Applied Engineering Analysis
- slides for class teaching*

Chapter 4
Linear Algebra and Matrices

* Based on the book of "Applied Engineering Analysis", by Tai-Ran Hsu, published by John Wiley & Sons, 2018.
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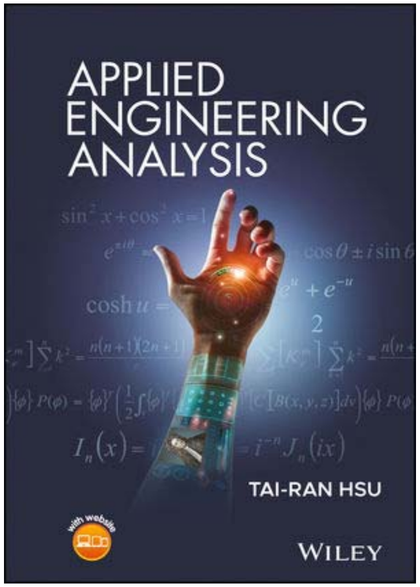


Chapter Learning Objectives

- Linear algebra and its applications
- Forms of linear functions and linear equations
- Expression of simultaneous linear equations in matrix forms
- Distinction between matrices and determinants
- Different forms of matrices for different applications
- Transposition of matrices
- Addition, subtraction and multiplication of matrices
- Inversion of matrices
- Solution of simultaneous equations using matrix inversion method
- Solution of large numbers of simultaneous equations using Gaussian elimination method
- Eigenvalues and Eigenfunctions in engineering analysis

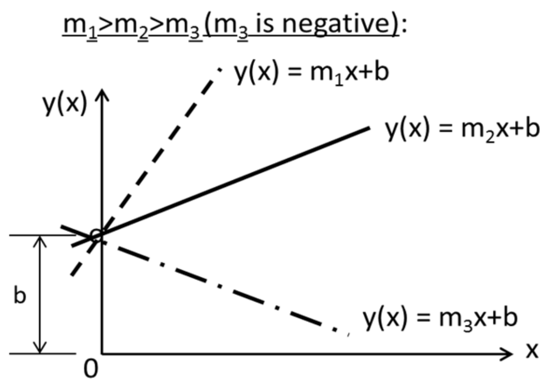
4.1 Introduction to Linear Algebra and Matrices

Linear algebra is concerned mainly with:

Systems of linear equations,
Matrices,
Vector space,
Linear transformations,
Eigenvalues, and eigenvectors.

Linear and Non-linear Functions and Equations:

Linear functions:



Examples of Nonlinear Equations:

$$-4x_1 + 3x_2 - 2x_3^2 + x_4^3 = 0$$

or $x^2 + y^2 = 1$

or $xy = 1$

or $\sin x = y$

Linear equations:

$-4x_1 + 3x_2 - 2x_3 + x_4 = 0$

where x_1, x_2, x_3 and x_4 are unknown quantities

Simultaneous linear equations:

$$\begin{aligned} 8x_1 + 4x_2 + x_3 &= 12 \\ 2x_1 + 6x_2 - x_3 &= 3 \\ x_1 - 2x_2 + x_3 &= 2 \end{aligned}$$

where x_1, x_2 and x_3 are unknown quantities

4.2 Determinants and Matrices

Both determinants and matrices are logical and convenient representations of large sets of real numbers or variables and vectors involved in engineering analyses.

These large sets of real numbers, variables and vector quantities are arranged in arrays of rows and columns:

$$\begin{array}{ccccccccc}
a_{11} & a_{12} & a_{13} & \bullet & \bullet & \bullet & \bullet & & a_{1n} \\
a_{21} & a_{22} & a_{23} & \bullet & \bullet & \bullet & \bullet & & a_{2n} \\
a_{31} & a_{32} & a_{33} & \bullet & \bullet & \bullet & \bullet & & a_{3n} \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & & \bullet \\
 & & & & & & & & \\
a_{m1} & a_{m2} & a_{m3} & \bullet & \bullet & \bullet & \bullet & & a_{mn}
\end{array}$$

in which a_{11}, a_{12},……………………., a_{mn} represent group of data, with m=row number, n = column number, and m = 1,2,3,….,m and n = 1,2,3,…..,n

There are different ways to express the Determinants and Matrices as shown below:

Determinant A is expressed with A placed between two vertical bars:

$$|A| = |a_{ij}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \bullet & \bullet & \bullet & \bullet & a_{1n} \\ a_{21} & a_{22} & a_{23} & \bullet & \bullet & \bullet & \bullet & a_{2n} \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ a_{m1} & a_{m2} & a_{m3} & \bullet & \bullet & \bullet & \bullet & a_{mn} \end{vmatrix}$$

whereas Matrix A is expressed by placing A in square brackets:

$$[A] = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \bullet & \bullet & \bullet & \bullet & a_{1n} \\ a_{21} & a_{22} & a_{23} & \bullet & \bullet & \bullet & \bullet & a_{2n} \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ a_{m1} & a_{m2} & a_{m3} & \bullet & \bullet & \bullet & \bullet & a_{mn} \end{bmatrix}$$

Difference between the determinants or matrices

The same data set in "determinants" can be evaluated to a single number, or a scalar quantity.

Matrices **cannot** be evaluated to single numbers or variables.

Matrices represent arrays of data and they remain so in mathematical operations in all engineering analyses.

Evaluation of determinants:

A determinant can be evaluated by sequential reduction in sizes, for example, a 2x2 determinant can be reduced to the size of 2-1=1 - a single number as in Example 4.1, whereas a 3x3 determinant can be reduced by two consequential reductions to reach a single value as illustrated in the Example 4.2. A general rule for the size reduction process is to use the following formula:

$$\left|C^{n}\right| = \left|c_{ij}^{n}\right| = \sum_{n=1}^{s} \left(-1\right)^{i+j} \left|c_{ij}^{n}\right| \qquad (4.9a)$$

where the superscript n denotes the reduction step number. Determinant $\left|C^{n}\right|$ is the determinant $\left|A\right|$ after the n-step reduction in size. The elements in these matrices c_{ij}^{n} are in the determinants that exclude the elements in i^{th} row and j^{th} column in the previous form of the determinant.

Matrices in engineering analysis:

As mentioned before, matrices cannot be evaluated to a single number or data. Rather, they will always be in the form of matrices.

4.3 Different Forms of Matrices

4.3.1 Rectangular matrices:

The general form of rectangular matrices is shown below:

$$[A] = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \bullet & \bullet & \bullet & \bullet & a_{1n} \\ a_{21} & a_{22} & a_{23} & \bullet & \bullet & \bullet & \bullet & a_{2n} \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ a_{m1} & a_{m2} & a_{m3} & \bullet & \bullet & \bullet & \bullet & a_{mn} \end{bmatrix} \qquad (4.11)$$

with the "elements" of this matrix designated by a_{ij} with the first subscript i indicating the row number and the second subscript j indicating the column number.

The rectangular matrices have the number of rows i ≠ number of columns j.

4.3.2 Square matrices:

This type of matrices with i = j, and are common in engineering analysis. Following is a typical square matrix of the size 3x3:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad (4.12)$$

4.3.3 **Row matrices:**

In this case, the total number of row I = m = 1 with the total number of columns = n:

$$\{A\} = \{a_{11} \quad a_{12} \quad a_{13} \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad a_{1n}\} \qquad (4.13)$$

4.3.4 **Column matrices:**

These matrices have only one column, i.e. n = j = 1 but with m rows.

$$\{A\} = \begin{Bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ a_{m1} \end{Bmatrix} \qquad (4.14)$$

Column matrices are used to express the components of vector quantities, such as the expression of a force vector:

$$\{\mathbf{F}\} = \begin{Bmatrix} \mathbf{F_x} \\ \mathbf{F_y} \\ \mathbf{F_z} \end{Bmatrix}$$

with $\mathbf{F_x}$, $\mathbf{F_y}$ and $\mathbf{F_z}$ to be the components of the force vector along the x, y and z coordinate respectively.

