Uncertainty Analysis

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Error

- **Error** is the difference between the measured value and the true value, and every measurement is subject to error.
- The error can not actually be known until after the measurement, and—depending on whether or not the true value is actually known—it may never be known exactly.

Uncertainty

- **Uncertainty** is an estimate of the magnitude of error, typically expressed in terms of a confidence interval within which the error lies.
- "An uncertainty statement assigns credible limits to the accuracy of a reported value, stating to what extent that value may differ from its reference value"
  [http://www.itl.nist.gov/div898/handbook/mpc/section5/mpc52.htm#ISO, September 2008]
- Uncertainty analysis considers both systematic error and random error.

Propagation of Uncertainties

- When a result y is a function of variables x_i, a first-order variation equation can be used to estimate a change Δy in terms of small changes in each of the variables x_i.

$$y = f\{x_1, x_2, \ldots x_n\} \qquad \Delta y = \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \cdots + \frac{\partial f}{\partial x_n}\Delta x_n$$

- Here the change Δy in output is expressed as a sum of contributing sources of uncertainty Δx_i, weighted by sensitivity coefficients.
- A "worst-case" uncertainty u from multiple uncertainties u_i could be computed by:

$$u = \sum_{i=1}^{n} \left| \frac{\partial f}{dx_i} u_i \right|$$

- Is there a better way to express the combined uncertainty?

Square Root of Sum-of-Squares

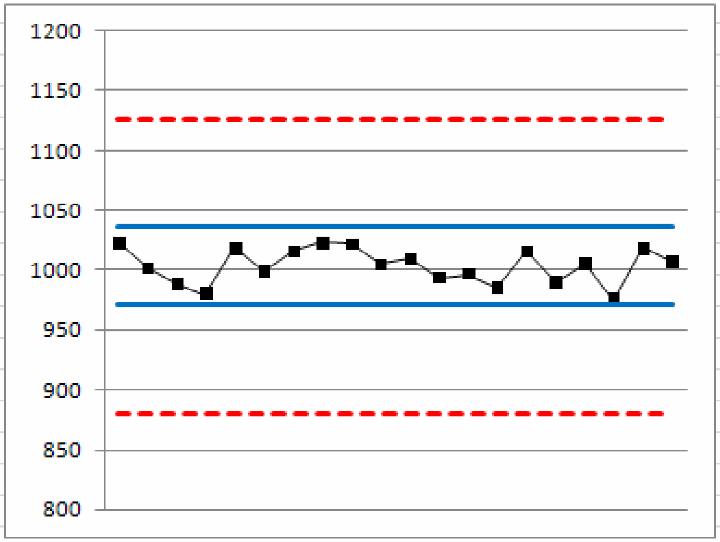
- Taking the square root of the sum-of-squares is an effective way to combine uncertainties into one value, and squaring each contributing term before taking the sum has some important advantages:
  - Positive and negative contributors to the uncertainty do not accidentally "cancel out".
  - Larger error sources are magnified compared to smaller ones, and this is desirable for identifying severe problems.
  - Sum-of-squares does not over-estimate uncertainty as an extreme worst-case scenario.

$$u = \sqrt{\left(\frac{\partial f}{\partial x_1} u_1\right)^2 + \left(\frac{\partial f}{\partial x_2} u_2\right)^2 + \cdots + \left(\frac{\partial f}{\partial x_n} u_n\right)^2}$$

Why Not Sum of Absolute Differences?

- The sum of absolute differences would be meaningful as a worst-case scenario in which all contributors were positive or all were negative, but in general it severely overestimates the error.

x	Average	x - Avg	|x - Avg|	(x - Avg)²
1023	1004	19.72	19.72	388.86
1002	1004	-1.90	1.90	3.60
988	1004	-15.42	15.42	237.75
980	1004	-23.23	23.23	539.74
1018	1004	14.80	14.80	219.00
999	1004	-4.52	4.52	20.40
1016	1004	12.39	12.39	153.48
1023	1004	19.07	19.07	363.74
1022	1004	18.58	18.58	345.23
1005	1004	1.25	1.25	1.57
1009	1004	5.63	5.63	31.66
993	1004	-10.10	10.10	102.05
997	1004	-7.04	7.04	49.53
985	1004	-18.50	18.50	342.22
1016	1004	12.38	12.38	153.16
990	1004	-14.05	14.05	197.54
1005	1004	1.76	1.76	3.10
975	1004	-28.50	28.50	812.34
1018	1004	14.51	14.51	210.46
1007	1004	3.18	3.18	10.11
	Sum:	0.00	**247**	4186
		Too Low	Too High	Wrong Units
Square Root of Sum-of-Squares:				**65**



