

# Applied Engineering Analysis

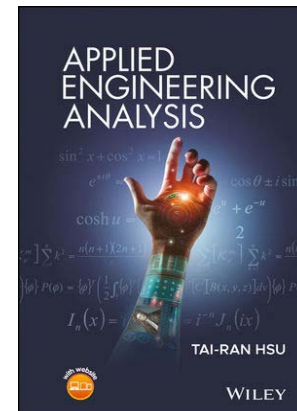
## - slides for class teaching\*

### Chapter 5

# Overview of Fourier Series

- \* Based on the book of “Applied Engineering Analysis”, by Tai-Ran Hsu, published by John Wiley & Sons, 2018 (ISBN 9781119071204)

(Chapter 5 Fourier series)  
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## **Chapter learning objectives:**

- Appreciate that the Fourier series are the mathematical form for periodic physical phenomena.
- Learn to use Fourier series to represent periodical physical phenomena in engineering analysis.
- Learn the required conditions for deriving Fourier series.
- Appreciate the principle of using Fourier series derived from the function for one period to apply the same Fourier series for other periods.
- Derive the mathematical expressions of Fourier series representing common physical phenomena.
- Understand the convergence of Fourier series of continuous periodic functions.
- Understand the convergence of Fourier series of piecewise continuous functions.
- Understand the convergence of Fourier series at discontinuities.

## Introduction of Fourier Series

The inventor: **Jean Baptiste Joseph Fourier** (1749-1829), a French mathematician.



### Major contributions to engineering analysis:

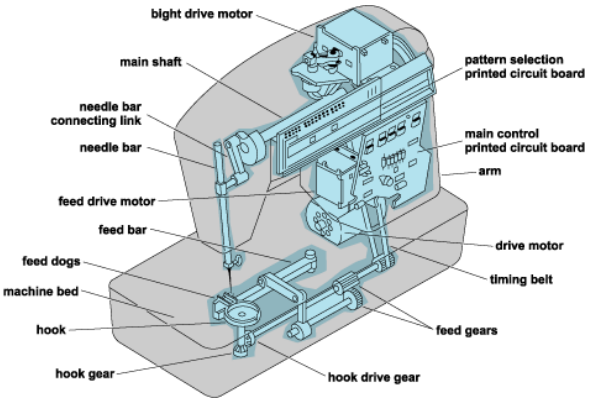
- Mathematical theory of heat conduction (Fourier law of heat conduction in Chapter 7).
- Fourier series representing periodical functions.
- Fourier transform: Similar to Laplace transform, but for transforming functions with variables in the range of  $(-\infty$  and  $+\infty)$  - a powerful tool in engineering analysis.

The function that describes a specific physical quantity by the Fourier series can be used to represent the same ***periodic*** physical quantity in the *entire spectrum* of which the variable of the function covers.

# Periodic Physical Phenomena are common in our day-to-day lives:



Motions of ponies in a go-round

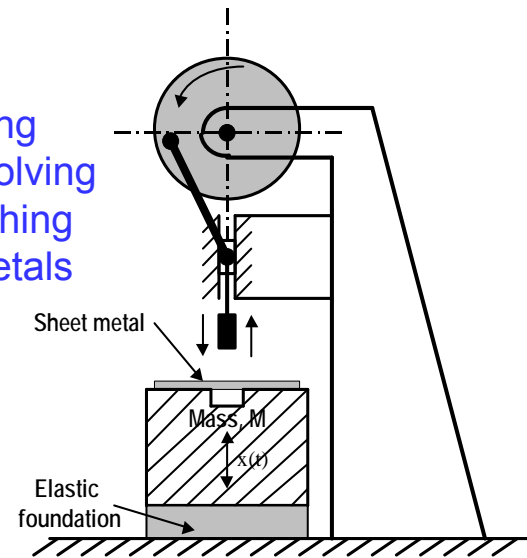


Forces on the needle in a sewing machine

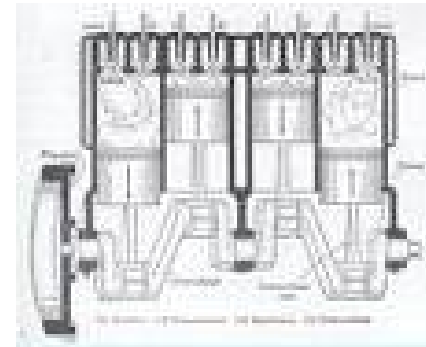
## Machines with Periodic Physical Phenomena – Cont'd



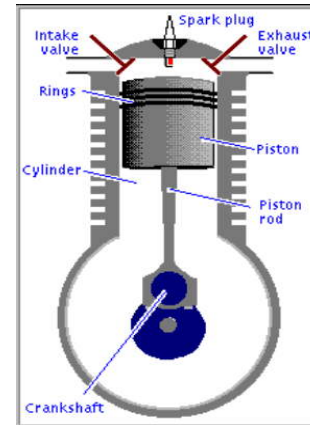
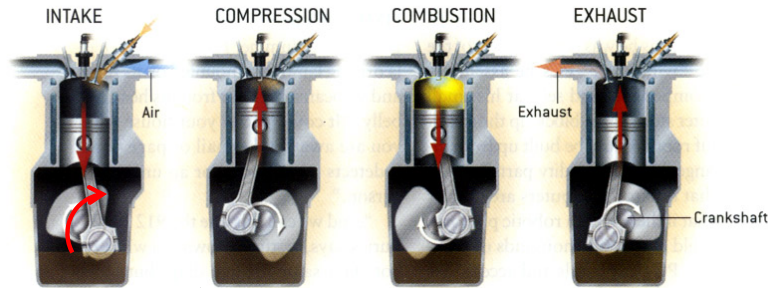
A stamping machine involving cyclic punching of sheet metals



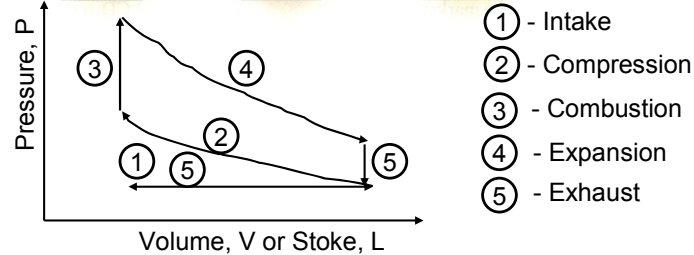
In a 4-stroke internal combustion engine:  
Cyclic gas pressures on cylinders,  
and forces on connecting rod and crank shaft



## The periodic variation of gas pressure in the cylinder head of a 4-stroke internal combustion engine:

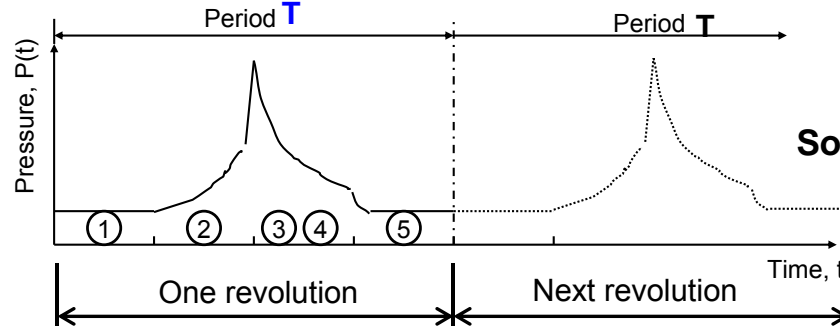


The P-V Diagram  
 $P$  = gas pressure  
in cylinders



- ① - Intake
- ② - Compression
- ③ - Combustion
- ④ - Expansion
- ⑤ - Exhaust

But the stroke  $l$  varies with time of the rotating crank shaft, so the time-varying gas pressure is illustrated as:



So,  $P(t)$  is a periodic function with period  $T$

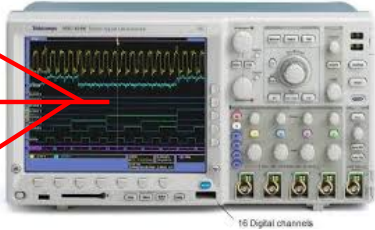
**Mathematical expressions for different kinds of periodical signals from an oscilloscope:**



Continuous function

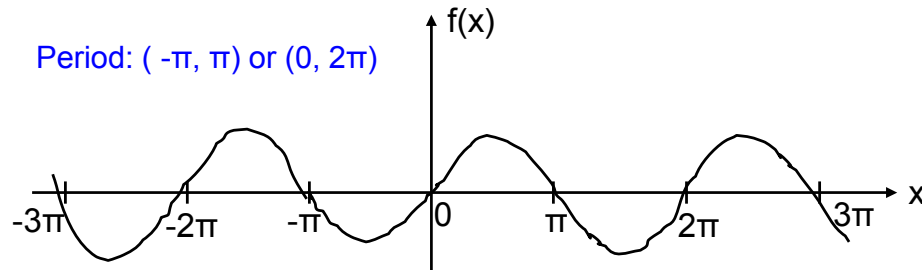


Piece-wise continuous function)

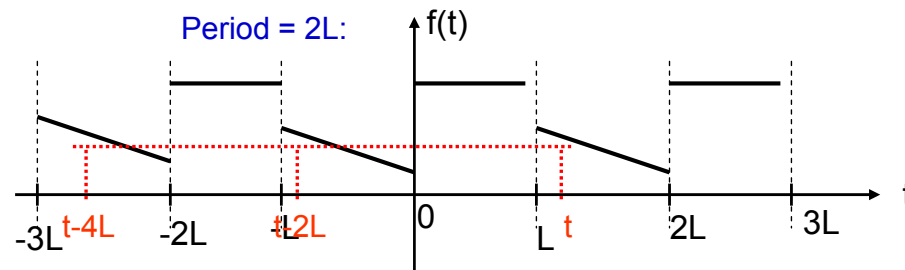


## FOURIER SERIES – The mathematical representation of periodic physical phenomena

- Mathematical expression for periodic functions:
  - If  $f(x)$  is a periodic function with variable  $x$  in **ONE period  $2L$**
  - Then  $f(x) = f(x \pm 2L) = f(x \pm 4L) = f(x \pm 6L) = f(x \pm 8L) = \dots = f(x \pm 2nL)$  where  $n = \text{any integer number}$



(a) Periodic function with period  $(-\pi, \pi)$



(b) Periodic function with period  $(-L, L)$



## Mathematical Expressions of Fourier Series

- Required conditions for deriving Fourier series:
  - The mathematical expression of the periodic function  $f(x)$  in one period must be available
  - The function in one period is defined in an interval  $(c < x < c+2L)$  in which  $c = 0$  or any arbitrarily chosen value of  $x$ , and  $L =$  half period
  - The function  $f(x)$  and its first order derivative  $f'(x)$  are either continuous or piece-wise continuous in  $c < x < c+2L$
- The mathematical expression of Fourier series for periodic function  $f(x)$  is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) = f(x \pm 2L) = f(x \pm 4L) = \dots \quad (5.1)$$

where  $a_0$ ,  $a_n$  and  $b_n$  are **Fourier coefficients**, to be determined by the following integrals:

$$a_n = \frac{1}{L} \int_c^{c+2L} f(x) \cos \frac{n\pi x}{L} dx \quad n = 0, 1, 2, 3, \dots \quad (5.2a)$$

$$b_n = \frac{1}{L} \int_c^{c+2L} f(x) \sin \frac{n\pi x}{L} dx \quad n = 1, 2, 3, \dots \quad (5.2b)$$

Occasionally the coefficient  $a_0$ , as a special case of  $a_n$  with  $n = 0$  in Equation (5.2a) needs to be determined separately by the following integral:

$$a_0 = \frac{1}{L} \int_c^{c+2L} f(x) dx \quad (5.2c)$$

**Example 5.1** (p.155)

Derive a Fourier series for a periodic function with period  $(-\pi, \pi)$ :

We realize that the period of this function  $2L = \pi - (-\pi) = 2\pi$ . The half period is  $L = \pi$ .  
If we choose  $c = -\pi$ , we will have  $c+2L = -\pi + 2\pi = \pi$

Thus, by using Equations (5.1) and (5.2), we will have:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L = \pi} + b_n \sin \frac{n\pi x}{L = \pi} \right)$$

and 
$$a_n = \frac{1}{L = \pi} \int_{c=-\pi}^{c+2L=-\pi+2\pi} f(x) \cos \frac{n\pi x}{L = \pi} dx$$

$$b_n = \frac{1}{L = \pi} \int_{c=-\pi}^{c+2L=-\pi+2\pi} f(x) \sin \frac{n\pi x}{L = \pi} dx$$

Hence, the Fourier series is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \tag{5.3}$$

with 
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \quad n = 0, 1, 2, 3, \dots \tag{5.4a}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \quad n = 1, 2, 3, \dots \tag{5.4b}$$

We notice the period  $(-\pi, \pi)$  might not be practical, but it appears to be common in many applied math textbooks. Here, we treat it as a special case of Fourier series.

**Example 5.2 (p.155)**

Derive a Fourier series for a periodic function  $f(x)$  with a period  $(-\ell, \ell)$

Let us choose  $c = -\ell$ , and the period  $2L = \ell - (-\ell) = 2\ell$ , and the half period  $L = \ell$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L = \ell} + b_n \sin \frac{n\pi x}{L = \ell} \right)$$

$$a_n = \frac{1}{L = \ell} \int_{c=-\ell}^{c+2L=-\ell+2\ell} f(x) \cos \frac{n\pi x}{L = \ell} dx$$

$$b_n = \frac{1}{L = \ell} \int_{c=-\ell}^{c+2L=-\ell+2\ell} f(x) \sin \frac{n\pi x}{L = \ell} dx$$

Hence the Fourier series of the periodic function  $f(x)$  becomes:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right) \tag{5.5}$$

with

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell} dx \quad n = 0, 1, 2, 3, \dots \tag{5.6a}$$

$$b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx \quad n = 1, 2, 3, \dots \tag{5.6b}$$

**Example 5.3** (p.155)

Derive a Fourier series for a periodic function  $f(x)$  with a period  $(0, 2L)$ .

As in the previous examples, we choose  $c = 0$ , and the half-period to be  $L$ . We will have the Fourier series in the following form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L=L} + b_n \sin \frac{n\pi x}{L=L} \right)$$

$$a_n = \frac{1}{L=L} \int_{c=0}^{c+2L=-0+2L} f(x) \cos \frac{n\pi x}{L=L} dx$$

$$b_n = \frac{1}{L=L} \int_{c=0}^{c+2L=0+2L} f(x) \sin \frac{n\pi x}{L=L} dx$$

The corresponding Fourier series thus has the following form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \quad (5.7)$$

$$a_0 = \frac{1}{L} \int_0^{2L} f(x) dx \quad (5.8a)$$

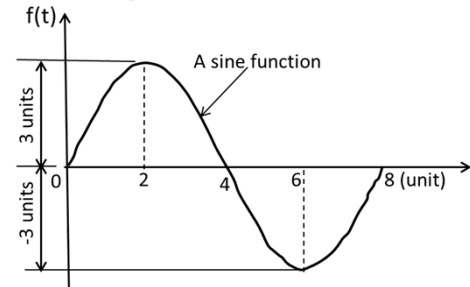
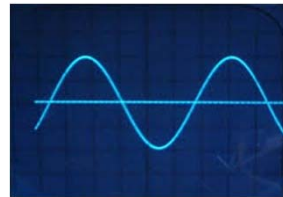
$$a_n = \frac{1}{L} \int_0^{2L} f(x) \cos \frac{n\pi x}{L} dx \quad n = 1, 2, 3, \dots \quad (5.8b)$$

$$b_n = \frac{1}{L} \int_0^{2L} f(x) \sin \frac{n\pi x}{L} dx \quad n = 1, 2, 3, \dots \quad (5.8c)$$

Periodic functions with periods  $(0, 2L)$  are more realistic. Equations (5.7) and (5.8) are thus more practical in engineering analysis.