4.3.5 **Upper triangular matrices:**

We realize that all SQUARE matrices have a "diagonal" line across the elements drawn from those at the first row and column. An upper triangular matrix has all the element in this matrix to be zero below its diagonal line, such as Illustrated in the form in the right for an upper triangular matrix of 3x3 with elements below the diagonal line: $a_{21}=a_{31}=a_{32}=0$:

Diagonal of a square matrix

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

4.3.6 Lower triangular matrices:

This is an opposite case to the upper triangular matrix, in which all elements above the diagonal lines are zero as shown below for a 3 x 3 square matrix:

$$[A] = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad (4.16)$$

4.3.7 Diagonal matrices:

In these matrices, the only non-zero elements are those on the diagonals. Example of a diagonal matrix of the size of 4x4 is shown below:

$$[A] = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \qquad (4.17)$$

4.3.8 Unity matrices [I]:

This type of matrices is similar to that of diagonal matrices, except that the non-zero elements on the diagonal lines have a value of unity, i.e. "1.0". A 4x4 unity matrix is shown in the right:

$$[I] = \begin{bmatrix} 1.0 & 0 \cdots 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1.0 \end{bmatrix}$$

Unity matrices have the following useful properties:

$$\alpha[I] = \alpha \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} = a\ diagonal\ martrix$$ and [A][I] = [I][A] (4.19b)

4.4 **Transposition of Matrices**

Transposition of a matrix [A] often is required in engineering analysis. It is designated by $[A]^T$.

Transposition of matrix [A] is carried out by interchanging the subscripts that define the locations of the elements in matrix [A].

Mathematical operations of matrix transposition will be followed by letting $[a_{ij}]^T = [a_{ji}]$.

Following are three such examples:

Case 1: Transposing a column matrix into a row matrix:

$$\{A\}^T = \begin{Bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ a_{m1} \end{Bmatrix}^T = \{a_{11} \quad a_{21} \quad a_{31} \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad a_{m1}\}$$

Case 2: Transposing a rectangular matrix into another rectangular matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

Case 3: Transposing a square matrix into another square matrix:

Diagonal of a square matrix

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(a) Original matrix

$$[A]^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

(b) Transposed matrix

4.5 Matrix Algebra

We are made aware of the fact that matrices are expressions of arrays of numbers or variables – but not single numbers. As such, addition/subtraction and multiplications of matrices need to follow certain rules.

4.5.1 Addition and subtraction of matrices:

Addition or subtraction of two matrices requires that both matrices having the *same size*, i.e., with equal number of rows and columns.

$$[A] \pm [B] = [C] \ \text{with} \ c_{ij} = a_{ij} \pm b_{ij} \tag{4.20}$$

in which a_{ij}, b_{ij} and c_{ij} are the elements of the matrices [A], [B] and [C] respectively.

4.5.2 Multiplication of matrices by a scalar quantity α

$$\alpha\,[C] = [\alpha\, c_{ij}] \tag{4.21}$$

4.5.3 Multiplication of two matrices:

Multiplication of two matrices requires the satisfaction of the following condition:

The total number of column in the first matrix =

the total number of rows in the second matrix

Mathematically, we must have:

$$\underset{(m \times p)}{[C]} = \underset{(m \times n)}{[A]} \times \underset{(n \times p)}{[B]} \tag{4.22}$$

sizes: (m x p) = (m x n) (n x p)

in which the notations shown in the parentheses below the matrices in Equation (4.22) denotes the number of rows and columns in each of these matrices.

The following recurrence relationship in Eqution (4.23) may be used to determine the elements in the product matrix [C] with $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$ (4.23)

where i=1,2,..m and j=1,2,3……..n.

4.5.3 **Multiplication of two matrices-**Cont'd

Following are four (4) examples on multiplications of matrices

Example 4.4

Multiply two matrices [A] and [B] defined as:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad [B] = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Solution:

$$[C] = [A][B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}$$

Example 4.5**:**

Multiply the following *rectangular* matrix and a *column* matrix

Solution:

We checked the number of column of the first matrix equals the number of rows of the second Matrix. We may thus have the flowing multiplication:

$$\{y\} = [C]\{x\} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} c_{11}x_1 + c_{12}x_2 + c_{13}x_3 \\ c_{21}x_1 + c_{22}x_2 + c_{23}x_3 \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}$$

4.5.3 **Multiplication of two matrices-**Cont'd

Example 4.6**:**

This example will show he difference in the results of multiplication of two matrices in the different OEDER of the matrices.

Case A: Multiply a *Row* matrix by a *Column* matrix, resulting in a ***scalar quantity***:

$$\{a_{11} \quad a_{12} \quad a_{13}\}\begin{Bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{Bmatrix} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

Case B: Multiply a *Column* matrix by a *Row* matrix, resulting in a ***Square matrix***!

$$\begin{Bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{Bmatrix}\{b_{11} \quad b_{12} \quad b_{13}\} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} \\ a_{21}b_{11} & a_{21}b_{12} & a_{21}b_{13} \\ a_{31}b_{11} & a_{31}b_{12} & a_{31}b_{13} \end{bmatrix} \qquad (a \ square \ matrix)$$

Example 4.7**:**

We will show that multiply a *square* matrix by a *column* matrix will result in another *column* matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{Bmatrix} \qquad (a \ columnmatrix)$$

4.5.5 Additional rules on multiplication of matrices:

- Distributive law: [A]([B] + [C]) = [A][B] + [A][C]

- Associative law: [A]([B][C]) = [A][B]([C])

- Caution: Different order of multiplications of two matrices will result in different forms, i.e. the order of matrices in multiplication is very IMPORTANT. Always Remember the following relations:

$$[A][B] \neq [B][A]$$

- Product of two transposed matrices: $([A][B])^T = [B]^T[A]^T$

4.5.4 Matrix Representation of Simultaneous Equations

Matrix operations are powerful tools in modern-day engineering analysis. They are widely used in solving large numbers of simultaneous equations using digital computers. Following are the expressions on how matrices may be used to develop algorithms for the solution of large number of n-simultaneous equations:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \bullet & \bullet & \bullet & \bullet & a_{1n} \\ a_{21} & a_{22} & a_{23} & \bullet & \bullet & \bullet & \bullet & a_{2n} \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ a_{n1} & a_{n2} & a_{n3} & \bullet & \bullet & \bullet & \bullet & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \bullet \\ \bullet \\ x_n \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ \bullet \\ \bullet \\ r_n \end{Bmatrix}$$

from which, we may conveniently express these simultaneous linear equations in the following simplified form:

$$[A]\{x\} = \{r\} \tag{4.25}$$

where matrix [A] is usually referred to as the "coefficient matrix," {x} is the "unknown matrix," and {r} is the "resultant matrix."

Example: The matrix equation: $\begin{bmatrix} 8 & 4 & 1 \\ 2 & 6 & -1 \\ 1 & -2 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 12 \\ 3 \\ 2 \end{Bmatrix}$ represents the 3 simultaneous Equations:

$$\begin{aligned} 8x_1 + 4x_2 + x_3 &= 12 \\ 2x_1 + 6x_2 - x_3 &= 3 \\ x_1 - 2x_2 + x_3 &= 2 \end{aligned}$$

where x_1, x_2 and x_3 are the unknowns to be solved by these 3 simultaneous equations

4.6 Matrix Inversion $[A]^{-1}$

Since matrices are used to represent ARRAYS of numbers or variables in engineering analysis (but not single numbers of variables), there is no such thing as the division of two matrices. The closest to "divisions" in matrix algebra is *matrix inversion*. We define the inversion of matrix [A], i.e. $[A]^{-1}$ to be:

$$[A][A]^{-1} = [A]^{-1}[A] = [I] \qquad (4.26)$$

where [I] is a unity matrix defined by Equation (4.18) on P. 125.

