ the component of instantaneous velocity along the θ-coordinate

The magnitude of the instantaneous velocity vector at time t is thus equal to:

$$v(t) = \sqrt{\left[r(t)\dot{\theta}(t)\right]^2 + [\,\dot{r}(t)]^2}$$

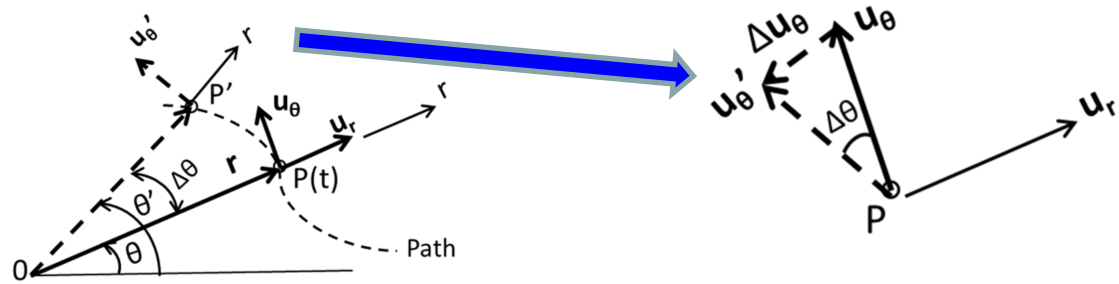
The instantaneous acceleration vector of the rigid body **a**(t) is expressed in the following way:

$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \frac{d}{dt}\left[\mathrm{r(t)}\dot{\theta}\mathrm{(t)}\mathbf{u_\theta} + \dot{r}(t)\mathbf{u_r}\right]$$

$$= \left[\dot{r}(t)\dot{\theta}(t)\mathbf{u_\theta} + r(t)\ddot{\theta}(t)\mathbf{u_\theta} + r(t)\dot{\theta}(t)\dot{\mathbf{u}}_\theta\right] + \left[\ddot{r}(t)\mathbf{u_r} + \dot{r}(t)\dot{\mathbf{u}}_\mathbf{r}\right]$$

Formulation of ??

Variation of unit vector $\mathbf{u_\theta}$ with motion of rigid body from P to P' ::



Note from the above diagram that although the magnitude of the unit vector $\mathbf{u_\theta}$ remains unchanged, its direction has varied from θ to θ' by an amount of Δθ. The shift is equal to $\Delta\mathbf{u_\theta} = \mathbf{u_\theta}' - \mathbf{u_\theta}$ or $\Delta\mathbf{u_\theta} \approx (\Delta\theta)\,\mathbf{u_\theta}$ for small Δθ, and also the magnitudes of these two unit vector to be 1.0. Consequently, we will have the following relationship:

$$\left|\Delta\mathbf{u}_\theta\right| = \Delta u_\theta \approx (\Delta\theta)\left|\mathbf{u}_\theta\right| = (\Delta\theta)(1) = (\Delta\theta)$$

$$\left|\Delta\mathbf{u}_\theta\right| = \Delta u_\theta \approx (\Delta\theta)\left|\mathbf{u}_\theta\right| = (\Delta\theta)\left|\mathbf{u}_r\right|$$ and $$\Delta\mathbf{u}_\theta = -(\Delta\theta)\mathbf{u}_r$$

A negative sign is added to the right-hand side of the last equation makes the vector $\Delta\mathbf{u_\theta}$ in opposite direction of the positive direction of vector $\mathbf{u_r}$ as indicated in the above figure.

3.7.4 **Plane Curvilinear Motion in Cylindrical Coordinates –** Cont

We are now ready to derive the expression for the rate of change of the unit vector $\mathbf{u_\theta}$, i.e. $\dot{\boldsymbol{u}}_{\boldsymbol{\theta}}$:

Realize that:

$$\lim_{\Delta t \to 0} \frac{\Delta \mathbf{u_\theta}}{\Delta t} = \dot{\mathbf{u}}_{\boldsymbol{\theta}} = \left(-\lim_{\Delta t \to o} \frac{\Delta \theta(t)}{\Delta t} \right) \mathbf{u_r} = -\dot{\theta}(t)\mathbf{u_r}$$

By substituting the above relationship into the following equation:

$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \left[\, \dot{r}(t)\dot{\theta}(t)\mathbf{u_\theta} + r(t)\ddot{\theta}(t)\mathbf{u_\theta} + r(t)\dot{\theta}(t)\dot{\mathbf{u}}_{\boldsymbol{\theta}} \,\right] + \left[\, \ddot{r}(t)\mathbf{u_r} + \dot{r}(t)\dot{\mathbf{u}}_{\mathbf{r}} \right]$$

We will get the instantaneous acceleration vector function: $\mathbf{a}(t) = a_r \mathbf{u_r} + a_\theta \mathbf{u_\theta}$
with the magnitudes:

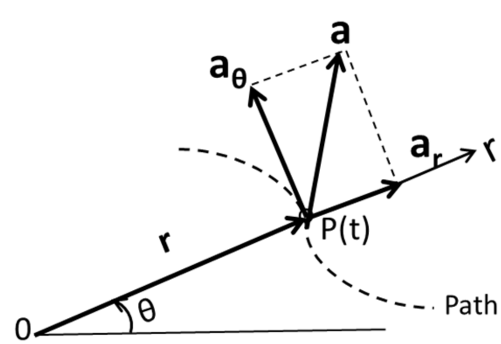
$a_r = \ddot{r}(t) - r\left[\, \dot{\theta}(t) \right]^2$ along the radial direction, and

$a_\theta = r\left[\, \ddot{\theta}(t) \right] + 2\left[\, \dot{r}(t)\dot{\theta}(t) \right]$ along the tangential direction

The magnitude of the instantaneous acceleration vector is thus obtained as:

$$a = |a| = \sqrt{a_r^2 + a_\theta^2}$$

$$= \sqrt{\left\{ \ddot{r}(t) - r\left[\, \dot{\theta}(t) \right]^2 \right\}^2 + \left\{ r\left[\, \ddot{\theta}(t) \right] + 2\left[\, \dot{r}(t)\dot{\theta}(t) \right] \right\}^2}$$

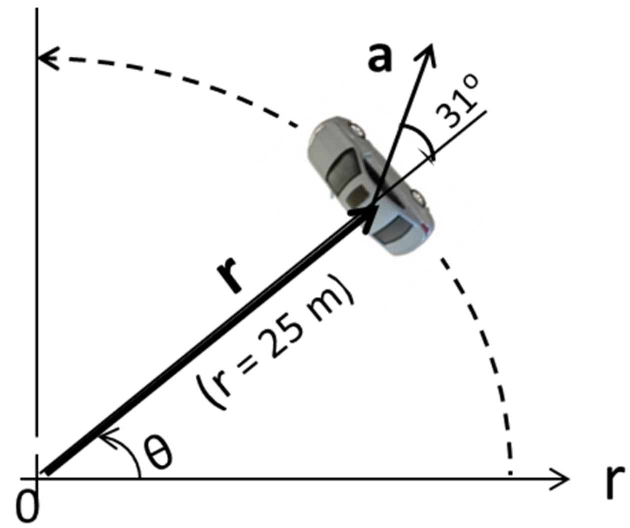


Example 3.18

An automobile is traveling along the section of circular curve of the road with a radius r = 25 m at the location and instant as shown in figure.

The rate of change of its angular displacement is $\dot{\theta} = 0.5\frac{rad}{s}$, with an angular acceleration of $\ddot{\theta} = 0.15\ rad/s^2$.

Determine the magnitude of the vehicle's velocity and acceleration at this instant.



Solution:

The position vector of the vehicle at time t is: $\mathbf{r}(t) = r\mathbf{u}_r + \theta\mathbf{u}_\theta = 25\mathbf{u}_r + \theta\mathbf{u}_\theta$

in which r and θ are the components of the position vector **r**(t) in the radial distance and angle from the reference line (the r-coordinate) in diagram. Vector of the vehicle

We will obtain the magnitude of the velocity vector of the vehicle by computing its magnitudes of its components using the expressions that we already derived to be:

$$v_r(t) = \frac{dr(t)}{dt} = \frac{d(25)}{dt} = 0 \quad \text{and} \quad v_\theta = r\dot{\theta}\ (t) = 25 \times 0.5 = 12.5\ rad/s$$

The magnitude of the velocity thus equals to: $v = \sqrt[2]{v_r^2 + v_\theta^2} = \sqrt{0^2 + 12.5^2}$ = 12.5 m/s

Example 3.18- Cont'd

The magnitudes of the components of the acceleration vector can b e computed by the following two equations as we have already derived:

$$a_r = \ddot{r}(t) - r\left[\dot{\theta}(t)\right]^2 \quad \text{and} \quad a_\theta = r\left[\dot{\theta}(t)\right] + 2\left[\dot{r}(t)\dot{\theta}(t)\right]$$

