Reactance and Impedance

AC Voltage and Current

In a DC circuit, we learned that the relationship between voltage and current was V=IR, also known as Ohm's law. We need to find a similar law for AC circuits, but now there is an important complication. In an AC circuit, we expect the voltage and current to fluctuate, and the fluctuations may be out of phase. Therefore, there are now <u>two</u> relationships between current and voltage. One is the ratio of the peak voltage to the peak current; Vp/Ip. Another is the phase angle between the voltage and the current.

In order to keep track of both of these quantities in as simple a manner as possible, we will use the mathematical shorthand known as complex numbers.

Complex Numbers

A complex number W consists of *real* and *imaginary* parts "a" and "b" respectively, and the imaginary constant "j" which is the square root of negative one.[*]

$$W = a + jb \qquad j = \sqrt{-1} \qquad \qquad \text{(note that } j^2 = -1 \text{ and } \frac{1}{j} = -j\text{)}$$

One can specify the real and imaginary parts of W explicitly: $Re(W) = a, Im(W) = b$.

Because a complex number has two components, it can be plotted on a two dimensional graph. One calls the x-axis the "real axis", and the y-axis the "imaginary axis". The diagram on the right shows a graph of the complex number $\vec{W} = a + jb$. Note that a complex number is really a vector, with both a magnitude and a direction.

The magnitude of W is the length of the vector, $|W|$, and can be found using the Pythagorean theorem: $|W| = \sqrt{a^2 + b^2}$. The angle θ signifies the direction of the complex vector, and can be found using trigonometry: $\theta = \tan^{-1}\left(\dfrac{b}{a}\right)$

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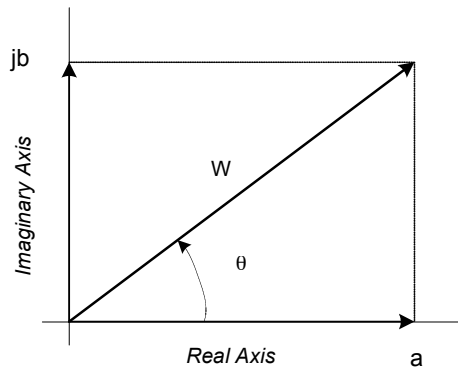
[*] Since "i" is used for current, "j" is traditionally used instead.

Further trigonometry yields two more relationships:

$$|W|\cos\theta = a \qquad |W|\sin\theta = b$$

which allows the complex number W to be re-written:

$$W = a + jb = |W|\cos\theta + j|W|\sin\theta = |W|(\cos\theta + j\sin\theta)$$

This form is very useful, because of Euler's formula: $(\cos\theta + j\sin\theta) = e^{j\theta}$

This formula allows us to write any complex number in an exponential form:

$$W = |W|e^{j\theta}$$

where as before: $|W| = \sqrt{a^2 + b^2}$ and: $\theta = \tan^{-1}\dfrac{b}{a}$

Finally, since $e^{j\theta}$ is a *unit vector* in the direction of W, the following **polar** form *notation* is used, where $|W|$ is magnitude and θ is angle with respect to the positive real axis.

$$W = |W|e^{j\theta} = |W|\angle\theta$$

A vector specified as: $1.74\angle 32°$, for example, has magnitude 1.74 and is oriented 32 degrees with respect to the real axis. Note that positive angles open (rotate) *counter-clockwise* from the positive real axis and negative angles open *clockwise* as shown.

♦ Example 4.1   (Complex Vector Notation)

Determine the four notational forms associated with the vector of magnitude 2.45 and angle 38 degrees.

In polar form: $W = 2.45\angle 38°$

In exponential form: $W = 2.45e^{j38°}$

In trigonometric form:
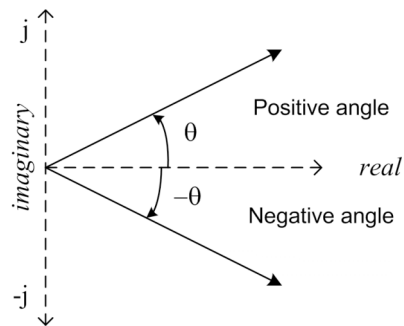 $W = 2.45(\cos 38° + \sin 38°)$

Now since:

$a = |W|\cos\theta = 2.45\cos 38° = 2.45(0.788) = 1.93$

And: $b = |W|\sin\theta = 2.45\sin 38° = 2.45(0.616) = 1.51$

The rectangular form is: $W = 1.93 + j1.51$

Regarding complex vector division:

$$\frac{M_1\angle\theta_1}{M_2\angle\theta_2} = \frac{M_1}{M_2}\cdot\frac{\exp(j\theta_1)}{\exp(j\theta_2)} = \frac{M_1}{M_2}\exp(j\theta_1 - j\theta_2) = \left(\frac{M_1}{M_2}\right)\angle(\theta_1 - \theta_2)$$

And the conversion process is simplified to:

$$\frac{M_1\angle\theta_1}{M_2\angle\theta_2} = M_{12}\angle(\theta_1 - \theta_2) \quad \text{where } M_{12} = \frac{M_1}{M_2}$$

Complex Voltage and Current

      Using this complex notation, we can now write the voltage and current in a manner that keeps track of both amplitude and phase. We have already seen one way to write a sinusoidal voltage: $V(t) = V_0\cos(\omega t + \theta_1)$. Euler's formula tells us that:

$$\text{Re}\left\{V_0 e^{j(\omega t + \theta_1)}\right\} = \text{Re}\left\{V_0\cos(\omega t + \theta_1) + jV_0\sin(\omega t + \theta_1)\right\} = V_0\cos(\omega t + \theta_1)$$

Therefore, we may also write a sinusoidal voltage as follows: $V(t) = \text{Re}\left\{V_0 e^{j(\omega t + \theta_1)}\right\}$

Here the imaginary part of V(t) is simply ignored. This sinusoidal-to-exponential *transformation* is done because it is easier perform multiplication and division with exponentials rather than sines and cosines. Moreover, it is common to make the transformation *implicit* by not specifying the "Re" operator. This is indicated below for both voltage and current, each with a different phase.

$$V(t) = V_0\cos(\omega t + \theta_1) \rightarrow V_0 e^{j(\omega t + \theta_1)}$$

$$I(t) = I_0\cos(\omega t + \theta_2) \rightarrow I_0 e^{j(\omega t + \theta_2)}$$

From the complex equations above, it is a simple manner to find the ratio of the peak voltage to the peak current; this ratio is simply V_0/I_0. It is also a simple matter to find the relative phase θ between the voltage and the current, simply by taking the difference $\theta = \theta_1 - \theta_2$. These two values are all one needs to determine the AC current from the AC voltage, or vice-versa, and they can be combined into one complex term known as the *impedance*, which is signified by "Z":

$$Z = \frac{V_0}{I_0} e^{j(\theta_1 - \theta_2)}$$

From the above equations, one can find that the general relationship between the complex voltage, the complex current, and the complex impedance, is simply:

$$V(t) = I(t)\, Z$$

This equation looks much like Ohm's law, but be warned: these values are all complex. The rest of this chapter will concern how to calculate the impedance for a given circuit. Once this is accomplished, the real and imaginary parts of the impedance fully determine the magnitude and phase relationships between the current and the voltage for any AC circuit.

Impedance

In general, a complex impedance vector will take the following form:

$$Z = Re(Z) + jIm(Z)$$

To find the ratio of the peak voltage to the peak current, one simply finds the magnitude of the impedance. To find the relative phase between the current and the voltage, one must find the phase angle of the impedance. Both of these equations are given here:

$$\frac{V_o}{I_o} = |Z| = \sqrt{Re(Z)^2 + Im(Z)^2} \qquad \theta = \tan^{-1}\frac{Im(Z)}{Re(Z)}$$

Therefore, the two (real) numbers Re(Z) and Im(Z) fully determine the relationship between the current and the voltage. These values are typically given the symbols "R" and "X" respectively, yielding the equation:

$$Z = R + jX$$

The value R is simply the *resistance* of the circuit. The value X is something known as the *reactance* of the circuit. Resistance, as we already know, determines the ratio between the voltage and current magnitudes without contributing to the relative phase. Reactance, however, contributes to both the voltage/current ratio as well as the phase between them. For example, a capacitor (as we will learn in the next section) contributes a negative reactance X_C.