One must note a fact that inversion of a matrix [A] is possible only if the equivalent determinant of [A], i.e. $|A| \neq 0$

The matrix [A] is called "singular matrix" if $|A| = 0$

Following are the 4 steps to invert the matrix [A]:

Step 1: Evaluate the equivalent determinant of the matrix [A], and make sure that $|A| \neq 0$

Step 2: If the elements of matrix [A] are a_{ij}, we may determine the elements of the co-factor matrix [C] to be: $c_{ij} = (-1)^{i+j}|A'|$ in which $|A'|$ is the equivalent determinant of a matrix [A'] that has all elements of [A] excluding those in the i^{th} row and j^{th} column

Step 3: Transpose the co-factor matrix from [C] to $[C]^T$ following the procedure outlined in Section 4.4 on p. 125

Step 4: The inverse matrix $[A]^{-1}$ for matrix [A] may be established by the following expression:

$$[A]^{-1} = \frac{1}{|A|}[C]^T \qquad (4.28)$$

Example 4.8 (p.130)

We will invert the following 3x3 matrix [A] following the 4 steps specified in the proceeding slide:

$$[A] = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ -2 & 5 & -3 \end{bmatrix}$$

Step 1: Evaluate the equivalent determinant of [A]:

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ -2 & 5 & -3 \end{vmatrix} = 1\begin{vmatrix} -1 & 4 \\ 5 & -3 \end{vmatrix} - 2\begin{vmatrix} 0 & 4 \\ -2 & -3 \end{vmatrix} + 3\begin{vmatrix} 0 & -1 \\ -2 & -3 \end{vmatrix} = -39\,(\neq 0)$$

Step 2: determine the elements of the co-factor matrix, [C]:

$$c_{11} = (-1)^{1+1}[(-1)(-3)-(4)(5)] = -17$$
$$c_{12} = (-1)^{1+2}[(0)(-3)-(4)(-2)] = -8$$
$$c_{13} = (-1)^{1+3}[(0)(5)-(-1)(-2)] = -2$$
$$c_{21} = (-1)^{2+1}[(2)(-3)-(3)(5)] = 21$$
$$c_{22} = (-1)^{2+2}[(1)(-3)-(3)(-2)] = 3$$
$$c_{23} = (-1)^{2+3}[(1)(5)-(2)(-2)] = -9$$
$$c_{31} = (-1)^{3+1}[(2)(4)-(3)(-1)] = 11$$
$$c_{32} = (-1)^{3+2}[(1)(4)-(3)(0)] = -4$$
$$c_{33} = (-1)^{3+3}[(1)(-1)-(2)(0)] = -1$$

We thus have the co-factor matrix, [C] in the form:

$$[C] = \begin{bmatrix} -17 & -8 & -2 \\ 21 & 3 & -9 \\ 11 & -4 & -1 \end{bmatrix}$$

Step 3: Transpose the [C] matrix is:

$$[C]^T = \begin{bmatrix} -17 & 21 & 11 \\ -8 & 3 & -4 \\ -2 & -9 & -1 \end{bmatrix}$$

which leads to the inverted matrix [A] to be:

Step 4: Determine the inverse matrix, $[A]^{-1}$ following Equation (4.28):

$$[A]^{-1} = \frac{[C]^T}{|A|} = \frac{1}{-39}\begin{bmatrix} -17 & 21 & 11 \\ -8 & 3 & -4 \\ -2 & -9 & -1 \end{bmatrix} = \frac{1}{39}\begin{bmatrix} 17 & -21 & -11 \\ 8 & -3 & 4 \\ 2 & 9 & 1 \end{bmatrix}$$

One may verify the correct inversion of matrix [A] by:

$$[A][A]^{-1} = [I]$$

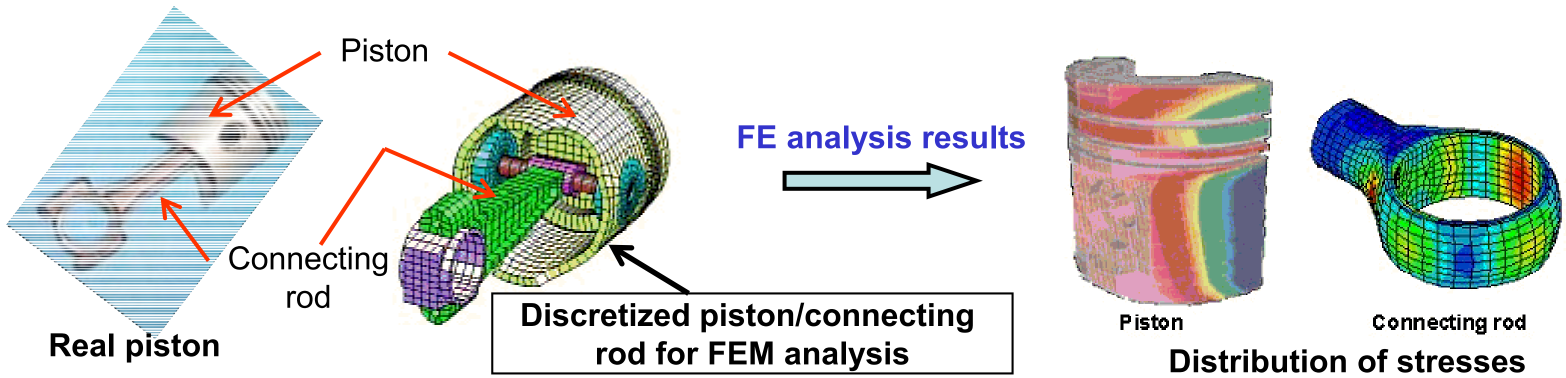
where [I] is a unit matrix defined in Equation (4.18)

Solution of Large Number of Simultaneous Equations Using Matrix Algebra

A vital tool for solving very large number of simultaneous equations in engineering analysis using digital computers

Why huge number of simultaneous equations in advanced engineering analyses?

- Numerical analyses, such as the **finite element method** (FEM) and **finite difference method** (FDM) are two effective and powerful analytical tools for engineering analysis in in real but complex situations in:
  - Mechanical stress and deformation analyses of machines and structures,
  - Thermofluid analyses for temperature distributions in solids, and fluid flow behavior requiring solutions in pressure drops and local velocity, as well as fluid-induced forces.
- The essence of FEM and FDM is to **DISCRETIZE the continua** of "real structures" or "flow patterns" of complex configurations and loading/boundary conditions into FINITE number of sub-components (called elements) inter-connected at common NODES.
- Analyses are performed in individual ELEMENTS instead of entire continua of complex solid or flow patterns.
- Example of discretization of a piston in an internal combustion engine and the results in stress distributions in piston and connecting rod are depicted in the following images:



http://www.npd-solutions.com/feaoverview.html

- FEM or FDM analyses requires the derivation of one algebraic equation for every NODE in the discretized model – One will readily appreciate the need for solving a huge number of simultaneous equations, in view of the huge number of elements (and nodes) involved in the analysis as illustrated in the 2 left images on real solid structure and the analytical model for the piston and the connecting rod !!
- Many analyses using FEM requiring solutions of tens of thousands simultaneous equations are not unusual in advanced engineering analyses.

4.7 Solution of Simultaneous Linear Equations

We have demonstrated in Section 4.7.1 and the case in the proceeding slide on the need for using commercial finite element computer codes (see detailed description of the finite element method and commercial code in Chapter 11) require the solutions of very large number of simultaneous equations (often in hundreds or thousands in the numbers).

Required time and efforts in solving these huge number of simultaneous equations obviously are much beyond human capability. These tasks apparently require the use of digital computers with proper algorithms. Matrix algebra is the only viable way for developing algorithms for digital computers to do this job.

There are generally two methods suitable for such applications:

(1) The inverse matrix technique, and (2) The Gaussian elimination technique, as will be presented in the following formulations.

4.7.2 Solution of Large Number of Simultaneous Linear Equations Using Inverse matrix technique:

We have demonstrated how the following simultaneous equations may be expressed in matrix form:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots\dots\dots\dots + a_{1n}x_n &= r_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots\dots\dots\dots\dots + a_{2n}x_n &= r_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots\dots\dots\dots\dots + a_{3n}x_n &= r_3 \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots\dots\dots\dots\dots + a_{nn}x_n &= r_n
\end{aligned}$$

or by a compact form of: [A]{x} = {r}

4.7 **Solution of Simultaneous Linear Equations –** Cont'd

4.7.2 Solution of Large Number of Simultaneous Linear Equations Using Inverse Matrix Technique – Cont'd:

The unknown matrix {x} in the above equation may be solved by multiplication of an inverse matrix of [A] on both sides of the equation as follows:

$$[A]^{-1}[A]\{x\} = [A]^{-1}\{r\} \qquad \text{or} \qquad [I]\{x\} = [A]^{-1}\{r\}$$

We may thus determine the unknown matrix $\{x\} = [A]^{-1}\{r\}$ to be the solution of the simultaneous equations.