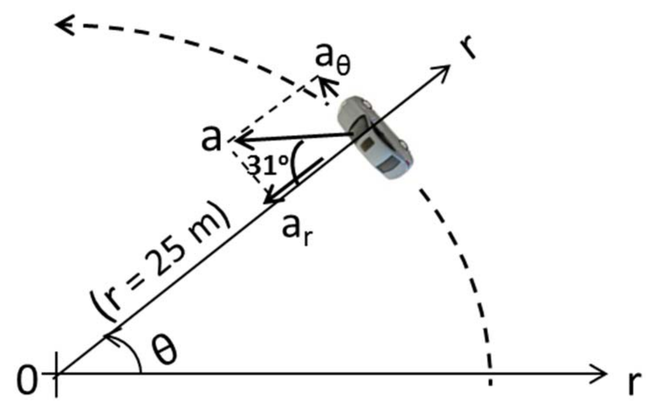
We thus have: $a_r = 0 - 25(0.5)^2 = -6.25\ m/s^2$ and $a_\theta = 25 \times 0.15 - 2 \times 0 \times 0.5 = 3.75\ m/s^2$

The magnitude of the acceleration vector of the vehicle is thus equals to:

$$a = \sqrt{(-6.25)^2 + (3.75)^2} = 7.29\ m/s^2$$

The angle ϕ that the direction of the acceleration vector **a** with the radial direction r in the figure is:

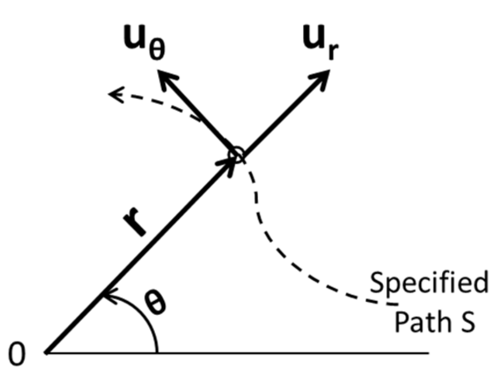
$$\emptyset = tan^{-1}\frac{a_\theta}{a_r} = tan^{-1}\frac{3.75}{(-6.25)} = -31^o$$

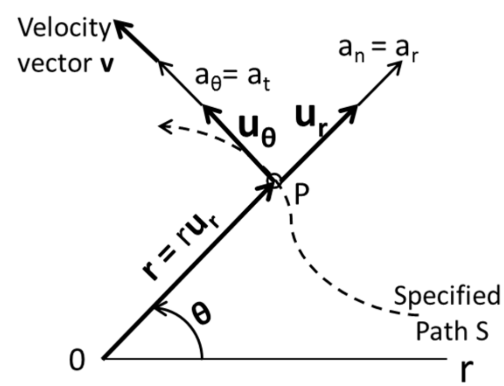


3.7.5 Plane Curvilinear Motion with Normal and Tangential Components

A well-known fact is that acceleration of moving rigid body is a well-sought physical quantity by engineers in order to determine the associated dynamic forces in engineering analysis. It requires engineers to express the acceleration vectors with their components in both the radial and tangential directions in a kinematic analysis of moving rigid bodies in circular paths. We may derive expressions for such components from the acceleration vector function in general curvilinear motion in cylindrical coordinates as presented in the proceeding slides.

The figure on the lower left shows the cylindrical coordinates (r,θ) and the two unit vectors along the linear coordinate r and the angular coordinate θ. We will derive expressions for the magnitudes of acceleration vectors along the linear radial direction r, and the tangential component normal to the r-coordinate as illustrated in the right figure.

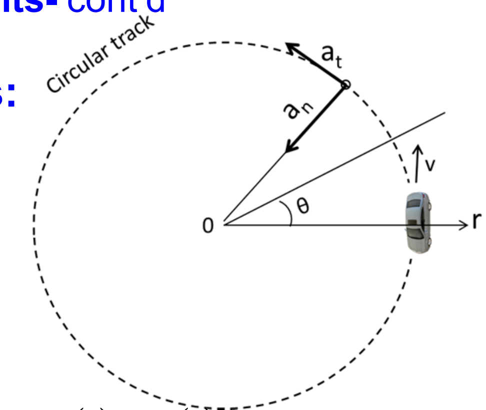




The acceleration vector of the moving solid can be expressed as:

$$\mathbf{a}(t) = a_r(t)\mathbf{u_r} + a_\theta(t)\mathbf{u_\theta}$$

where $a_r = \ddot{r} - r[\dot{\theta}(t)]^2$ *and* $a_\theta = r[\ddot{\theta}(t)] + 2r[\ddot{\theta}(t)]$ *are radial and tangential directrion rspecively*