The impedance diagram below corresponds to the RC circuit on the right. Resistance vector R represents the resistor and reactance vector X_C represents the capacitor. Because the circuit elements are in series, the impedance vector Z is the vector sum of these two vectors; $Z = R + X_C$. Since impedance is a vector quantity, it is specified in terms of *magnitude* and *direction angle*. A two-dimensional plane with *real* and *imaginary* axes is used to depict impedance vectors., where resistance R corresponds to the real (horizontal) axis and reactance X corresponds to the imaginary (vertical) axis. .X_C is directed along the negative imaginary axis because it is a negative quantity.
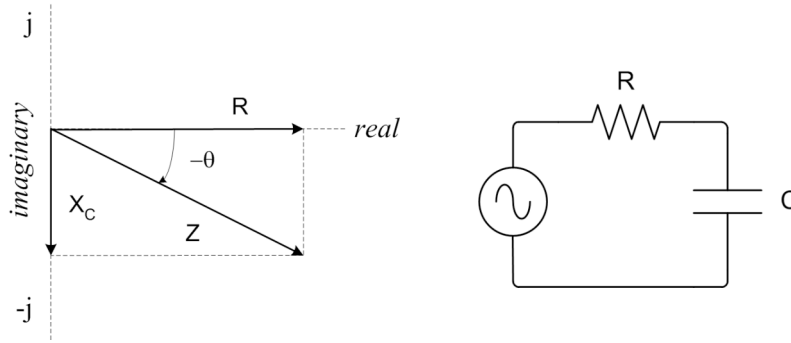
Magnitude is calculated by use of the Pythagorean relation and **angle** is calculated using the inverse tangent relation. Note that angle is referenced to the positive real axis, and negative angles rotate in the *clockwise* direction.

$$|Z| = \sqrt{R^2 + X_C^2} \qquad\qquad \theta = \tan^{-1}\frac{X_C}{R}$$

The impedance vector allows for calculation of associated voltage and current quantities, as shown above. Current through the circuit corresponds to current through the resistor, which is the same current that flows through the capacitor because it is a series circuit. Note that this is consistent with the fact that capacitor current changes instantaneously.

The left diagram (large angle) corresponds to a circuit that is dominated by capacitive reactance, in which the current is nearly 90 degrees ahead of the voltage. The right diagram (small angle) corresponds to a circuit that is dominated by resistance, in which the current is nearly in phase with the voltage. The middle diagram corresponds to equality between capacitive reactance and resistance, in which the current lead is *midway* between the 90 and zero degree limits.
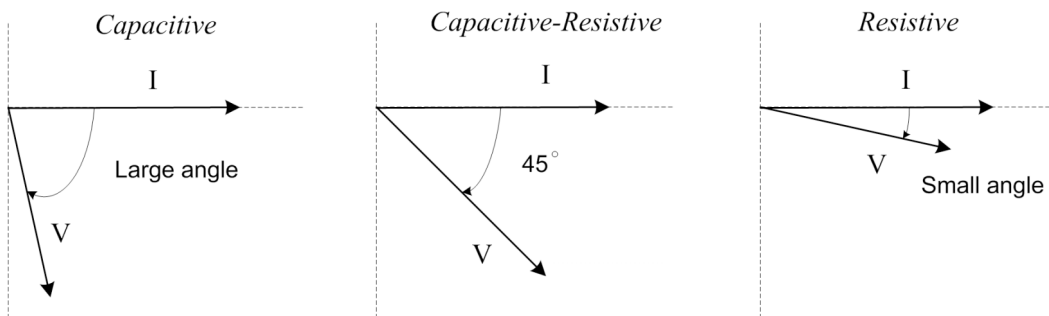
Capacitive Reactance

One way of arriving at capacitive reactance is to examine the current through a capacitor in relation to the voltage across it. In doing so; however, we first discover a *phase difference* between the applied voltage and resulting current. This phase difference is graphed below. Note that current *leads* voltage (by 90 degrees).
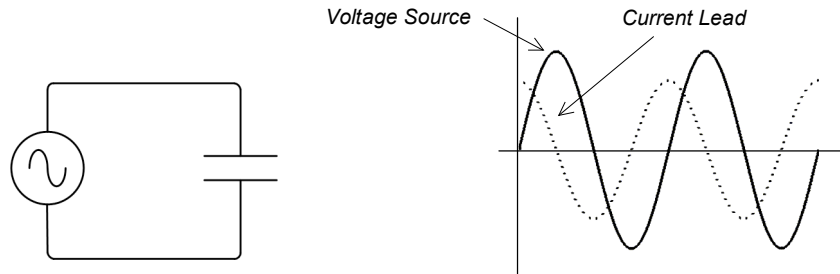
For the voltage source: $V(t) = V_0 \sin \omega t$

And for the capacitor: $I(t) = C \dfrac{dv}{dt} = C(V_0 \omega) \cos \omega t = I_0 \cos \omega t$

Where: $I_0 = CV_0 \omega$

Thus, a *sinewave* voltage across a capacitor results in *cosine* current through the device. Note that both signals have the *same* frequency (ω).