Example 4.9 (p.134)

Solve the following simultaneous equations using the inverse matrix method:

$$4x_1 + x_2 = 24 \qquad \text{(a)}$$

$$x_1 - 2x_2 = -21 \qquad \text{(b)}$$

Solution:

We may express the above simultaneous equations into a matrix form: [A]{x} = {r}.

where the matrices $[A] = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}$ $\{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$ *and* $\{r\} = \begin{Bmatrix} 24 \\ -21 \end{Bmatrix}$

We found the inverse of [A] matrix to be:

$$[A]^{-1} = \frac{[C]^T}{|A|} = -\frac{1}{9}\begin{bmatrix} -2 & -1 \\ -1 & 4 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 2 & 1 \\ 1 & -4 \end{bmatrix}$$

which leads to the solution of x_1=3 and x_2=12 as follows:

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = [A]^{-1}\{r\} = \frac{1}{9}\begin{bmatrix} 2 & 1 \\ 1 & -4 \end{bmatrix}\begin{Bmatrix} 24 \\ -21 \end{Bmatrix} = \frac{1}{9}\begin{Bmatrix} 2x24 - 1x21 = 27 \\ 1x24 + (-4)(-21) = 108 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 12 \end{Bmatrix}$$

The use of Inverse matrix technique in solving simultaneous equations is usually limited to moderate number of simultaneous equations in engineering analysis, say less than 100.

4.7.3 Solution of large number of simultaneous equations using Gaussian elimination method

The inventor of this method was Johann Carl Friedrich Gauss (1777–1855)

A German astronomer (planet orbiting),
Physicist (molecular bond theory, magnetic theory, etc.), and
Mathematician (differential geometry, Gaussian distribution in statistics, etc.)

- Gaussian elimination method and its derivatives, e.g., Gaussian-Jordan elimination method and Gaussian-Siedel iteration method are widely used in solving large number of simultaneous equations as required in many modern-day numerical analyses, such as engineering analyses involving FEM and FDM as mentioned earlier.

- The principal reason for Gaussian elimination method being popular in this types of applications is the required formulations in this method are in simple arithmetic expressions, and the solution procedure for the large number of unknown quantities can be readily programmed using current programming languages such as FORTRAN for digital computers with enormous memory capacities and incrediblly high computational efficiencies.

The essence of Gaussian elimination method:

1) To convert the square coefficient matrix [A] of a set of simultaneous equations into the form of "Upper triangular" matrix in Equation (4.15) by using an "*elimination procedure*"

$$[A]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{\text{Via "elimination process}} [A]^{upper} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

The superscripts attached to the elements of the right-hand-side matrix designate the step number in the elimination process, for instance: (') for the step 1 and ('') for step 2 after the elimination process.

2) The last unknown quantity in the converted upper triangular coefficient matrix and the corresponding changes in the resultant matrix in the simultaneous equations becomes immediately available, as shown below:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2' \\ r_3'' \end{Bmatrix} \quad \text{with} \Longrightarrow x_3 = r_3''/a_{33}''$$

3) The second last unknown quantity x_2 may be obtained by substituting the newly found numerical value of the last unknown quantity into the second last equation:

$$a_{22}' x_2 + a_{23}' x_3 = r_2' \Longrightarrow x_2 = \frac{r_2' - a_{23}' x_3}{a_{22}'}$$

4) The remaining unknown quantities (x_1) may be obtained by the similar procedure, which is termed as "*back substitution*"

4.7.3 **Math formulations of The Gaussian elimination process:**

We will present the math formulations of this process by the solution of the following three simultaneous equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = r_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = r_2 \qquad \text{(4.34 a,b,c)}$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = r_3$$

We will express this simultaneous equation in (4.34a,b,c) in a matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} \qquad \text{(4.35)}$$

or in a simpler form: $[A]\{x\} = \{r\}$

Step 1: We will express the unknown x_1 in Equation (4.34a) in terms of x_2 and x_3 as follows:

$$x_1 = \frac{r_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2 + \frac{a_{13}}{a_{11}}x_3$$

Now, if we substitute x_1 in Equation (4.34b and c) with: $x_1 = \frac{r_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2 + \frac{a_{13}}{a_{11}} x_3$

we will turn Equation (4.34a,b,c) from: → to a new form in Equation (4.36a,b,c):

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = r_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = r_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = r_3$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = r_1 \quad (4.36a)$$

$$0 + \left(a_{22} - a_{21}\frac{a_{12}}{a_{11}}\right)x_2 + \left(a_{23} - a_{21}\frac{a_{13}}{a_{11}}\right)x_3 = r_2 - \frac{a_{21}}{a_{11}} r_1 \quad (4.36b)$$

$$0 + \left(a_{32} - a_{31}\frac{a_{12}}{a_{11}}\right)x_2 + \left(a_{33} - a_{31}\frac{a_{13}}{a_{11}}\right)x_3 = r_3 - \frac{a_{31}}{a_{11}} r_1 \quad (4.36c)$$

You will not see x_1 in the new Equation (4.36b and c) anymore with this substitution –
So, x_1 is "eliminated" in Equations (4,36a,b,c) after Step 1 elimination

The new matrix form of the simultaneous equations has the form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a^1_{22} & a^1_{23} \\ 0 & a^1_{32} & a^1_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r^1_2 \\ r^1_3 \end{Bmatrix} \quad (4.37)$$

where $a^1_{22} = a_{22} - a_{21}\frac{a_{12}}{a_{11}}$ $\quad a^1_{23} = a_{23} - a_{21}\frac{a_{13}}{a_{11}}$

$a^1_{32} = a_{32} - a_{31}\frac{a_{12}}{a_{11}}$ $\quad a^1_{33} = a_{33} - a_{31}\frac{a_{13}}{a_{11}}$

$r^1_2 = r_2 - \frac{a_{21}}{a_{11}} r_1$ $\quad r^1_3 = r_3 - \frac{a_{31}}{a_{11}} r_1$

The superscript index numbers ("1") indicates "elimination step 1" in the above expressions

Step 2: Elimination involves the expression of x_2 in Equation (4.36b) in term of x_3:

from
$$0+\left(a_{22}-a_{21}\frac{a_{12}}{a_{11}}\right)x_2+\left(a_{23}-a_{21}\frac{a_{13}}{a_{11}}\right)x_3=r_2-\frac{a_{21}}{a_{11}}r_1 \qquad (4.36b)$$

to
$$x_2=\frac{r_2-\frac{a_{21}}{a_{11}}r_1-\left(a_{23}-a_{21}\frac{a_{13}}{a_{11}}\right)x_3}{\left(a_{22}-a_{21}\frac{a_{12}}{a_{11}}\right)}$$

and submitted it into Equation (4.36c), resulting in eliminating x_2 in that equation.

The matrix [A] form of the original simultaneous equations now takes the form of an Upper triangular matrix, and we have thus accomplished the Gaussian elimination process:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a^2_{22} & a^2_{23} \\ 0 & 0 & a^2_{33} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}=\begin{Bmatrix} r^2_1 \\ r^2_2 \\ r^2_3 \end{Bmatrix} \qquad (4.38)$$

We notice the coefficient matrix [A] now has become an “upper triangular matrix,” from which we have the solution from the las row of the elements to give:

$$x_3=\frac{r^2_3}{a^2_{33}}$$

The other two unknowns x_2 and x_1 may be obtained by the “back substitution process” from Equation (4.38),such that:

$$x_2=\frac{r^2_2-a^2_{23}x_3}{a^2_{22}}=\frac{r^2_2-a^2_{23}\frac{r^2_3}{a^2_{33}}}{a^2_{22}} \quad \text{and} \quad x_1=\frac{r_1}{a_{11}}-\frac{a_{12}}{a_{11}}x_2+\frac{a_{13}}{a_{11}}x_3$$

Recurrence relations for Gaussian elimination process:

We have learned that Gaussian elimination method requires the conversions of the original square coefficient matrix [A] into a upper triangular matrix form with the corresponding modifications of the Resultant matrices {r} in the given simultaneous equations.

For a set of 3-simultaneous equations, (3-1) = 2 steps of elimination would be sufficient for this process as we have demonstrated in the previous case. So, we may say that we need to perform (n-1) elimination steps to solve the n-simultaneous equations.