**Example 5.4** (p.156)

Find the Fourier series for the sinusoidal signals from an oscilloscope shown in the right of the figure below. Magnitudes of this sinusoidal signals are shown in the figure below:



Solution:

We will express the period of the sine function in the right of the above figure by letting the period  $2L = 8$  units ( $L = 4$ ), and the corresponding sine function in one period thus has the

form: 
$$f(t) = 3 \sin \frac{\pi}{4} t \quad \text{for } 0 \leq t \leq 8 \quad (\text{a})$$

The Fourier series for the sine function in Equation (a) can be expressed according to Equation (5.7) to be: 
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi t}{4} + b_n \sin \frac{n\pi t}{4} \right)$$

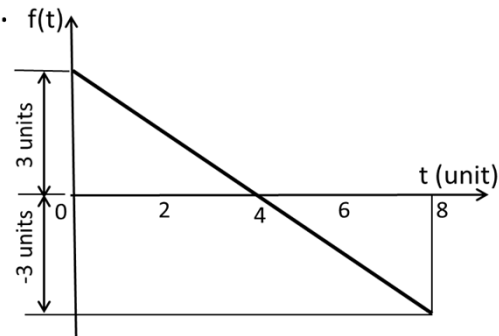
with the coefficients: 
$$a_0 = \frac{1}{4} \int_0^8 f(t) dt = \frac{1}{4} \int_0^8 \left( 3 \sin \frac{\pi}{4} t \right) dt = 0$$

$$a_n = \frac{1}{4} \int_0^8 \left( 3 \sin \frac{\pi t}{4} \right) \left( \cos \frac{n\pi t}{4} \right) dt = 0 \quad b_n = \frac{1}{4} \int_0^8 \left( 3 \sin \frac{\pi t}{4} \right) \left( \sin \frac{n\pi t}{4} \right) dt = \frac{3n}{\pi(1-n^2)} \quad \text{for } n \neq 1$$

$$b_1 = \frac{1}{4} \int_0^8 \left( 3 \sin \frac{\pi t}{4} \right) \left( \sin \frac{\pi t}{4} \right) dt = 3 \quad \text{Leads to the Fourier Series in the form: } f(t) = 3 + \frac{3}{\pi} \sum_{n=2}^{\infty} \left( \frac{n}{1-n^2} \right) \sin \frac{n\pi}{4} t$$

**Example 5.6** (p.158)

Find the Fourier series for the periodic piece-wise continuous linear signals shown on the screen of an oscilloscope shown in the figure in the left below. The figure in the right shows the numerical scale in one period.



Solution:

As seen from the diagram in the right of the figure, the period of the function  $f(t)$  remains to be  $2L = 8$  units, which gives  $L = 4$  units. The function  $f(t)$  for one period is:

$$f(t) = -\frac{3}{4}t + 3 \quad \text{for } 0 < t < 8$$

The Fourier series for the periodic function  $f(t)$  may be expressed in the following expression:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi t}{4} + b_n \sin \frac{n\pi t}{4} \right)$$

with the coefficients:

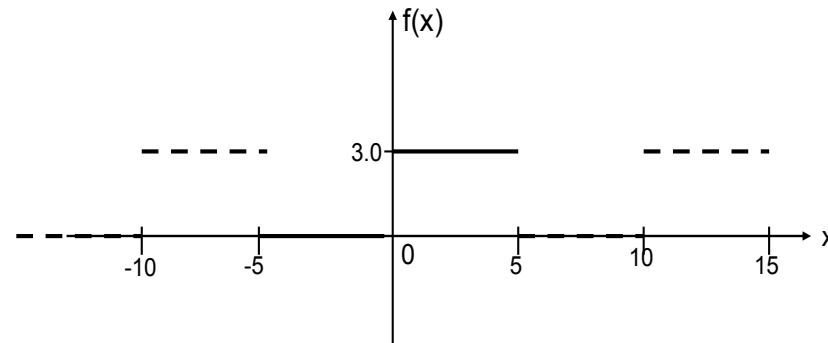
$$a_0 = \frac{1}{4} \int_0^8 \left( -\frac{3}{4}t + 3 \right) dt = 0 \quad a_n = \frac{1}{4} \int_0^8 \left( -\frac{3}{4}t + 3 \right) \cos \frac{n\pi t}{4} dt = -\frac{6}{n^2 \pi^2} \quad b_n = \frac{1}{4} \int_0^8 \left( -\frac{3}{4}t + 3 \right) \sin \frac{n\pi t}{4} dt = \frac{9}{n \pi}$$

The corresponding Fourier Series thus has the form:

$$f(t) = \sum_{n=1}^{\infty} \left[ \left( -\frac{6}{n^2 \pi^2} \right) \cos \frac{n\pi t}{4} + \frac{9}{n \pi} \sin \frac{n\pi t}{4} \right]$$

**Example 5.8** (p.160)

Here, we will offer another example of finding the Fourier series to represented by a piece-wise continuous function whining one period, as illustrated in the figure below:



Solution: We have the period,  $2L = 5 - (-5) = 10$ , and the half-period  $L = 5$ . We may choose the starting point  $c = -5$ .

The corresponding Fourier series in this case fits in that shown in Equations (5.5) and (5.6a,b) to be:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{5} + b_n \sin \frac{n\pi x}{5} \right)$$

Equation (5.8b) and (5.8c) are used to evaluate the following coefficients:

$$a_n = \frac{1}{5} \int_{-5}^5 f(x) \cos \frac{n\pi x}{5} dx = \frac{1}{5} \left[ \int_{-5}^0 (0) \cos \frac{n\pi x}{5} dx + \int_0^5 (3) \cos \frac{n\pi x}{5} dx \right] = \frac{3}{5} \int_0^5 \cos \frac{n\pi x}{5} dx$$

$$a_n = \frac{3}{5} \left( \frac{5}{n\pi} \sin \frac{n\pi x}{5} \right) \Big|_0^5 = 0 \quad \text{if } n \neq 0 \quad \text{from which we get: } a_0 = \frac{3}{5} \int_0^5 \cos \frac{(0)\pi x}{5} dx = \frac{3}{5} \int_0^5 dx = 3$$

**Example 5.8 – Con'd**

$$b_n = \frac{1}{5} \int_{-5}^5 f(x) \sin \frac{n\pi x}{5} dx = \frac{1}{5} \left[ \int_{-5}^0 (0) \sin \frac{n\pi x}{5} dx + \int_0^5 (3) \sin \frac{n\pi x}{5} dx \right] = \frac{3}{5} \int_0^5 \sin \frac{n\pi x}{5} dx$$

After integration: 
$$b_n = \frac{3}{5} \left( -\frac{5}{n\pi} \cos \frac{n\pi x}{5} \right) \Big|_0^5 = \frac{3(1 - \cos n\pi)}{n\pi} = \frac{3[1 - (-1)^n]}{n\pi}$$