Now for a sinewave and cosine wave of the *same* frequency, the cosine is essentially a sinewave that is *advanced* in phase by 90 degrees. This is based on the general relation:

$\cos \omega t = \sin(\omega t + \dfrac{\pi}{2})$

In summary, it is characteristic for capacitors that current *leads* voltage by a 90-degree phase-shift, or alternatively, that voltage *lags* current by 90-degrees.

But knowing the phase is not enough; we want to know the capacitive reactance. We are now in position to do so by way of the sinusoidal voltage and current equations above. The procedure is simple in that only the *amplitude coefficients* V_0 and I_0 are used. By finding the ratio between them, we arrive at the mathematical relation for capacitive reactance.

$$|X_C| = \frac{V_0}{I_0} = \frac{V_0}{CV_0 \omega} = \frac{1}{\omega C}$$

Combining the phase and magnitude information, we must add a minus sign (see example 4.8 for a full derivation), which gives us the reactance for a capacitor:

$$X_C = \frac{-1}{\omega C} = \frac{-1}{2\pi fC}$$

♦ Example 4.2   (Capacitive Reactance Calculation)

For the simple AC capacitor circuit, a 100-microfarad load has capacitive reactance of 26.5 ohms when subjected to a source frequency of 60 hertz.
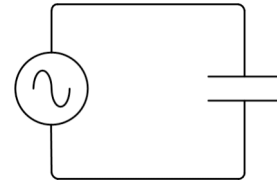
$$X_C = \frac{-1}{2\pi fC} = \frac{1}{2\pi(60 \cdot 100 \times 10^{-6})} = -26.5\Omega$$

♦ Example 4.3   (RC Circuit Impedance)

A 5-kilohertz sinewave voltage source is used to drive a series RC circuit comprised of a 270-ohm resistor and 0.068-microfarad capacitor. The associated impedance vector has magnitude 540 ohms and angle –60 degrees. The associated capacitive reactance, calculated first, is –468 ohms.

$$X_C = \frac{-1}{2\pi fC} = \frac{-1}{2\pi(5 \times 10^3)(0.068 \times 10^{-6})} = -468\Omega$$

$$|Z| = \sqrt{R^2 + X_C^2} = \sqrt{(270)^2 + (-468)^2} = 540\Omega$$

$$\theta = \tan^{-1}\frac{X_C}{R} = \tan^{-1}\frac{-468}{270} = -60°$$

♦ Example 4.4   (RC Circuit Voltage and Current)

Regarding the circuit of example 4.2, a 10-volt (rms) sinusoidal source results in 18.5 milliamps (rms) of current. Since the voltage vector is along the impedance vector, which has an angle of –60 degrees, current is *advanced* in phase by 60 degrees relative to the source voltage. Note that RMS amplitudes are used.

$$I_{RMS} = \frac{V_{RMS}}{|Z|} = \frac{10V_{RMS}}{540\Omega} = 18.5mA$$

Inductive Reactance

An inductor is the electronic *complement* of a capacitor. The simple AC inductor circuit (on the right) can be used to formulate the following expression for inductive reactance. (See example 4.9 for a derivation.)
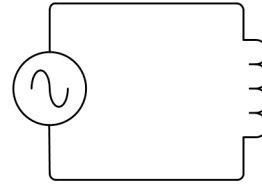
$$X_L = \omega L = 2\pi f L$$

Inductive reactance is directly proportional to frequency and is also measured in ohms. As with capacitors, one can derive the important *inductor* characteristic; that current *lags* voltage by a 90-degree phase-shift, or alternatively, that voltage *leads* current by 90-degrees.

At this point, we can summarize the impedances of *individual* resistors, capacitors, and inductors.

$$Z_R = R$$

$$Z_C = jX_C = -\frac{j}{2\pi f C} = \frac{1}{j\omega C}$$

$$Z_L = jX_L = j2\pi f L = j\omega L$$

However, knowing the impedance for a single circuit element is not enough if the circuit has more than one element. Fortunately, impedances add together very much like resistors; they add in series and add reciprocally in parallel.