In reality, for example, we often need to perform (50,000 – 1) = 49,999 elimination steps to solve 50,000 given simultaneous equations in an engineering analysis. Such task is by no means realistic if all such computations are performed by human efforts.

A realistic way to perform such tasks is to use digital computers with their horrendous capacity of storage memories and super fast computation of arithmetic operations. Gaussian elimination method that provides eliminations of elements in the coefficient matrices [A], and the corresponding revisions of the Resultant matrices {r} can all be done with arithmetic operations as shown in the previous example appear to be viable for having them used in developing the algorithm for programming for most available digital computers. The following slide will show the recurrence relations that can achieve the above set goals.

Recurrence relations for Gaussian elimination process-Cont'd:

Given a general form of n-simultaneous equations:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots\dots\dots\dots\dots\dots\dots + a_{1n}x_n &= r_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots\dots\dots\dots\dots\dots\dots\dots + a_{2n}x_n &= r_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots\dots\dots\dots\dots\dots\dots\dots + a_{3n}x_n &= r_3 \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots& \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots& \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots\dots\dots\dots\dots\dots\dots\dots + a_{mn}x_n &= r_n
\end{aligned}
\qquad (4.30)
$$

The following recurrence relations can be used in Gaussian elimination process:

For elimination:

with i > n and j>n, in which
n = elimination step number

$$a_{ij}^{n} = a_{ij}^{n-1} - a_{in}^{n-1}\frac{a_{nj}^{n-1}}{a_{nn}^{n-1}} \qquad (4.39a)$$

$$r_{i}^{n} = r_{i}^{n-1} - a_{in}^{n-1}\frac{r_{n}^{n-1}}{a_{nn}^{n-1}} \qquad (4,39b)$$

Solution of unknowns from back substitution:

$$x_i = \frac{r_i - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}} \quad with \;\; i = n-1, n-2, \dots\dots, 1 \qquad (4.40)$$

where a_{ij}, r_i and x_i are the elements in the final matrices at the conclusion of the elimination process.

Example 4.10 (p. 138):

Solve the following simultaneous equations using Gaussian elimination method .

$$4x_1 + x_2 = 24 \qquad \text{(a)}$$
$$x_1 - 2x_2 = -21 \qquad \text{(b)}$$

<u>Solution</u>:
We may express these simultaneous equations into matrix form as:

$$\begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 24 \\ -21 \end{Bmatrix}$$

Recognize that: $a_{11}^0 = a_{11} = 4, a_{12}^0 = a_{12} = 1, a_{21}^0 = a_{21} = 1, a_{22}^0 = a_{22} = -2, r_1 = 24 \text{ and } r_2^0 = r_2 = -21$

We are now ready to use the recurrent relationships shown in Equations (4.39 a,b) for the Gaussian elimination process. We realize that only 2-1= one step is required to convert the coefficient matrix for these 2 simultaneous equations.

<u>Step 1</u> with n = 1, i > n = 2 and j > n = 2:

$$a_{22}^1 = a_{22}^0 - a_{21}^0 \frac{a_{12}^0}{a_{11}^0} = -2 - 1x\frac{1}{4} = -2 - \frac{1}{4} = -\frac{9}{4} \quad \text{and} \quad r_2^1 = r_2^0 - a_{21}^0 \frac{r_1^0}{a_{11}^0} = -21 - 1x\frac{24}{4} = -21 - 6 = -27$$

The coefficient matrix [A] after this step of elimination becomes:

$$\begin{bmatrix} 4 & 1 \\ 0 & -\frac{9}{4} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 24 \\ -27 \end{Bmatrix}$$

we have the solution for x_2 from the last (the 2nd) equation as: $-\frac{9}{4}x_2 = -27 \rightarrow x_2 = 12$

and use the back substitution to compute the other unknown:

$$x_1 = \frac{r_1 - \sum_{j=2}^{2} a_{1j} x_j}{a_{11}^o} = \frac{r_1 - a_{12}^o x_2}{a_{11}^o} = \frac{24 - 1x12}{4} = 3$$

Example 4.11 (p. 139):

Solve the following 3 simultaneous equations using Gaussian elimination method:

$$8x_1 + 4x_2 + x_3 = 12 \quad \text{(a)}$$
$$2x_1 + 6x_2 - x_3 = 3 \quad \text{(b)}$$
$$x_1 - 2x_2 + x_3 = 2 \quad \text{(c)}$$

Solution:

We will first express Equations (a,b and c) in the following matrix form:

$$\begin{bmatrix} 8 & 4 & 1 \\ 2 & 6 & -1 \\ 1 & -2 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 12 \\ 3 \\ 2 \end{Bmatrix} \quad \text{(d)}$$

Because there are 3 simultaneous equations in this example, we will need to perform 3-1 = 2 steps of elimination for the solutions:

Step 1 with n = 1, i > n = 2 and j > n = 2:

with i = 2, j = 2:

$$\overset{1}{a}_{22} = \overset{0}{a}_{22} - \overset{0}{a}_{21} \frac{\overset{0}{a}_{12}}{\overset{0}{a}_{11}} = a_{22} - a_{21} \frac{a_{12}}{a_{11}} = 6 - 2x\frac{4}{8} = 5$$

and with i = 2, j = 3:

$$\overset{1}{a}_{23} = \overset{0}{a}_{23} - \overset{0}{a}_{21} \frac{\overset{0}{a}_{13}}{\overset{0}{a}_{11}} = a_{23} - a_{21} \frac{a_{13}}{a_{11}} = -1 - 2x\frac{1}{8} = -1.25$$

$$\overset{1}{r}_2 = \overset{0}{r}_2 - \overset{0}{a}_{21} \frac{\overset{0}{r}_1}{\overset{0}{a}_{11}} = r_2 - a_{21} \frac{r_1}{a_{11}} = 3 - 2x\frac{12}{8} = 0$$

Example 4.11-Cont'd

with i = 3 and j = 2:

$$a_{32}^{1} = a_{32}^{0} - a_{31}^{0}\frac{a_{12}^{0}}{a_{11}^{0}} = a_{32} - a_{31}\frac{a_{12}}{a_{11}} = -2 - 1x\frac{4}{8} = -2.5$$

$$a_{33}^{1} = a_{33}^{0} - a_{31}^{0}\frac{a_{13}^{0}}{a_{11}^{0}} = a_{33} - a_{31}\frac{a_{13}}{a_{11}} = 1 - 1x\frac{1}{8}\ 0.875$$

$$r_{3}^{1} = r_{3}^{0} - a_{31}^{0}\frac{r_{1}^{0}}{a_{11}^{0}} = r_{3} - a_{31}\frac{r_{1}}{a_{11}} = 2 - 1x\frac{12}{8} = 0.5$$

We may thus express Equation (d) after Step 1 of elimination to take the form:

$$\begin{bmatrix} 8 & 4 & 1 \\ 0 & 5 & -1.25 \\ 0 & -2.5 & 0.875 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 12 \\ 0 \\ 0.5 \end{Bmatrix} \qquad \text{(e)}$$

We are now ready to perform Step 2 (the last step) in the elimination process to convert the coefficient matrix in Equation (e) into an upper triangular matrix:

Step 2 with n = 2, i > n= 3 and j > n = 3:

We realize that $a_{21}^{2} = a_{31}^{2} = a_{32}^{2} = 0$ because the subscripts i and j of these matrix elements are less than n = 2. by using the recurrence relations, we compute the following:

$$a_{33}^{2} = a_{33}^{1} - a_{32}^{1}\frac{a_{23}^{1}}{a_{22}^{1}} = 0.875 - (-2.5)x\frac{(-1.25)}{5} = 0.25 \quad \text{and}$$

$$r_{3}^{2} = r_{3}^{1} - a_{32}^{1}\frac{r_{2}^{1}}{a_{22}^{1}} = 0.5 - (-2.5)x\frac{0}{5} = 0.5$$

Example 4.11-Cont'd

We have completed the conversion of the matrix equation in Equation (e) to a new form of upper triangular coefficient matrix [A] with modified resultant matrix {r} in Equation (f) after Step 2 elimination:

$$\begin{bmatrix} 8 & 4 & 1 \\ 0 & 5 & -1.25 \\ 0 & 0 & 0.25 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 12 \\ 0 \\ 0.5 \end{Bmatrix} \qquad \text{(f)}$$