The Fourier series for the function for this piece-wise continuous function in one period thus has the form:

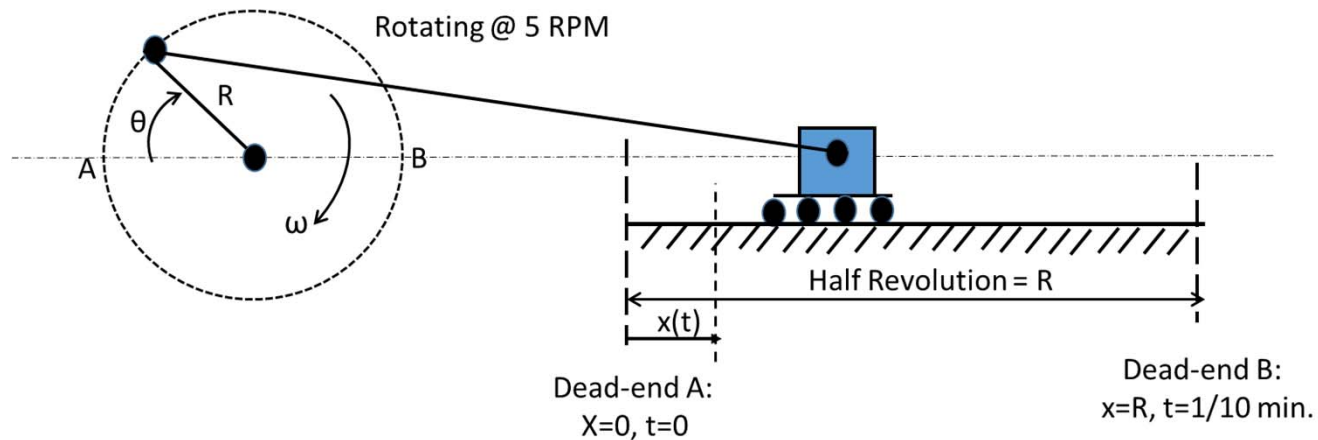
$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{3(1 - \cos n\pi)}{n\pi} \sin \frac{n\pi x}{5} = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{3[1 - (-1)^n]}{n\pi} \sin \frac{n\pi x}{5}$$



**Special Example on deriving the Fourier series in describing the motion of the “slider” in a “crank-slider” mechanism – a common mechanical engineering design problem:**

Derive a function describing the position of the sliding block M in one period in a slide mechanism as illustrated below. If the crank rotates at a constant velocity of 5 rpm.

- (a) Illustrate the periodic function in three periods, and
- (b) Derive the appropriate Fourier series describing the position of the sliding block  $x(t)$  in which  $t$  is the time in minutes



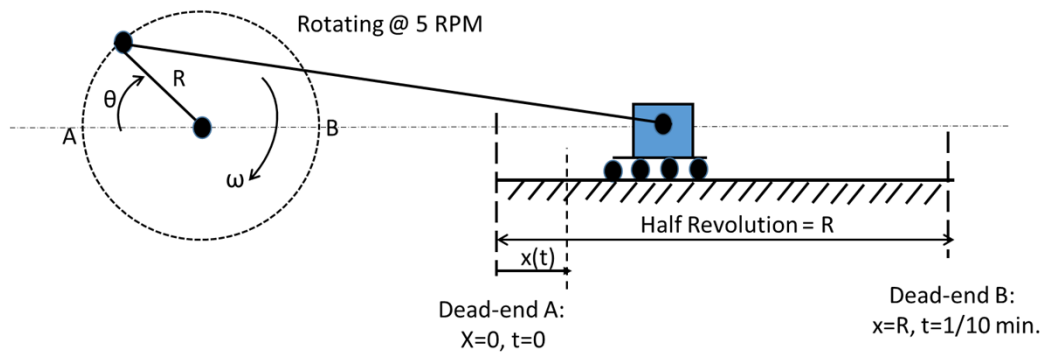
**Solution:**

(a) Graphic Illustration on this periodic physical phenomenon in three periods:

Determine the angular displacement of the crank R:

We realize the relationship: rpm  $N = \omega/(2\pi)$ , and  $\theta = \omega t$ , where  $\omega$  = angular velocity, and  $\theta$  = angular displacement relating to the position of the sliding block

For  $N = 5$  rpm, we have:  $\frac{\theta}{2\pi} = \frac{t}{\frac{1}{5}} = 5t$  Based on one revolution ( $\theta=2\pi$ ) corresponds to 1/5 min. We thus have  $\theta = 10\pi t$



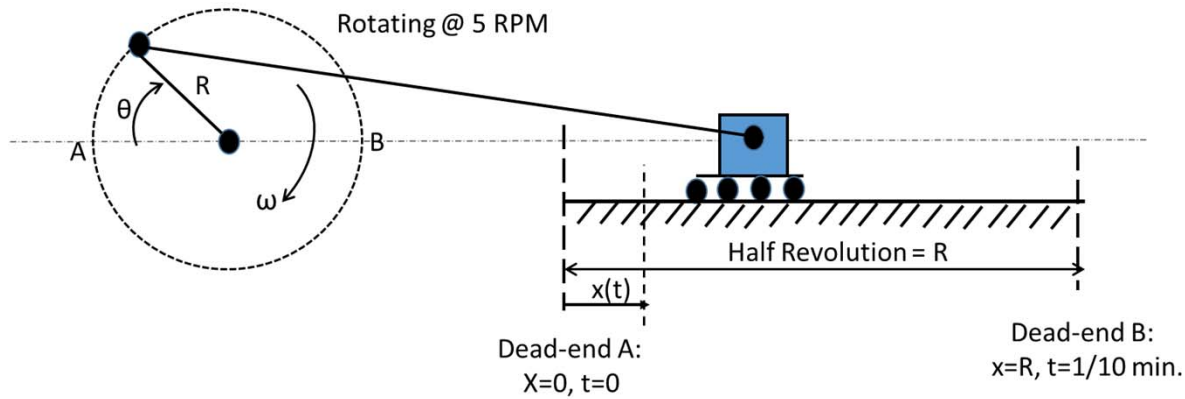
Position of the sliding block along the x-direction can be determined by:

$$x = R - R\cos\theta$$

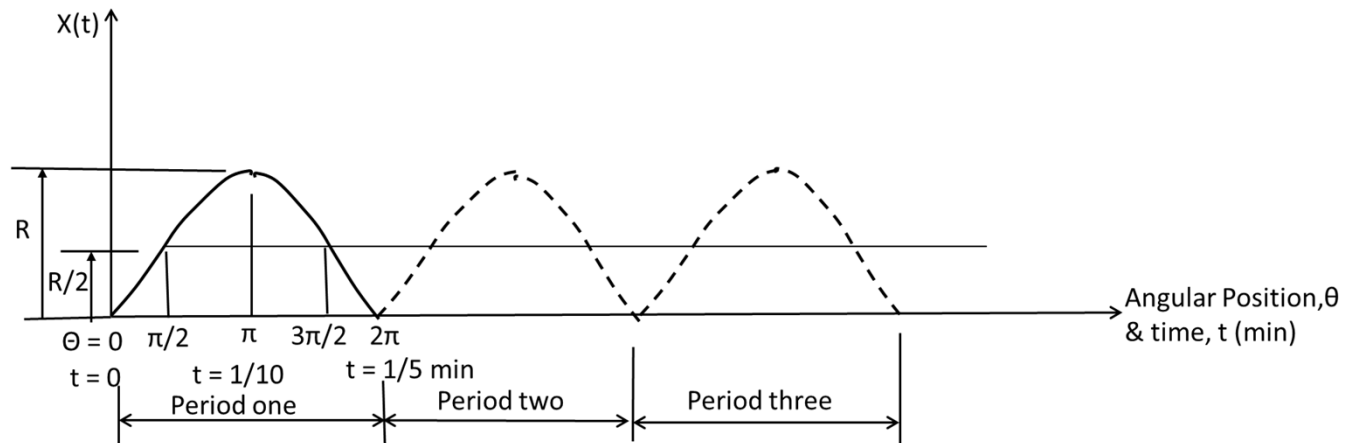
or  $x(t) = R - R\cos(10\pi t) = R[1 - \cos(10\pi t)] \quad 0 < t < 1/5 \text{ min}$