For two circuit elements in series, the impedances are simply added together to find the equivalent total impedance: $Z_T = Z_1 + Z_2$
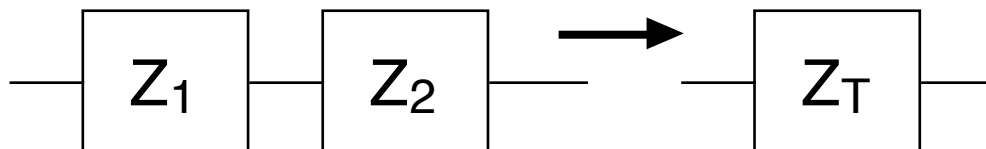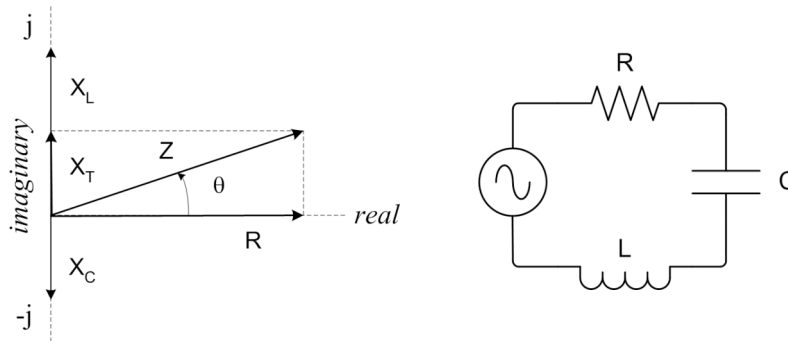
For two circuit elements in parallel, one adds reciprocals: $\dfrac{1}{Z_T} = \dfrac{1}{Z_1} + \dfrac{1}{Z_2}$
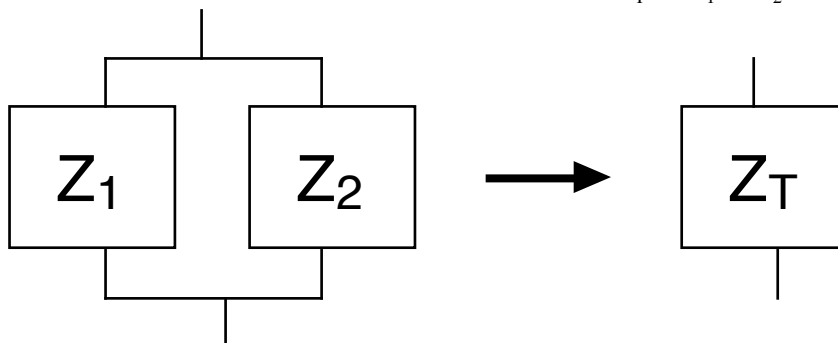
♦ Example 4.5 (RCL Circuit Analysis)

The series RCL circuit on the right consists of a 270-ohm resistor, 0.068-microfarad capacitor, and 1.1-millihenry inductor. It is driven by a 20-volt (rms) sinewave at 26-kilohertz. The aim is to calculate associated impedance and current vectors.

Because this is a series circuit, capacitive and inductive reactances are individually calculated, and then added together to obtain the total reactance X_T.

$$X_C = \frac{-1}{2\pi fC} = \frac{-1}{2\pi(26\text{x}10^3)(0.068\text{x}10^{-6})} = -90\Omega$$

$$X_L = 2\pi fL = 2\pi(26\text{x}10^3)(1.1\text{x}10^{-3}) = 180\Omega$$

$$X_T = X_C + X_L = -90 + 180 = 90\Omega$$

The impedance vector comprises resistance and total reactance, from which magnitude and angle are calculated. Refer to impedance diagram above.

$$Z = R + jX = 270 + j90$$

$$|Z| = \sqrt{R^2 + X^2} = \sqrt{270^2 + 90^2} = 285\Omega$$

$$\theta = \tan^{-1}\frac{X}{R} = \tan^{-1}\frac{90}{270} = 18.4°$$

Current through the circuit corresponds to current through the resistor, and so the current vector lies along the positive real axis, as does the impedance vector for the resistor. The magnitude (rms) of the current vector is calculated using Ohm's law.