One may readily see from the last line in Equation (f) for the solution of x_3 to be:

$$x_3 = 0.5/0.25 = 2$$

The values of the remaining two unknowns, x_2 and x_1, may be obtained by using the recurrence relation of back substitution as given in Equation (4.39b) as follows:

We begin with n = 3 in Equation (4.39b) with:

$$x_i = \frac{r_i - \sum_{j=i+1}^{3} a_{ij} x_j}{a_{ii}} \qquad with\ i = 2, 1$$

Hence, with i = 2: $x_2 = \dfrac{r_2 - \sum_{j=3}^{3} a_{2j} x_j}{a_{22}} = \dfrac{r_2 - a_{23} x_3}{a_{22}} = \dfrac{0 - (-1.25)x2}{5} = 0.5$

and to determine x_1 with i = 1: $x_1 = \dfrac{r_1 - \sum_{j=2}^{3} a_{1j} x_j}{a_{11}} = \dfrac{r_1 - (a_{12} x_2 + a_{13} x_3)}{a_{11}} = \dfrac{12 - (4x0.5 + 1x2)}{8} = 1$

Additional Example:

Solve the following simultaneous equations using the Gaussian elimination method:

$$\begin{aligned} x + z &= 1 \\ 2x + y + z &= 0 \\ x + y + 2z &= 1 \end{aligned} \qquad \text{(a)}$$

Express the above equations in a matrix form:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} \qquad \text{(b)}$$

If we compare Equation (b) with the following typical matrix expression of 3-simultaneous equations:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix}$$

we will have the following matrix elements at Zeroth step:

$$\begin{array}{llllll} a_{11} = 1 & a_{12} = 0 & a_{13} = 1 & & & r_1 = 1 \\ a_{21} = 2 & a_{22} = 1 & a_{23} = 1 & & \text{and} & r_2 = 0 \\ a_{31} = 1 & a_{32} - 1 & a_{33} = 2 & & & r_3 = 1 \end{array}$$

Let us use the recurrence relationships for the elimination process in Equation (8.25):

$$a_{ij}^{n} = a_{ij}^{n-1} - a_{in}^{n-1}\frac{a_{nj}^{n-1}}{a_{nn}^{n-1}} \qquad r_{i}^{n} = r_{i}^{n-1} - a_{in}^{n-1}\frac{r_{n}^{n-1}}{a_{nn}^{n-1}}$$ with i >n and j> n

Step 1 n = 1, so i = 2,3 and j = 2,3

For i = 2, j = 2 and 3:

i = 2, j = 2: $$a_{22}^{1} = a_{22}^{0} - a_{21}^{0}\frac{a_{12}^{0}}{a_{11}^{0}} = a_{12} - a_{21}\frac{a_{12}}{a_{11}} = 1 - 2\frac{0}{1} = 1$$

i = 2, j = 3: $$a_{23}^{1} = a_{23}^{0} - a_{21}^{0}\frac{a_{13}^{0}}{a_{11}^{0}} = a_{23} - a_{21}\frac{a_{13}}{a_{11}} = 1 - 2\frac{1}{1} = -1$$

i = 2: $$r_{2}^{1} = r_{2}^{o} - a_{21}^{0}\frac{r_{1}^{0}}{a_{11}^{0}} = r_{2} - a_{21}\frac{r_{1}}{a_{11}} = 0 - 2\frac{1}{1} = -2$$

For i = 3, j = 2 and 3:

i = 3, j = 2: $$a_{32}^{1} = a_{32}^{0} - a_{31}^{0}\frac{a_{12}^{0}}{a_{11}^{0}} = a_{32} - a_{31}\frac{a_{12}}{a_{11}} = 1 - 1\frac{0}{1} = 1$$

i = 3, j = 3: $$a_{33}^{1} = a_{33}^{0} - a_{31}^{0}\frac{a_{13}^{0}}{a_{11}^{0}} = a_{33} - a_{31}\frac{a_{13}}{a_{11}} = 2 - 1\frac{1}{1} = 1$$

i = 3: $$r_{3}^{1} = r_{3}^{0} - a_{31}^{0}\frac{r_{1}^{0}}{a_{11}^{0}} = r_{3} - a_{31}\frac{r_{1}}{a_{11}} = 1 - 1\frac{1}{1} = 0$$

So, the original simultaneous equations after Step 1 elimination have the form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{1} & a_{23}^{1} \\ 0 & a_{32}^{1} & a_{33}^{1} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2^{1} \\ r_3^{1} \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -2 \\ 0 \end{Bmatrix}$$

We now have: $a_{21}^{1} = 0 \quad a_{22}^{1} = 1 \quad a_{23}^{1} = -1$

$a_{31}^{1} = 0 \quad a_{32}^{1} = 1 \quad a_{33}^{1} = 1$

$r_2^{1} = -2 \quad r_3^{1} = 0$

Step 2 n = 2, so i = 3 and j = 3 (i > n, j > n)

i = 3 and j = 3: $$a_{33}^2 = a_{33}^1 - a_{32}^1 \frac{a_{23}^1}{a_{22}^1} = 1 - 1\frac{(-1)}{1} = 2$$

$$r_3^2 = r_3^1 - a_{32}^1 \frac{r_2^1}{a_{22}^1} = 0 - 1\frac{(-2)}{1} = 2$$

The coefficient matrix [A] has now been triangularized, and the original simultaneous equations has been transformed into the form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^1 & a_{23}^1 \\ 0 & 0 & a_{33}^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2^1 \\ r_3^2 \end{Bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -2 \\ 2 \end{Bmatrix}$$

We get from the last equation with (0) x_1 + (0) x_2 + 2 x_3 = 2, from which we solve for x_3 = 1. The other two unknowns x_2 and x1 can be obtained by back substitution of x_3 using Equation (8.26):

$$x_2 = \left(r_2 - \sum_{j=3}^{3} a_{2j} x_j \right) / a_{22} = (r_2 - a_{23} x_2) / a_{22} = [-2 - (-1)(1)]/1 = -1$$

and $$x_1 = \left(r_1 - \sum_{j=2}^{3} a_{1j} x_j \right) / a_{11} = [r_1 - (a_{12} x_2 + a_{13} x_3)] / a_{11}$$

$$= \{1 - [0(-1) + 1(1)]\}/1 = 0$$

We thus have the solution: x = x_1 = 0; y = x_2 = -1 and z = x_3 = 1

4.8 Eigenvalues and Eigenfunctions (p.141)

Eigenvalue – a German term meaning "characteristic value" that appears in some math operations.
Eigenfunction – a "characteristic function" present in a math operation

Both eigenvalues and eigenfunctions, in general, are used in the following two areas in engineering analysis:

(1) In transform of a geometry from one space to another for the same or desired enlargement or reduced magnitudes and orientations to simplify the analysis, and

(2) In the form of parameters appearing in, and characterizing the solutions of certain equations in the analysis.

We will introduce the application of eigenfunctions in geometric transformation first, to be followed by the second type of application as mentioned above.

4.8 Eigenvalues and Eigenfunctions in geometric transformation:

Geometric transformation often is performed in engineering analysis involving complex geometry of solids or fluids. The purpose of using this technique is to transform these solids and fluids of complex geometry to a much simpler geometry that can be handled by available analytical techniques.

Geometric Transformation-Linear transformation:

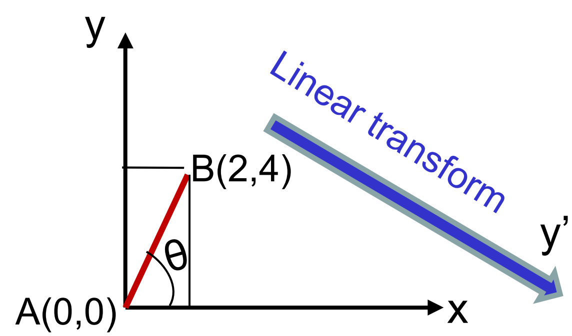
It is a rule for changing one geometric figure (or matrix or vector) into another, using a formula with a specified format. This format is a linear combination, in which the original components (e.g., the x and y coordinates of each point of the original figure) are changed via the formula $ax + by$ to produce the coordinates of the transformed figure.