Plane Circular Motion with Normal and Tangential Components:

The corresponding acceleration vector function **a**(t) of the vehicle in a circular motion as illustrated in the figure can be obtained by the following derivative of the velocity vector function **v**(t) with respect to time as:

$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \frac{d[v(t)\mathbf{u}_\theta(t)]}{dt} = \frac{dv(t)}{dt}\mathbf{u}_\theta(t) + v(t)\frac{d\mathbf{u}_\theta(t)}{dt} = \frac{dv(t)}{dt}\mathbf{u}_\theta(t) + v(t)\,\dot{\mathbf{u}}_\theta(t)$$

or $$\mathbf{a}(t) = \frac{dv(t)}{dt}\mathbf{u}_\theta(t) + v(t)\left[-\dot{\theta}\mathbf{u}_r(t)\right] = \dot{v}\mathbf{u}_\theta(t) - v(t)(\dot{\theta})\mathbf{u}_r(t) = a_r\mathbf{u}_r(t) + \dot{v}\mathbf{u}_\theta(t) \qquad (3.54)$$

with the relationship of: **v** (t)= v(t)$\mathbf{u}_\theta$(t) in which v(t) is the magnitude of the velocity vector **v**(t) and $\mathbf{u}_\theta$(t) is the unit vector in the θ-coordinate.

The acceleration vector function **a**(t) expressed in Equation (3.54) may be written in the following typical vector form: $\mathbf{a} = a_r\mathbf{u}_r + a_t\mathbf{u}_\theta$

where $a_r = a_n$ = the magnitude of normal component of the acceleration

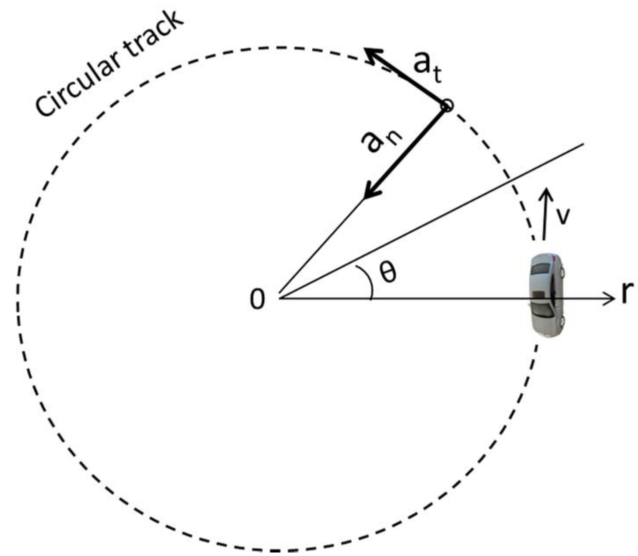
$a_t = a_\theta$ = the magnitude of tangential component of the acceleration vector.

By using the relationship: $\dot{\theta} = \frac{v_\theta}{r} = \frac{v}{r}$ and Equation (3.54), we will derive both the tangential and normal components of the acceleration vector to be:

$$a_t = \dot{v} = \frac{dv(t)}{dt} \quad \text{and} \quad a_n = -\frac{v^2}{r} \qquad (3.55a,b)$$

Example 3.19

An automobile travels on a circular track with a radius of 25 m as shown in figure to the roght. If the magnitude of the velocity vector function of the vehicle is v(t) = 1+2t^2 m/s, and the starting location of the vehicle is P in the figure, determine the following:



(a) The magnitude of the acceleration of the vehicle at 4 seconds from standstill location
(b) The distance and the number of laps that the car has traveled in 10 seconds.