We have now derived the periodic function describing the instantaneous position of the sliding block as:

$$x(t) = R[1 - \cos(10\pi t)] \quad (a)$$



Graphical representation of function in Equation (a) can be produced as:



(b) Formulation of Fourier Series:

We have the periodic function:  $x(t) = R[1 - \text{Cos}(10\pi t)]$  with a period:  $0 < t < 1/5$  min

If we choose  $c = 0$  and period  $2L = 1/5$ , we will have the Fourier series expressed in the following form by using Equations (6.7) and (6.8):

$$\begin{aligned} x(t) &= \frac{a_o}{2} + \sum_{n=1}^{\infty} \left[ a_n \text{Cos} \frac{n\pi t}{L=1/10} + b_n \text{Sin} \frac{n\pi t}{L=1/10} \right] \\ &= \frac{a_o}{2} + \sum_{n=1}^{\infty} [a_n \text{Cos} 10n\pi t + b_n \text{Sin} 10n\pi t] \end{aligned} \quad (b)$$

with 
$$a_n = \frac{1}{1/10} \int_0^{1/5} x(t) \text{Cos} 10n\pi t dt = -\frac{R}{2\pi} \left[ \frac{\text{Sin} 2(1-n)\pi}{1-n} + \frac{\text{Sin} 2(1+n)\pi}{1+n} \right] \quad (c)$$

We may obtain coefficient  $a_o$  from Equation (c) to be  $a_o = 0$ :

The other coefficient  $b_n$  can be obtained by:

$$\begin{aligned} b_n &= 10 \int_0^{1/5} x(t) \text{Sin} 10n\pi t dt = 10 \int_0^{1/5} R(1 - \text{Cos} 10\pi t) \text{Sin} 10n\pi t dt \\ &= \frac{R}{2(n-1)\pi} [\text{Cos} 2(n-1)\pi - 1] + \frac{R}{2(n+1)\pi} [\text{Cos} 2(n+1)\pi - 1] \end{aligned} \quad (d)$$

## 5.4 Convergence of Fourier Series (p.161)

We have learned, with amazement, how periodic functions with a given form in ONE period can be expressed by an **infinite series** called the **Fourier series** such as shown in Equation (5.1) (p.154). In theory, it requires to include INFINITE number of terms in the series i.e.  $n=1,2,3,4,5,6,\dots$  in Equation (5.1) to get such correlation, which obviously is neither REALISTIC nor PRACVTICAL in any engineering analysis.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) = f(x \pm 2L) = f(x \pm 4L) = \dots \quad (5.1)$$

The question then is “How **MANY**” terms in the infinite series one needs to include in Equation (5.1) in order to reach an accurate representation of the required periodic function [e.g.,  $f(x)$  in one period]?”

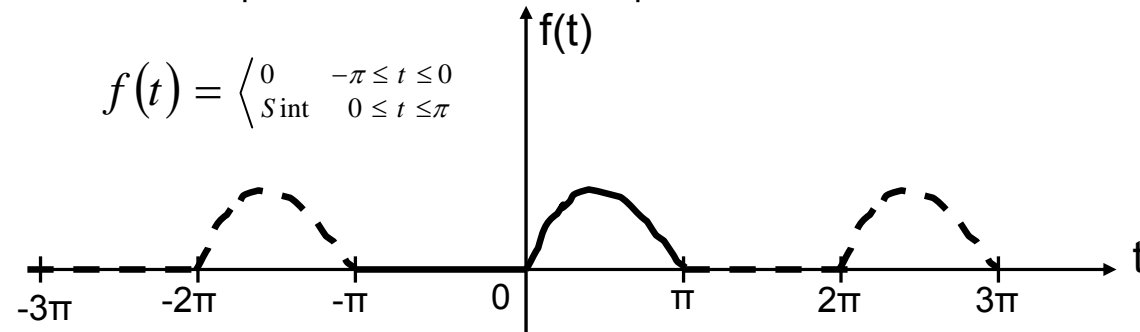
The following case study will give us some idea in the answering the above question, that is: “How many terms in the Fourier series in Equation (5.1) will give users reasonably accurate results.” This issue is called the “**Convergence of Fourier series**.”

This case involves the derivation of a Fourier series for the following periodic function in one period, and find how many terms in the series that one will need to include in order to get accurate representation of the function in the specified periods:

$$f(t) = \begin{cases} 0 & -\pi \leq t \leq 0 \\ \sin t & 0 \leq t \leq \pi \end{cases}$$

## 5.4 Convergence of Fourier Series - Cont'd

The following is a graphic illustration of the periodic function in three periods:



We identified the period to be:  $2L = \pi - (-\pi) = 2\pi$ , and according to Equation (5.3), we have:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nt) + b_n \sin(nt)] \quad (\text{a})$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt = \frac{1}{\pi} \int_{-\pi}^0 (0) \cos nt dt + \frac{1}{\pi} \int_0^{\pi} \text{Sint} \cos nt dt = \frac{1 + \cos n\pi}{(1-n^2)\pi} \quad \text{for } n \neq 1 \quad (\text{b})$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt = \frac{1}{\pi} \int_{-\pi}^0 (0) \sin nt dt + \frac{1}{\pi} \int_0^{\pi} \text{Sint} \sin nt dt \quad n = 1, 2, 3, \dots \quad (\text{c})$$

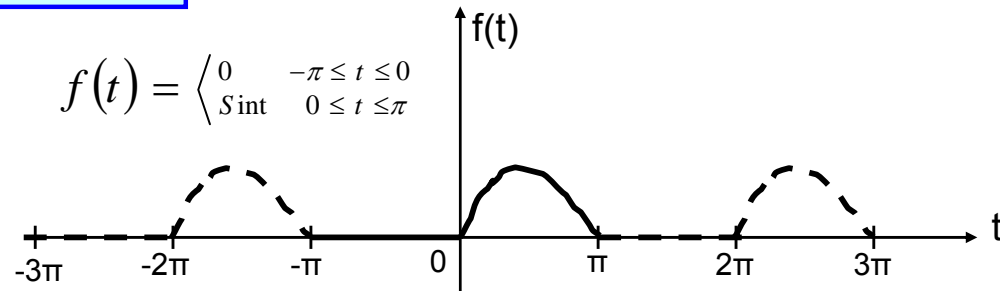
or

$$b_n = \frac{1}{\pi} \left\{ \frac{1}{2} \left[ \frac{\sin(1-n)t}{1-n} - \frac{\sin(1+n)t}{1+n} \right] \right\} \Bigg|_0^{\pi} = 0 \quad \text{for } n \neq 1$$

For the case  $n = 1$ , the two coefficients  $a_1$  and  $b_1$  become:

$$a_1 = \frac{1}{\pi} \int_0^{\pi} \text{Sint} \text{Cost} dt = \frac{\text{Sin}^2 t}{2\pi} \Bigg|_0^{\pi} = 0 \quad \text{and} \quad b_1 = \frac{1}{\pi} \int_0^{\pi} \text{Sint} \text{Sint} dt = \frac{1}{2} \quad (\text{d}), (\text{e})$$