Variant on Textbook Example 7.1

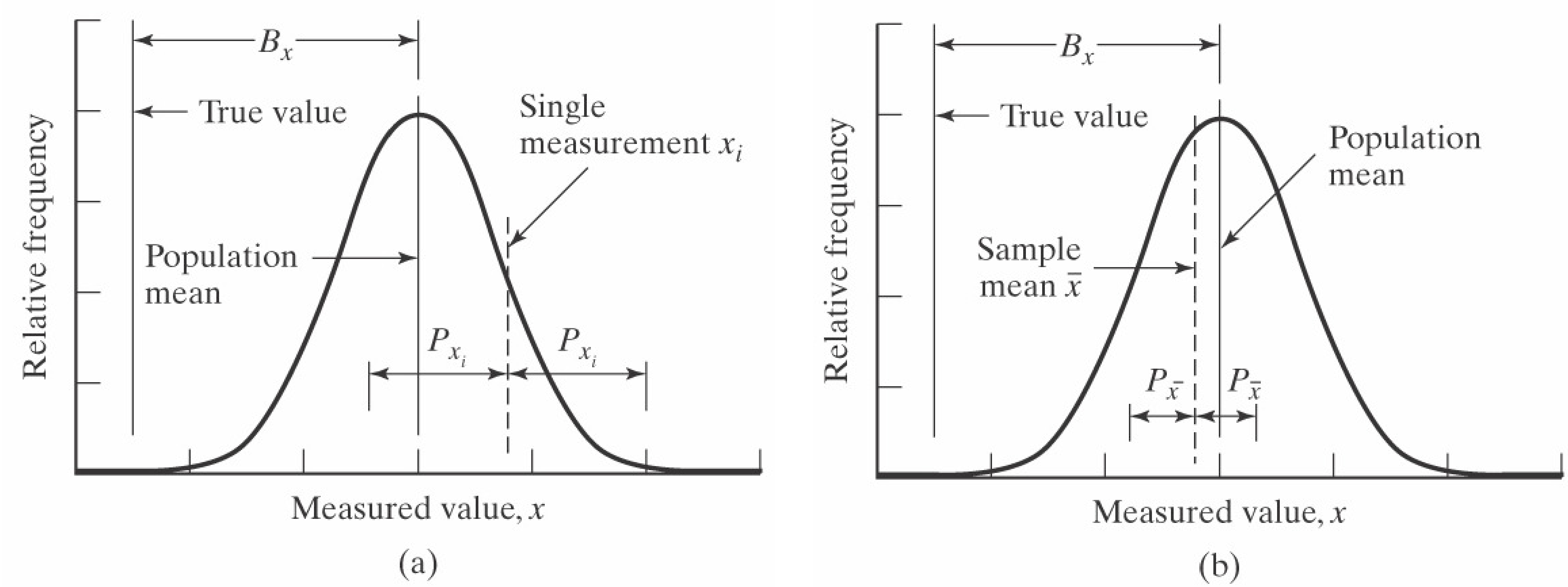
- (In class)

Questions for Conducting Uncertainty Analysis

- Is the evaluation applied to random errors or systematic errors?
- Can the uncertainty be based on statistical probability distributions or not?
- Is the uncertainty being estimated for a single measurement or a sample mean?

- For more comprehensive discussion (as of September 2008), see [http://www.itl.nist.gov/div898/handbook/mpc/section5/mpc5.htm]

Random and Systematic Uncertainties



- Quantifying uncertainty differs for single measurements versus sample means.
- Systematic (or bias B) uncertainty is the same in both cases, but random (or precision P) uncertainty is reduced by increased sample size.
- Random uncertainty for a sample mean is estimated from the standard deviation, scaled by the *t*-distribution and the sample size.

$$P_{\bar{x}} = \pm t \frac{s_x}{\sqrt{n}}$$

For large sample size ($n > 30$), $t \approx 2$.

Methodology for Uncertainty Analysis

- Define the relevant variables and exact method of measurement.
- List all contributing elemental sources of systematic error and random error, and estimate their respective magnitudes.
- Quantify standard deviations S_x for random uncertainties. For complex or single-value measurements, S_x is not obvious and may need to come from auxiliary measurements.
- Calculate the systematic uncertainty B and random uncertainty P separately, then combine to calculate the total uncertainty.

$$B_x = \sqrt{\sum_{i=1}^{k} B_i^2} \qquad S_x = \sqrt{\sum_{i=1}^{m} S_i^2}$$

$$P_{\bar{x}} = t\frac{S_x}{\sqrt{n}} \qquad u_{\bar{x}} = \sqrt{B_x^2 + P_{\bar{x}}^2}$$

$$P_x = tS_x \qquad u_x = \sqrt{B_x^2 + P_x^2}$$

Which Errors are Systematic vs. Random?

TABLE 7.1 Guidelines for assigning elemental error

ERROR	ERROR TYPE
accuracy	systematic
common-mode volt	systematic
hysteresis	systematic
installation	systematic
linearity	systematic
loading	systematic
noise	random*
repeatability	random*
resolution/scale/ quantization	random*
spatial variation	systematic
thermal stability (gain, zero, etc.)	random*

*Assume that the number of samples is greater than 30 unless specified otherwise.