Solution:

We are given the velocity vector function to be: **v**(t) = 1+2t^2 m/s, from which we may obtain the magnitude of the tangential component of the acceleration vector function to be:

$$a_t(t) = |\mathbf{a}_t(t)| = \frac{dv(t)}{dt} = \frac{d(1+2t^2)}{dt} = 4t$$

(a) Acceleration at time t = 4s:

$$a_n = -\frac{v(t)^2}{r}\bigg|_{t=4} = -\frac{\left[1+2(4)^2\right]^2}{25} = -43.56\ m/s^2$$

Example 3.19 – Cont'd

The magnitude of the acceleration vector of the vehicle at t=4s can thus be computed from the tow components in Equations (3.55a and b) as:

$$a(4) = \sqrt{a_t^2 + a_n^2} = \sqrt{16^2 + 43.56^2} = 46\ m/s^2$$

(b) The distance traveled by the vehicle at time t = 10s:

The distance S(t) that the vehicle has traveled after time t can be computed by using the following relationship: $v(t) = \dfrac{dS(t)}{dt}$ and thus: $\int_0^S dS(t) = \int_0^t v(t)dt$

Consequently, we will get:

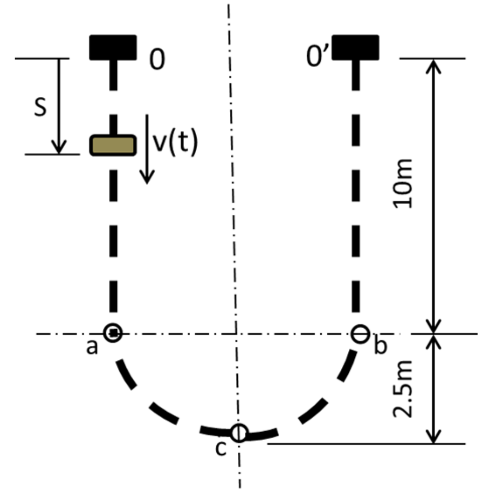
$$S(10) = \int_0^{10} (1 + 2t^2)dt = \left(t + \frac{2}{3}t^3 \right)\Bigg|_0^{10} = 676.67\ m$$

which is equivalent to the number of laps that the vehicle has traveled to be:

$$N = \frac{S(10)}{2\pi r} = \frac{676.67}{2x3.14x25} = 4.31\ laps$$

Example 3.20

The figure in the right is a baggage conveyor in the Frankfurt-Hann Airport in Germany.



We assume a box shown in grey in the associate diagram is being transported from the exit of the collection station by the moving conveyor. Approximate dimensions and the motion of the conveyor are illustrated in the diagram.

We further assume that the conveyor is designed to move the baggage from zero initial velocity at entry location at Point 0 with acceleration $a(t) = 0.001t$ m/s^2 between location 0 to Point C, but it moves the baggage at a constant velocity with no acceleration thereafter. Determine the following:

(a) The velocity of the box baggage at locations a and c,
(b) The acceleration and its direction of the baggage at the same locations.
(c) The time required for the baggage to reach Point c,
(d) The time required for the baggage to return to the collection station in 0' if it is not picked up by any passenger, and
(e) The time for a complete excursion.

Example 3.20- Cont'd

Solution:

We need to find the velocity vector function of the baggage that will lead us to the computation of the distance the box that has traveled from location 0 to c.
The magnitude of the tangential velocity vector function $\mathbf{v_t}(t)$ that can be derived in the following way:

Since we are given the magnitude of the acceleration vector function from given acceleration function: **a(t) = a_t(t) = 0.001t**, but this magnitude of the acceleration component is related to the rate of change of the velocity v_t(t) by the given relationship of:

$$\mathbf{a_t}(t) = \frac{d\mathbf{v_t}(t)}{dt}$$

We may thus obtain the velocity vector function by the following integration:

$$\int_0^t d\mathbf{v_t}(t)dt = \int_0^t \mathbf{a_t}(t)dt = \int_0^t (0.001t)dt$$

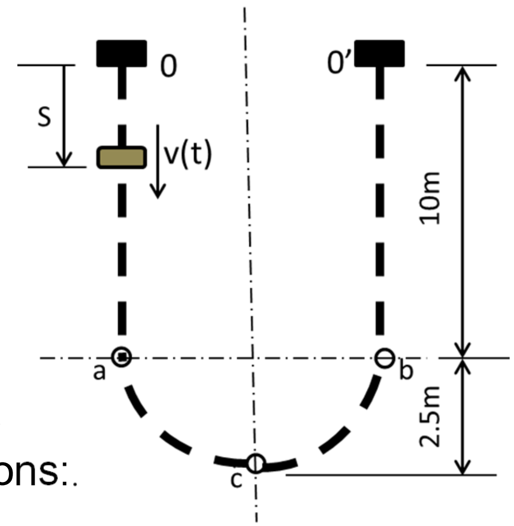
from which, we get the velocity vector function to be: **v_t(t) = 0.0005 t^2**

The above velocity vector function will lead us to derive the distance of the motion of the box traveling on the convey, as will show in the next slide.