## Convergence of Fourier Series-Cont'd



The Fourier series for the periodical function  $f(t)$  with the coefficients become:

$$f(t) = \frac{1}{\pi} + \frac{\text{Sint}}{2} + \sum_{n=2}^{\infty} (a_n \text{Cos } nt + b_n \text{Sin } nt) \quad (5.10)$$

The Fourier series in Equation (b) can be expanded into the following infinite series:

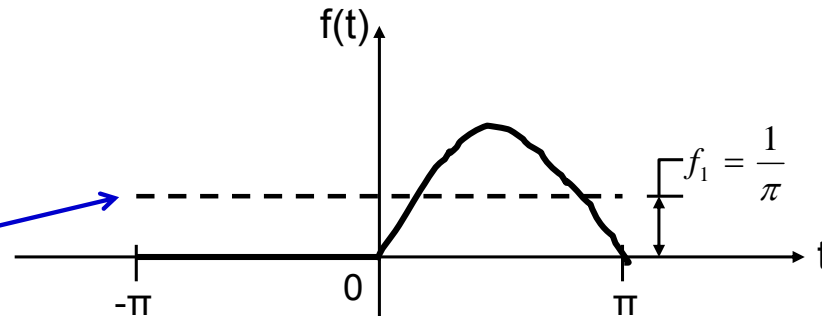
$$f(t) = \frac{1}{\pi} + \frac{\text{Sint}}{2} - \frac{2}{\pi} \left( \frac{\text{Cos } 2t}{3} + \frac{\text{Cos } 4t}{15} + \frac{\text{Cos } 6t}{35} + \frac{\text{Cos } 8t}{63} + \dots \right) \quad (5.11)$$

Let us now examine what the function would look like by including different number of terms in expression (5.11):

Case 1: Include only one term in Equation (5.11):

$$f(x) = f_1 = \frac{1}{\pi} \quad \longrightarrow$$

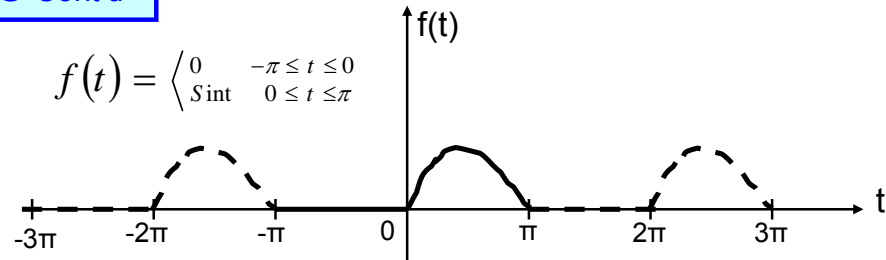
Graphically it will look like:



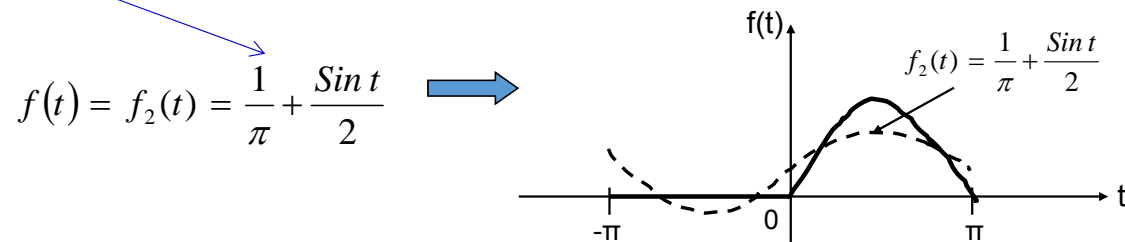
**Observation:** the function with one term does Not even closely resemble the original shape of the function!!

- We thus conclude that the Fourier series in Equation (5.10) with one term does not converge to the function!

## Convergence of Fourier Series-Cont'd

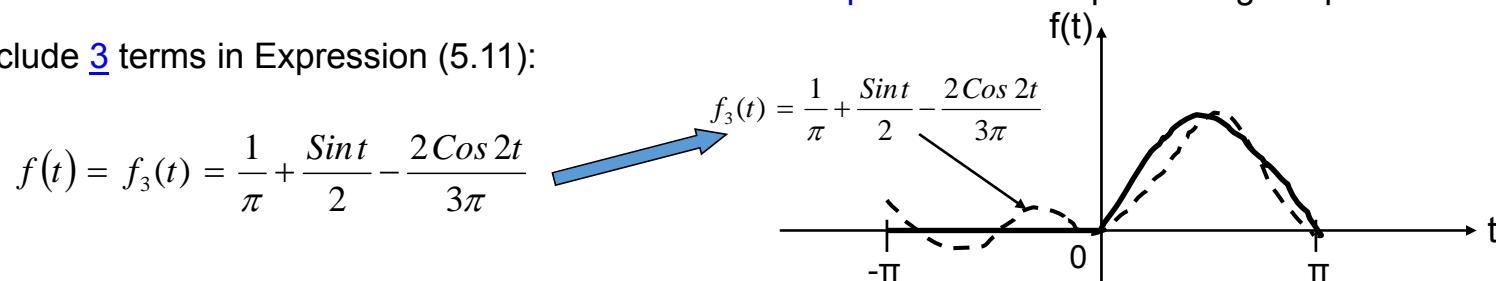


Case 2: Include 2 terms in Expression (5.11):  $f(t) = \frac{1}{\pi} + \frac{\text{Sint}}{2} - \frac{2}{\pi} \left( \frac{\text{Cos } 2t}{3} + \frac{\text{Cos } 4t}{15} + \frac{\text{Cos } 6t}{35} + \frac{\text{Cos } 8t}{63} + \dots \right)$



**Observation:** A Fourier series with 2 terms has shown some improvement in representing the periodic function.

Case 3: Include 3 terms in Expression (5.11):



**Observation:** A Fourier series with 3 terms represent the function much better than the two previous cases with 1 and 2 terms in the infinite series.

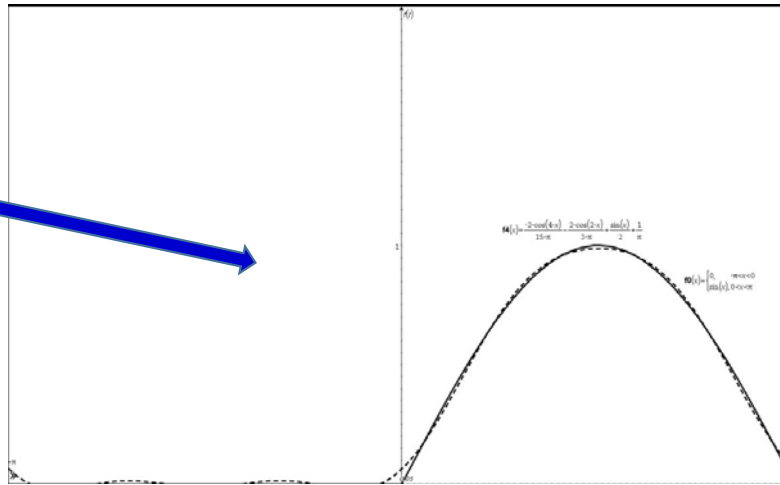


## Convergence of Fourier Series-Cont'd

**Case 4: Use four terms in Equation (5.11):**

**Solid lines** = graphic display of the periodic function in one period.

**Dotted lines** = graphic display of the Fourier series with 4 terms in Equation (5.11):



**Conclusion:** Fourier series **converges better** to the periodic function that it represents with **more** terms included in the series.