$$I_{RMS} = \frac{V_{RMS}}{|Z|} = \frac{20V_{RMS}}{285\Omega} = 70.2\text{mA}$$

The voltage vector corresponds to the driving source, and lies along the impedance vector Z, which is 18.4 degrees above the real axis. The source voltage, therefore, is *advanced* in phase by 18.4 degrees with respect to current. This corresponds to an inductive circuit, which is consistent in that the inductive reactance is larger than the capacitive reactance. Refer to the diagram on the right.
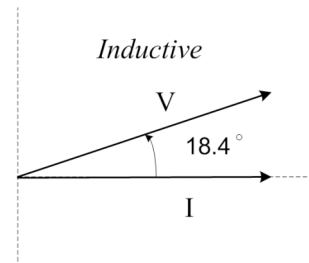
◆ Example 4.6   (Parallel RC Circuit Impedance)

The following impedance vector is obtained for the parallel RC circuit with source frequency 63.7 hertz, resistance 500 ohms, and capacitance 10 microfarads.
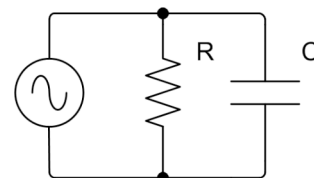
$$Z = 100 - j200$$

Because this is a parallel circuit, this expression is obtained by calculation of the parallel combination of individual impedances.

$$Z_R = R = 500\Omega$$

$$Z_C = jX_C = -j\frac{1}{2\pi(63.7)(10 \times 10^{-6})} = -j250\Omega$$

$$Z_T = \frac{Z_R \cdot Z_C}{Z_R + Z_C} = \frac{(500)(-j250)}{500 - j250} = \frac{125000\angle -90°}{559\angle -26.6°} = 224\angle -63.4° = 100 - j200$$
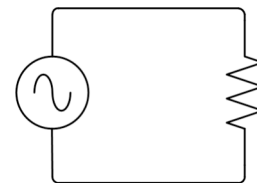

◆ Example 4.7   (Impedance of a Resistor)

Regarding a resistive load driven by a sinusoidal source:

$$V(t) = V_0 e^{j\omega t}$$

By Ohm's law: $$I(t) = \frac{V(t)}{R} = \frac{V_0 e^{j\omega t}}{R} = i_0 e^{j\omega t}$$
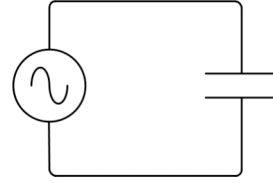
where: $$I_0 = \frac{V_0}{R}$$

A resistor's impedance, therefore, is equal to its resistance: $Z_R = R$

◆ Example 4.8   (Capacitive Reactance Derivation)
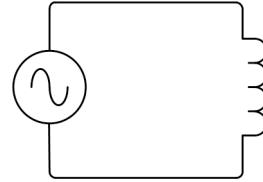
Regarding a capacitor load driven by a sinusoidal source:

$$V(t) = V_0 e^{j\omega t} \quad \text{and:} \quad I = C\frac{dv}{dt}$$

So: $I(t) = C \cdot \dfrac{d}{dt}\left(V_0 e^{j\omega t}\right) = C \cdot j\omega \cdot V_0 e^{j\omega t} = \dfrac{V_0}{\left(\dfrac{1}{j\omega C}\right)} e^{j\omega t} = \dfrac{V_0}{Z_C} e^{j\omega t} = I_0 e^{j\omega t}$

Where: $Z_C = \dfrac{1}{j\omega C} = j\dfrac{-1}{\omega C} = j\dfrac{-1}{2\pi f C} = jX_C \quad \text{and:} \quad I_0 = \dfrac{V_0}{|Z_C|}$

◆ Example 4.9   (Inductive Reactance Derivation)

Regarding an inductor load driven by a sinusoidal source:

$$V(t) = V_0 e^{j\omega t} \quad \text{and:} \quad V = L\frac{dI}{dt}$$

So: $I(t) = \dfrac{1}{L}\int V(t) \cdot dt = \dfrac{1}{L}\int V_0 e^{j\omega t} \cdot dt = \dfrac{V_0}{j\omega L} e^{j\omega t} = \dfrac{V_0}{Z_L} e^{j\omega t} = I_0 e^{j\omega t}$

Where: $Z_L = j\omega L = j2\pi f L = jX_L \quad \text{and:} \quad I_0 = \dfrac{V_0}{|Z_L|}$