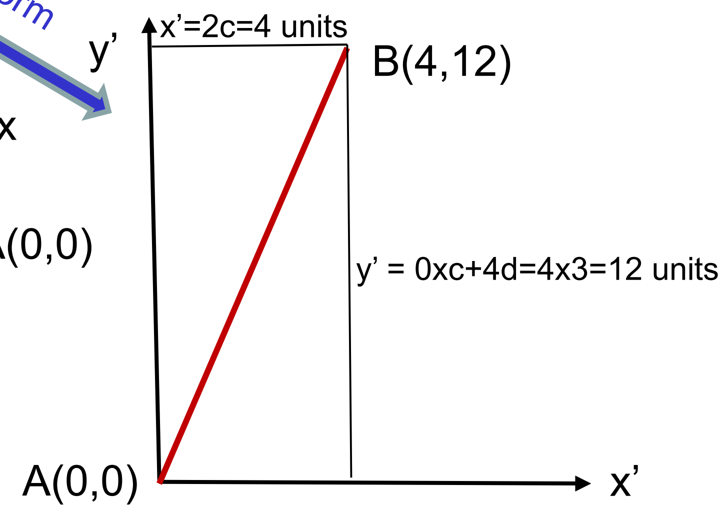
Linear transformation of a geometric figure such as a straight line can be stretched or compressed, and rotate subject to the values of coefficients a and b in the formula **ax+by** in the x-y plane as will be seen in the following example. We also recognize that some such transformations have an inverse, which undoes their effects.

Geometric Transformation - Linear transformation – Cont'd:

A simple example of linear transformation of a straight line AB from x-y plane to x'-y' plane:

This simple case involves the transformation of a straight line AB from a plane defined by x-y coordinate system to that in a x'-y' coordinate system via a <u>linear function of</u>: cx + dy in which c = 2 and d= 3 units.





Changes in length from $\sqrt{20}$ to $\sqrt{80}$ with a rotation after the transformation

The coordinates of terminal points of the line A and B in the original x-y plane are: x_a=0 and y_a=0 at Point A(0,0) and x_B=2 and y_B=4 at Point B.

The coordinates of A and B in the new transformed Plane (x'-y' plane) is obtained by the following the linear transformation function: 2x+3y. We thus obtain: x_A' = 0x2 =0 for Point A in x'-y' plane, and y_B' = 2x2+4x3=16 for Point B in the x'-y' plane.

We may also obtain the coordinates of A and B in the transformed plane by the following a matrix in the form: $\begin{Bmatrix} x' \\ y' \end{Bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$ from which we get: x_A'=0, y'_A = 0; x_B'=4, y_B'=12 units

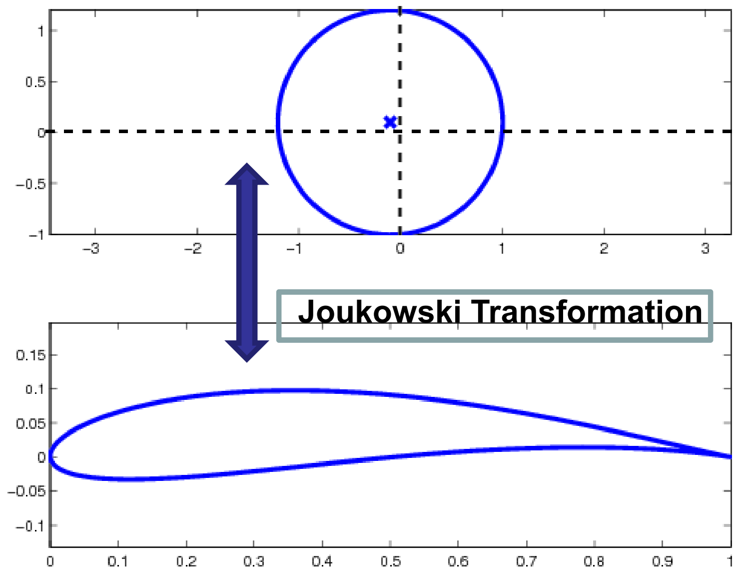
We notice that the line AB has changed the length from $\sqrt{20}$ units to $\sqrt{160}\ units$, but also changed the Angle θ from 63.43° to 71.56° – a significant amount of rotation after the transformation.

Example on Geometric Transformation - Nonlinear transformation:

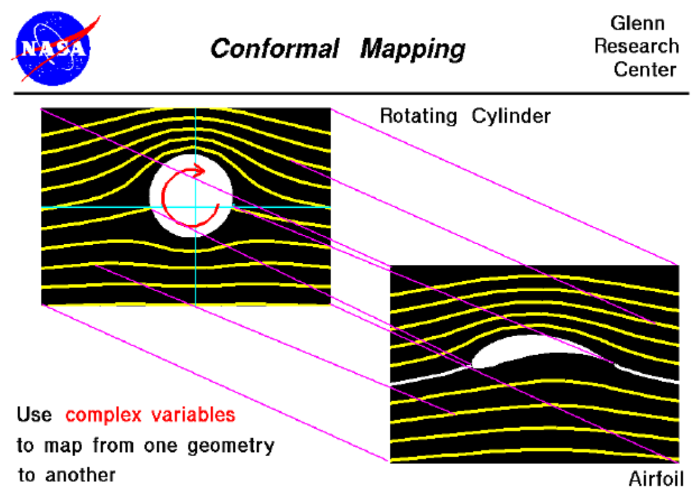
A well-known nonlinear geometric transformation is the **Joukowski's** Transformation in the design analysis of airfoils.

Joukowski was a Russian mathematician who invented this transformation. By using this technique, the fluid flow around the geometry of an airfoil can be analyzed as the flow around a rotating cylinder whose geometric symmetry simplifies the needed computations of the non-symmetric airfoil geometry.

Example of Joukowski's transformation:



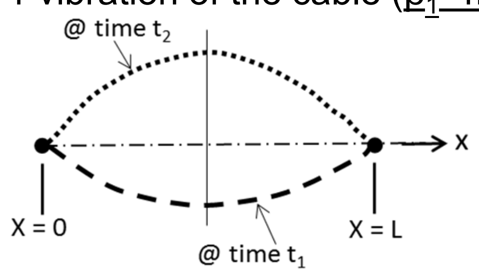
Aerodynamic Analysis of Airfoil using Joukowski's transformation:



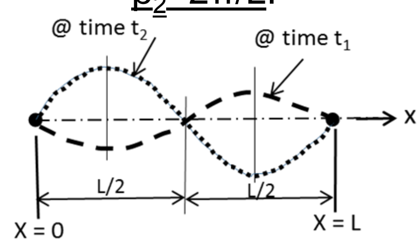
Eigenvalues and Eigenfunctions with Characteristic Functions:

"Characteristic functions" appear in "Characteristic equations" which often appear in the solutions in certain engineering analyses.
For example, determining natural frequencies in modal analyses of structures, in which natural frequencies of the structures designated by ω_n with mode number n = 1,2,3,….are important design parameters of their vibrations by applied periodic excitation forces with frequencies ω. Uncontrollable, and often devastating, vibration called "resonant vibration" of the structure can occur with the excitation force frequency ω matching any one of the natural frequencies of the structure (ω_n). The governing differential equations used to determine the natural frequencies of cable structures are homogeneous differential equations as will be presented in Chapters 9. The characteristic equation associated with the solution of the amplitude of vibration y(x) from these equations would have a form of: $\sin\beta L=0$ in which L is the length of the cable and β is the eigenvalues of the eigenfunction $\sin\beta L$. We realize the there a great many number of non-zero eigenvalues β = π/L, 2π/L, 3π/L,……, nπ/L, and each of these β-value will "characterize" the way how the cable would vibrate (we call the modes of vibration). For instance:

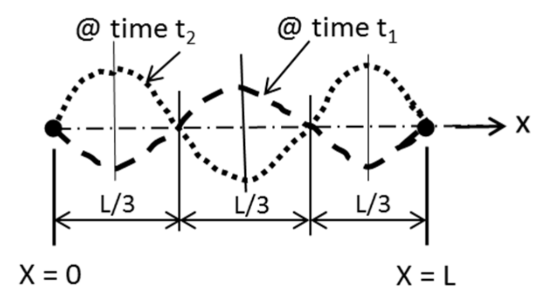
Mode 1 vibration of the cable ($\underline{\beta_1=\pi/L}$):



$\underline{\beta_2=2\pi/L}$:



$\underline{\beta_3=3\pi/L}$:



So, we can see that the eigenvalues β in eigenfunction $\sin\beta L=0$ characterize the shapes of the cable in various modes of vibration.