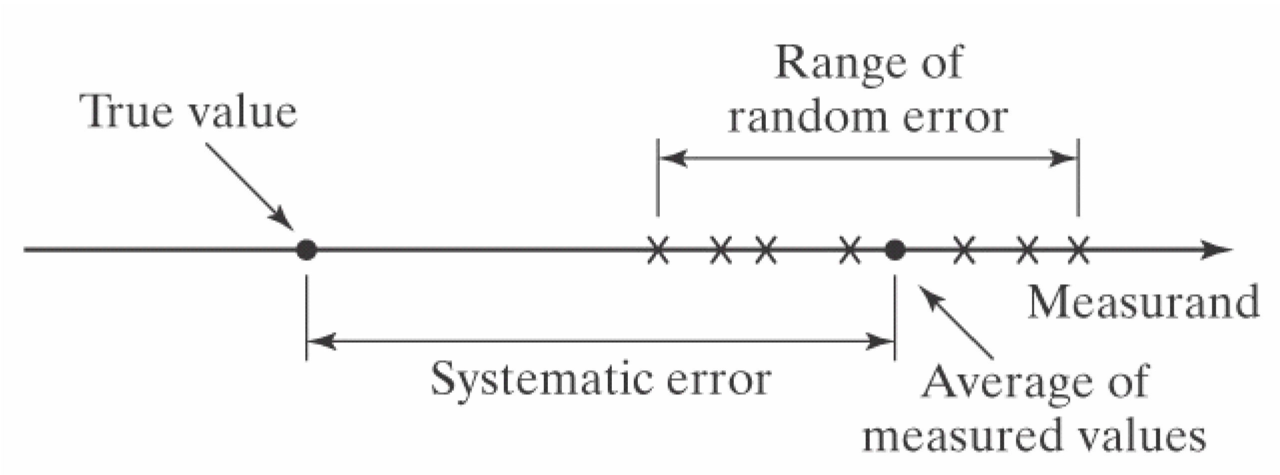
- In general, any random uncertainties assume large sample size (n > 30).
- If in doubt, for the purposes of uncertainty analysis assume systematic error.
- To combine random uncertainties, the same confidence level must apply to each elemental uncertainty.

Variant on Textbook Example 7.7

- (In class)

Systematic Error and Random Error (Review)

- **Systematic error** (or "bias" error) is repeatable.
  - e.g. imperfect calibration, residual loading, intrusive measurements, spatial bias
- **Random error** (or "precision" error) is not predictable.
  - e.g. environmental variability, noise, vibration



Example

- What is the uncertainty in the $P = iv$ power of a resistive circuit, if the voltage is measured to be $v = 100 \pm 1$ V and the current is measured to be $i = 10 \pm 0.1$ A?
- How much difference is there between “worst-case scenario” and “best estimate”?

Example

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$$\frac{\partial P}{dv} = i = 10\text{ A} \qquad u_v = 1\text{ V} \qquad \frac{\partial P}{di} = v = 100\text{ V} \qquad u_v = 0.1\text{ V}$$

$$u = \left|\frac{\partial P}{dv} u_v\right| + \left|\frac{\partial P}{di} u_i\right| \qquad u = \sqrt{\left(\frac{\partial P}{dv} u_v\right)^2 + \left(\frac{\partial P}{di} u_i\right)^2}$$

$$u = 10(1) + 100(0.1)\text{ W} = 20\text{ W} \qquad u = \sqrt{(10 \cdot 1)^2 + (100 \cdot 0.1)^2}\text{ W} \approx 14\text{ W}$$

Example

- A pressure transducer has full-scale (FS) range 1000 kPa.
- Linearity uncertainty is ±0.2% FS.
- Hysteresis uncertainty is ±0.1% FS.
- The repeatability uncertainty, expressed in this case as standard deviation over a large number of repeated measurements at a fixed typical setting is 10 kPa.
- The transducer is subject to uncertainties from temperature, that affects measurements with a standard deviation of 3 kPa.
- What is the total uncertainty of pressure measurement with this transducer?

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$$B_L = 0.002(1000) \text{ kPa} = 2 \text{ kPa}$$

$$B_H = 0.001(1000) \text{ kPa} = 1 \text{ kPa}$$

$$B_x = \sqrt{(B_L)^2 + (B_H)^2} = \sqrt{5} \text{ kPa}$$

$$S_x = \sqrt{(S_R)^2 + (S_T)^2} = \sqrt{109} \text{ kPa}$$

$$P_x = tS_x = 2\sqrt{109} \text{ kPa}$$

$$u_x = \sqrt{B_x^2 + P_x^2} = \sqrt{5 + 4(109)} \text{ kPa} = 21 \text{ kPa}$$