**Practical consideration:** It is not realistic to include infinite number of terms in the Fourier series for complete convergence. Normally an approach with 20 terms would be sufficiently accurate in representing most periodic functions

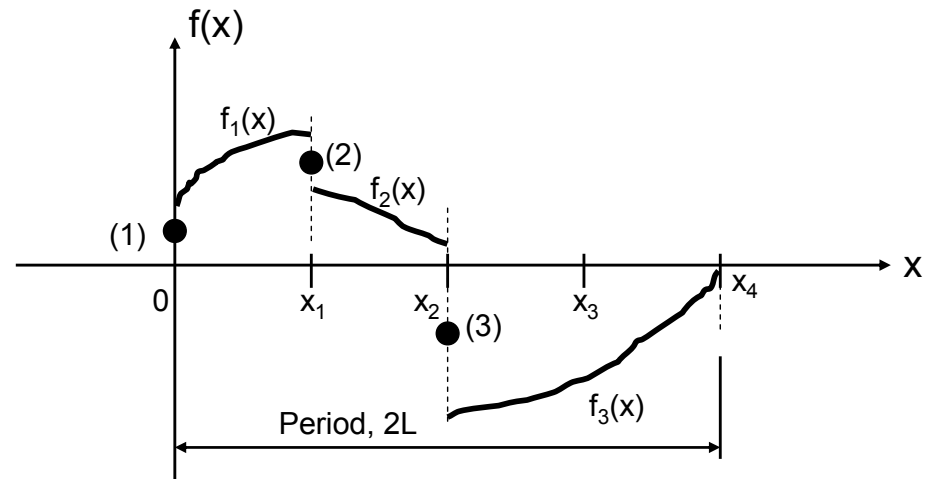
## 5.5 Convergence of Fourier Series at Discontinuities (p.164)

Fourier series in Equations (5.1) and (5.2a,b,c) converges to periodic functions **everywhere except** at **discontinuities** of piece-wise continuous function such as illustrated below:

A piece-wise discontinuous function in one period:

$$f(x) = \begin{cases} = f_1(x) & 0 < x < x_1 \\ = f_2(x) & x_1 < x < x_2 \\ = f_3(x) & x_2 < x < x_4 \end{cases}$$

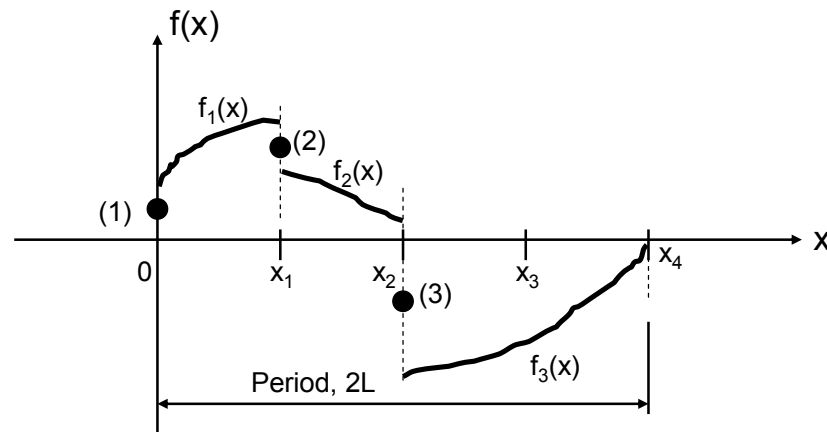
The periodic function  $f(x)$  has discontinuities at:  $x_0, x_1, x_2$  and  $x_4$



The Fourier series for this piece-wise continuous periodic function will **NEVER** converge at these discontinuous points even with inclusion of infinite number ( $\infty$ ) number of terms in the series

- The Fourier series in Equations (5.1) and (5.2a,b,c) will converge every where to the values of the function **except** at those **discontinuities** as described above, at which the series will converge **HALF-WAY** of the **function values** at these **discontinuities** as shown in **closed circled dots** in the above graph.

## Convergence of Fourier Series at Discontinuities of Piece-wise Continuous Functions in One Period



**Mathematical expressions on the convergence of Fourier series at HALF-WAY points:**

$$f(0) = \frac{1}{2} f_1(0) \quad \text{at Point (1)}$$

$$f(x_1) = \frac{1}{2} [f_1(x_1) + f_2(x_1)] \quad \text{at Point (2)}$$

$$f(x_2) = \frac{1}{2} [f_2(x_2) + f_3(x_2)] \quad \text{at Point (3)}$$

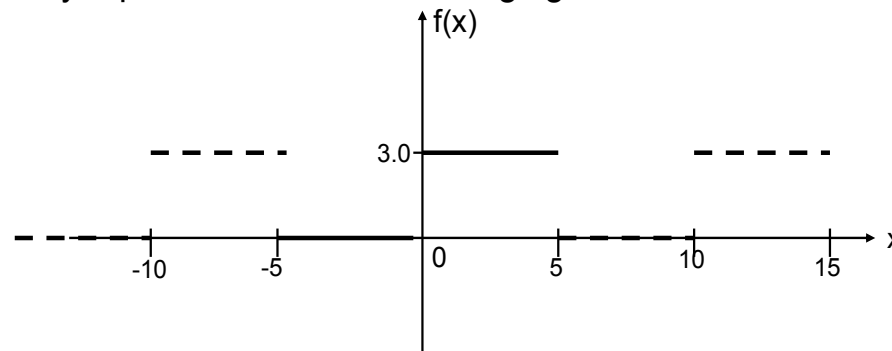
$$f(x_4) = f_3(x_4) = \frac{1}{2} f_1(0) \quad \text{same value as Point (1)}$$

**Example 5.9** (p.165)

Determine the value of Fourier series representing the periodic function in Example 5.8 at  $x = 0, 5, 10, 12.5, -5, -7.5$  and  $-10$ .

Solution:

The function in this example is graphically represented in the following figure:



(1) We will notice the function  $f(x)$  is a piece-wise continuous function in the above figure:

The piece-wise continuous periodic function  $f(x)$  in one period is defined as:  $f(x) = 0$  in the sub-period  $(-5 < x < 0)$ , and  $f(x) = 3$  in the other portion of the period  $(0 < x < 5)$ . We have the period,  $2L = 5 - (-5) = 10$ , and  $L = 5$ . We will choose the starting point,  $c = -5$ .

The corresponding Fourier series in this case fits in that shown in Example 5.8 on p.160, as follows:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{5} + b_n \sin \frac{n\pi x}{5} \right)$$

### Example 5.9 –Cont'd

With the coefficients in the above Fourier series to be:

$$a_n = \frac{1}{5} \int_{-5}^5 f(x) \cos \frac{n\pi x}{5} dx = \frac{1}{5} \left[ \int_{-5}^0 (0) \cos \frac{n\pi x}{5} dx + \int_0^5 (3) \cos \frac{n\pi x}{5} dx \right] = \frac{3}{5} \int_0^5 \cos \frac{n\pi x}{5} dx$$

$$\text{with } a_n = \frac{3}{5} \left( \frac{5}{n\pi} \sin \frac{n\pi x}{5} \right) \Big|_0^5 = 0 \quad \text{if } n \neq 0$$

For the case with  $n = 0$ , we may get:

$$a_0 = \frac{3}{5} \int_0^5 \cos \frac{(0)\pi x}{5} dx = \frac{3}{5} \int_0^5 dx = 3$$

The other coefficients are obtained as:

$$b_n = \frac{3}{5} \left( -\frac{5}{n\pi} \cos \frac{n\pi x}{5} \right) \Big|_0^5 = \frac{3(1 - \cos n\pi)}{n\pi} = \frac{3[1 - (-1)^n]}{n\pi}$$

The resultant Fourier series thus takes the form:

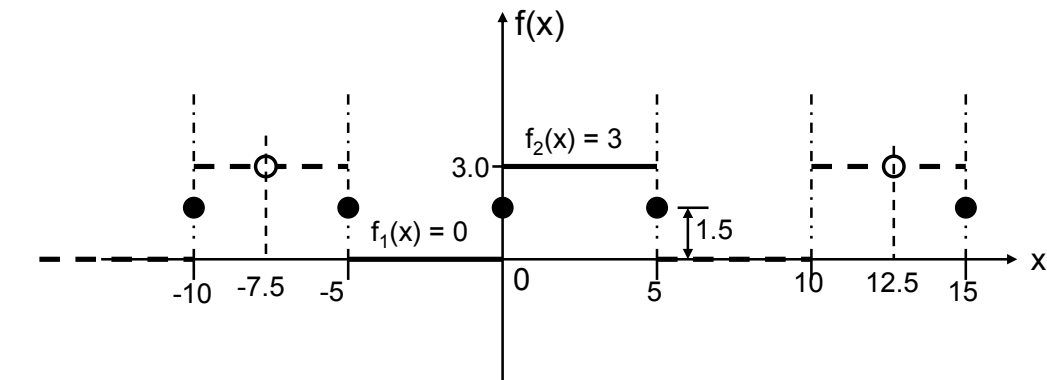
$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{3(1 - \cos n\pi)}{n\pi} \sin \frac{n\pi x}{5} = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{3[1 - (-1)^n]}{n\pi} \sin \frac{n\pi x}{5}$$

with  $n = 1, 2, 3, 4, 5, 6, \dots$

**Example 5.9 –Cont'd**

(2) We are now ready to show the convergence of the above Fourier series at the designated points:

By using the convergence criterion outline in Section 5.5 for discontinuous functions, we will find the values of the Fourier series in the above Fourier series converge at the points, i.e.  $x = 0, 5, 10, 12.5, -5, -7.5$  and  $-10$ , in both open and closed circles as depicted in the following figure following the following computations:



$$f(x)|_{x=0,5,10,-5,-10} = \frac{1}{2}[f_1(x) + f_2(x)]|_{x=0,5,10,-5,-10} = \frac{1}{2}(0 + 3) = 1.5$$

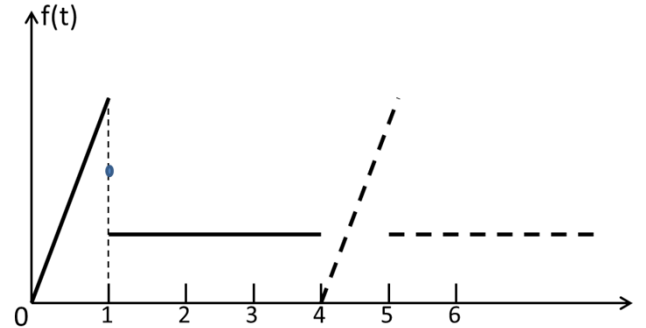
at all locations of  $x$  at which the functions discontinue (shown in closed circles), whereas the Fourier series will converge to the function values at  $x = -7.5$ , and  $x=12.5$  shown in open circles.

**Example 5.10** Determine the Fourier series and the convergence values for the piece-wise continuous function in one period (p.166):

The periodic function in one period is:

$$f(t) = \begin{cases} 3t & (0 < t < 1) \\ 1.0 & (1 < t < 4) \end{cases}$$

The function has a period of 4  
but is discontinuous at:  
 $t = 1$  and  $t = 4$



Solution:

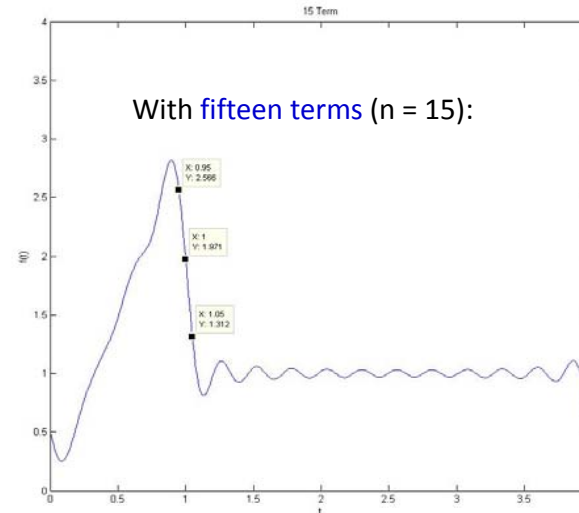
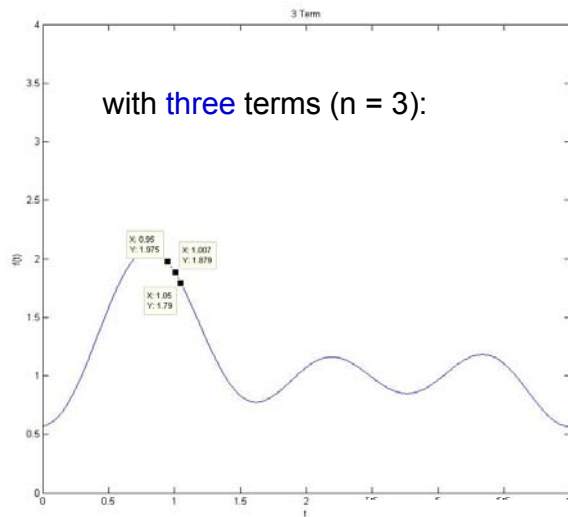
Let us first derive the Fourier series to be in the form of:  $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L} \right)$  (a)

with:  $a_0 = \frac{15}{4}$  (b)

and  $a_n = \frac{6}{n^2 \pi^2} \left( \cos \frac{n\pi}{2} - 1 \right) + \frac{1}{n^3 \pi^3} (12 - n^2 \pi^2) \sin \frac{n\pi}{2}$  with  $n = 1, 2, 3, 4, 5, \dots$  (c)

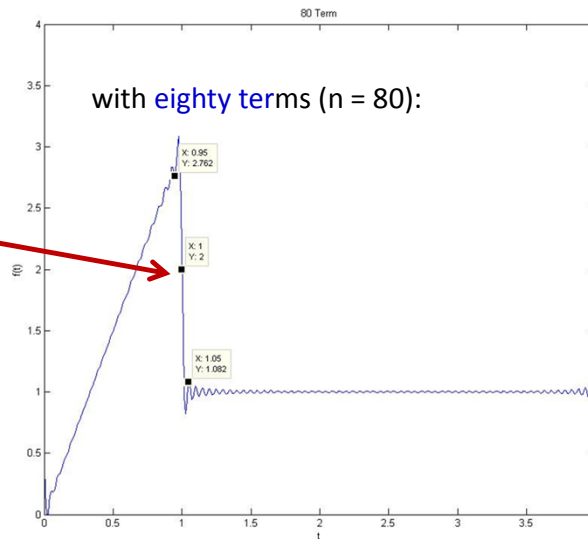
$b_n = \frac{6}{n^2 \pi^2} \sin \frac{n\pi}{2} - \frac{1}{n\pi} \left( 4 \cos \frac{n\pi}{2} + \cos 2n\pi \right)$  with  $n = 1, 2, 3, 4, 5, \dots$  (d)

We will then use a digital computer to compute the function values and draw the curves represented by the above Fourier series in Equation (a) to (d) with different number of terms and illustrate the convergence of the series as follows.



Converges well with 80 terms!!

Observe the convergence of  
Fourier series at  
DISCONTINUITIES



**We observed that the Fourier series representing these piece-wise continuous Functions converged at “half-way” of the function values at  $x = 1$  where the discontinuity is, as expected with  $n=80$  terms.** (This convergence actually began with 15 terms in the Fourier series, as illustrated in right of the top two figures)