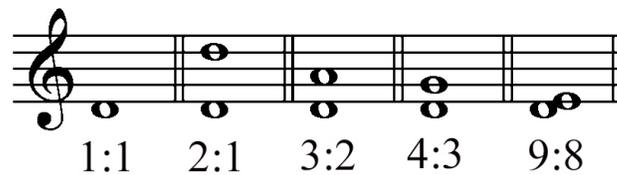


## Pythagoras and Greek Tuning

One of the first questions we might ask of Western music is where did the **notes**—the very foundation of our music—come from? While most music around the world derives its notes from the **overtone series**, Greek theorists, such as the mathematician **Pythagoras** (c. 500 BCE), discovered and explored the numeric basis behind the acoustics of these intervals. Legend has it that one day Pythagoras walked by a forge and noticed the different tones emanating from anvils of different sizes as they were struck by the blacksmiths, and conjectured that the different tones of the anvils had to do with their **ratios of one size to another**. Pythagoras and his followers elaborated this theory to generate a series of musical intervals—the so-called “perfect” intervals of the octave, fifth, fourth, and the second—with whose whole number **ratios** that could be demonstrated on the string of the **monochord**.



Greek theorists called the octave the **Diapason** (“across all”), the fifth the **Diapente** (“across five”), and the fourth the **Diatessaron** (“across four”), and these terms were used up through the Renaissance.

A strange phenomenon is revealed, however, when using the Pythagorean ratios. On an equal-tempered piano, the sum of 7 stacked octaves (CC to  $c^4$ ) and the sum of 12 superimposed fifths (CC to  $c^4$ )—the “circle of fifths”—sound equal.

A musical staff showing 12 superimposed fifths. The notes are: C, G, D, A, E, B, F#, C#, G#, D#, A#, E, B, F#. The intervals are labeled as 8<sup>va</sup> and 16<sup>va</sup>. To the right of the staff is the equation:  $= \left(\frac{3}{2}\right)^{12}$ .

A musical staff showing 7 stacked octaves. The notes are: C, C, C, C, C, C, C. The intervals are labeled as 8<sup>va</sup> and 16<sup>va</sup>. To the right of the staff is the equation:  $= \left(\frac{2}{1}\right)^7$ .

However, the pitches on a modern piano are **tempered**—that is, the mathematically “pure” intervals of Pythagorean intonation are slightly adjusted. Using the whole-number Pythagorean ratios of 2:1 for octaves and 3:2 for fifths, arithmetic shows that the sum of 7 octaves and of 12 fifths are *not* the same, and the sum of the circle of fifths overshoots getting back to “c” by about a quarter of a semitone, or more accurately, by 21.51 cents.

$$\frac{\left(\frac{3}{2}\right)^{12}}{\left(\frac{2}{1}\right)^7} = \frac{531441}{524288} = 1.01364\dots$$

This difference is called the **Pythagorean comma**,<sup>1</sup> and can be seen here in this table.

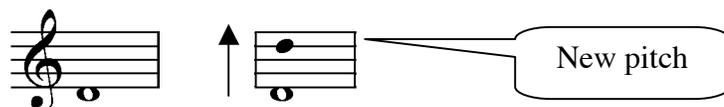
Note	Fifth	Frequency ratio	Equivalent ratio
C	0	1 : 1	= 1 : 1
G	1	2 : 3	= 1 : 1.5
D	2	4 : 9	= 1 : 2.25
A	3	8 : 27	= 1 : 3.375
E	4	16 : 81	= 1 : 5.0625
B	5	32 : 243	= 1 : 7.59375
F#	6	64 : 729	= 1 : 11.390625
C#	7	128 : 2187	= 1 : 17.0859375
G#	8	256 : 6561	= 1 : 25.62890625
D#	9	512 : 19683	= 1 : 38.443359375
A#	10	1024 : 59049	= 1 : 57.6650390625
E#	11	2048 : 177147	= 1 : 86.49755859375
B# (≈ C)	12	4096 : 531441	= 1 : 129.746337890625

The Pythagorean comma—which is the byproduct of acoustics—means that certain intervals generated from tuning using **mathematical ratios using whole numbers**—or **just tuning**—will sound “bad.” How then did the Greeks generate a set of musical pitches out of these intervals? And what problems does the comma cause?

<sup>1</sup> A comma is the **difference** between two seemingly enharmonic notes: an A flat tuned as a major third below C in **just intonation**, and a G sharp tuned as a major third above E, will not be exactly the same note.

## Generating the Greek “Phrygian” Scale with Pythagorean Intervals

With the whole-number ratio of 2:1 we can generate a tone and an octave.



From those two pitches (D, D) we can tune up and down by perfect fifths (3:2) to generate two new pitches, A and G.



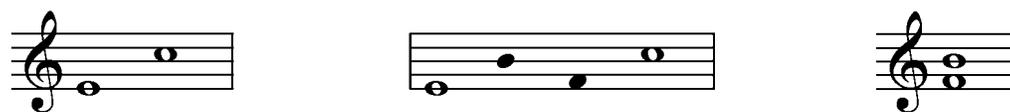
From those two pitches (G, A) we can tune up and down by perfect fourths (4:3) to generate two new pitches, C and E.



The resulting pitches—so far—produce a pentatonic scale.



Following this pattern we can again tune up and down by perfect fifths (3:2) from C and E to generate two new pitches, B and F, that are the last two pitches needed to complete the Greek Phrygian scale.



However, the interval between B and F resulting from tuning in this way with just intervals creates the dissonant interval of the **tritone** (729:512) that sounds **horribly bad**. Similarly, the acoustics of just intonation generates a comma—the **diesis**—between a perfect octave (2:1) and three stacked **pure** major thirds (5:4).<sup>2</sup> While three equal-tempered stacked thirds “add up” to an octave, three just-tuned major thirds add up to a slightly smaller interval than a perfect octave with a difference of 21.51 cents.



**The Comma of Didymus**

<sup>2</sup> Tunings are referred to as “pure” when they match the acoustic ratios of the overtone series.