4.8.1 Eigenvalues and Eigenvectors of Matrices (p. 142)

We mentioned at the beginning of this section that vector quantities in Chapter 3 may be transformed in 2-D or 3-D spaces via both linear and nonlinear transformations.

Such transformations can be accomplished by linear transformation functions in such forms as: ax+by for 2-D transformations, or ax+by+cz for 3-D transformation where a,b,c are real numbers.

Because vectors involve components, transformation functions are usually in the forms of matrices.

The eigenvalues (λ) for a eigenfunction matrix that associated with the linear transformation of a vector quantity may be expressed by a vector expressed in a matrix form of:

$\{\mathbf{x}\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$ for a vector with 2 components along the x- and y-coordinates, or

$\{\mathbf{x}\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$ for a vector with 3 components along the respective x-, y- and z-coordinates

[A] = the linear transformation matrix, a square matrix with real number elements

The eigenvalues (λ) of the eigenfunction matrix [A] may be defined as:

$$[A]\{x\} = \lambda\{x\} \qquad (4.44)$$

We realize that Equation (4.43) may be expressed in another form of:

$$([A] - \lambda[I])\{x\} = \{0\} \qquad (4.45)$$

The value of λ can be determined by:

$$\det \left| [A] - \lambda[I] \right| = 0 \qquad (4.46)$$

Example 4.13 (p. 143)

Find eigenvalues and eigenvectors of the matrix: $[A] = \begin{bmatrix} -1 & 2 \\ -7 & 8 \end{bmatrix}$

Solution:

We may use Equation (4.46) to obtain the following equation:

$$\|[A]-\lambda[I]\| = \begin{vmatrix} -1-\lambda & 2 \\ -7 & 8-\lambda \end{vmatrix} = 0$$

from which we solve for the two eigenvalues: $\lambda_1 = 1 \ and \ \lambda_2 = 6$

Next, we will determine the eigenvectors corresponding to these two eigenvalues.

For eigenvalue $\lambda_1 = 1$:

We will use Equation (4.45), with which: $([A]-\lambda[I])\{x\} = \{0\}$ to determine the eigenvector {x} corresponding to this eigenvalue λ_1= 1:

$$\left(\begin{bmatrix} -1 & 2 \\ -7 & 8 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} -1-1 & 2 \\ -7 & 8-1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

leading to the following simultaneous equations:

$$-2x_1 + 2x_2 = 0$$
$$-7x_1 + 7x_2 = 0$$

Solving for x_1=x_2=p= a real number, which leads to: $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} p \\ p \end{Bmatrix} = p\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

We thus conclude that eigenvalue λ_1=1 corresponding to eigenvector: $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

Example 4.13 – Cont'd

For eigenvalue $\lambda_2 = 6$:

We will follow the same procedure for the case with eigenvalue λ_1, with the following equation in matrix form:

$$\begin{bmatrix} -1-6 & 2 \\ -7 & 8-6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

leading to the following simultaneous equations:

$$-7x_1 + 2x_2 = 0$$
$$-7x_1 + 2x_2 = 0$$

If we assume $x_2 = p$ in the above, which will lead to $x_1 = 2/7$. We will thus obtain the eigenvector to be:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} \frac{2}{7}p \\ p \end{Bmatrix} = 7p \begin{Bmatrix} 2 \\ 7 \end{Bmatrix}$$

We thus conclude that the eigenvector corresponding to eigenvalue $\lambda_2 = 6$ is: $\begin{Bmatrix} 2 \\ 7 \end{Bmatrix}$

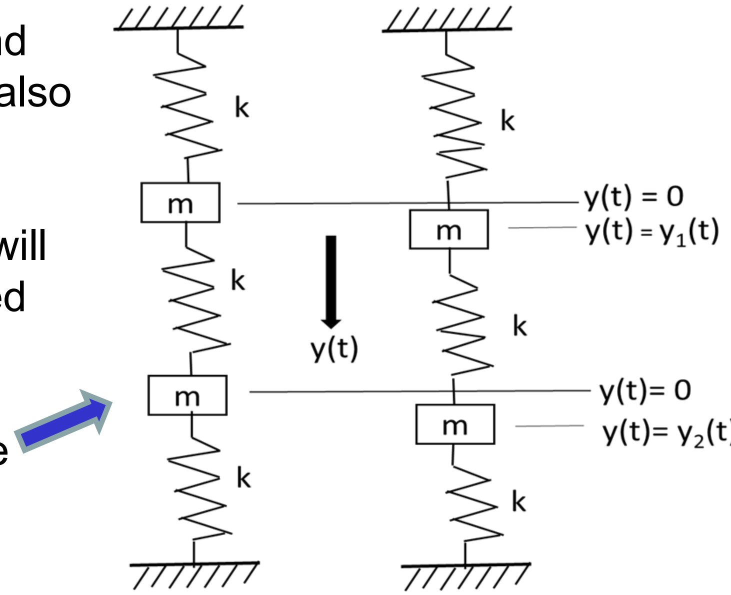
4.8.1 **Eigenvalues and Eigenvectors of Matrices-** Cont'd

Similar procedure will be followed for geometric transformations in 3-D space. In such cases, the transformation functions would be in 3x3 matrices, from which the eigenvalues and eigenfunction vectors may be determines in similar ways as illustrated in Example 4.14.

4.8.3 **Application of Eigenvalues and Eigenfunctions in Engineering Analysis** (p.146)

Engineering analysis often involve eigenvalues and eigenfunctions, as mentioned in Section 4.8, and also in the subsequent Chapters 8 and 9 of this book.

Following is one example that illustrate how they will be used in solving a complex problem that involved "coupled" simultaneous differential equations to determine the frequencies of the movements of the two masses in a system Illustrated in Figure 4.6 in the right:



We may derive the following two simultaneous differential equations for the amplitudes of both masses $y_1(t)$ and $y_2(t)$ from their initial equilibrium conditions:

$$m\frac{d^2y_1(t)}{dt^2} = -ky_1(t) + k\left[y_2(t) - y_1(t)\right] \qquad (4.47a)$$

$$m\frac{d^2y_2(t)}{dt^2} = -k\left[y_2(t) - y_1(t)\right] - ky_2(t) \qquad (4.47b)$$

We notice that the two unknowns, $y_1(t)$ and $y_2(t)$ in both Equations (4.47a) and (4.47b) This "coupling" effect makes the solution for both these quantities extremely difficult.

Fortunately, we realize that the motion of both masses m in the system follow simple harmonic motion pattern. Mathematically, this motion can be expressed in sine functions, or: $y_i(t) = Y_i\sin(\omega t)$ for i = 1,2, where Yi = the maximum amplitude of vibration of mass m, and ω is the frequency of vibration.

4.8.3 Application of Eigenvalues and Eigenfunctions in Engineering Analysis – Cont'd

Upon substitution of the relationship $y_i(t) = Y_i \sin(\omega t)$ into the simultaneous differential equations In Equations (4.47a,b), we get the following equations:

$$\left(\frac{2k}{m} - \omega^2\right)Y_1 - \frac{k}{m}Y_2 = 0 \qquad (4.49a)$$

$$-\frac{k}{m}Y_1 + \left(\frac{2k}{m} - \omega^2\right)Y_2 = 0 \qquad (4.49b)$$

We may express these equations in the following matrix form:

$$\begin{bmatrix} \left(\frac{2k}{m} - \omega^2\right) & -\frac{k}{m} \\ -\frac{k}{m} & \left(\frac{2k}{m} - \omega^2\right) \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (4.50a)$$

or in a different form:

$$\left[\begin{bmatrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (4.50b)$$

Matching Equation (4.50b) with (4.45) result in the following relations:

$$[A] = \begin{bmatrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} \end{bmatrix}, \quad \{x\} = \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix}, \text{ and } \lambda = \omega^2 \qquad (4.51)$$

We may thus obtain the frequency of the vibrating mass m to be: $\omega = \sqrt{\lambda}$

This is a speedy way to get this critical solution