Example 3.20- Cont'd

Let S(t) be the distance that the box has traveled from the starting point 0 after time t. The following relation: $v_t(t) = dS(t)/dt$ will lead to the corresponding distance of S(t) it has travelled with time t:

$$\int_0^t dS(t) = \int_0^t v_t(t)dt = \int_0^t (0.0005t^2)dt$$



from which, we obtain: **$S(t) = 1.67x10^{-4}\ t^3$**

The specific solutions to the example may thus proceed by the following steps:

(a) Determination of velocity of the box at locations at a and c and the required time for the box to reach these two locations:.

Now, let the time required for the box to reach location a = t_a.
From the distance Oa = 10 m, we may compute: $t_a = (10/1.67x10^{-4})^{1/3} = 39.12$ s

Likewise, we let the time for the box to reach location c from the from the entry location 0 = t_c, from which we solve for: $S(t_c) = 10 + 2\pi r/4 = 10 + 2x3.14x2.5/4 = 13.925$ m, resulting in $t_c = (13.925/1.67x10^{-4})^{1/3} = 46.69$s.

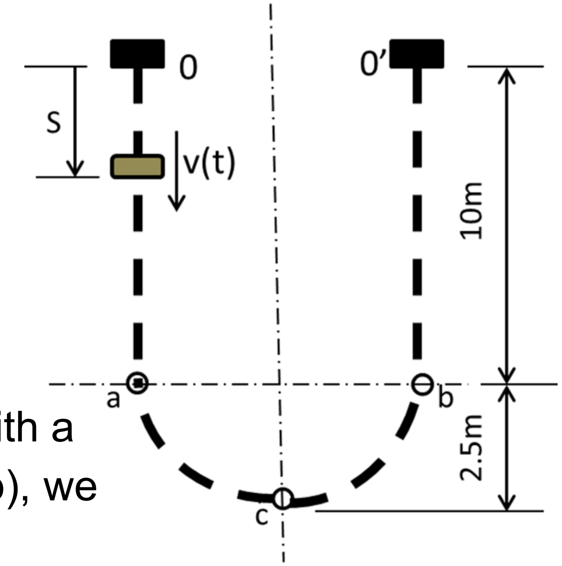
with t_a = 39.12s and t_c = 46.69s, we may compute the velocity at locations a and c by using: $v_t(t) = 0.0005\ t^2$, we compute $v_t(t_a) = 0.0005\ (39.12)^2 = 0.765$ m/s at location a, and $v_t(t_c) = 0.0005\ (46.69)^2 = 1.09$ m/s at location c.

Example 3.20- Cont'd

(b) The acceleration of the box at locations a and c :

Since we are given the magnitude of the acceleration vector function from given acceleration function: **$a(t) = a_t(t) = 0.001t$,** we may compute the following:

$a_t(t_a) = 0.001x39.12 = 0.04\ m/s^2$ at location a, and
$a_t(t_c) = a(t_c) = 0.001x46.69 = 0.0467\ m/s^2$ at location C.



However, we notice that location c is on a curvilinear path with a radius of curvature r = 2.5 m. Thus, by using Equation (3.55b), we may compute the magnitude of the normal component of the acceleration to be: $a_n(t_c) = [v_t(t_c)]^2/r^2 = (1.09)^2/2.5 = 0.4752\ m/s^2$.

Consequently, the magnitude of the acceleration at location C is thus:

$$a_c = \sqrt{a_t^2 + a_n^2} = \sqrt{(0.0467)^2 + (0.4752)^2} = 0.4776\ m/s^2$$

(c) The time required to pass location c is t_c = 46.69 s.

(d) We realize that the conveyor moves the baggage (and thus the box) with no acceleration beyond location C, i.e., *the box moves at constant speed from location C to the end location 0'*. The speed for the remaining portion of the movement follows the velocity at location c, or v = 1.09 m/s.

The time required to move the box from location c to 0' can be computed by the expression of $t_{c-0'} = S_{c-0'}/v = (10 + 2\pi r/4)/1.09 = 12.78$ s.

(e) The time for the entire excursion of the box movement by this particular conveyor is thus equal to: $t = t_c + t_{c-0'} = 46.69 + 12.78 = 59.47s \approx 1$ minute.