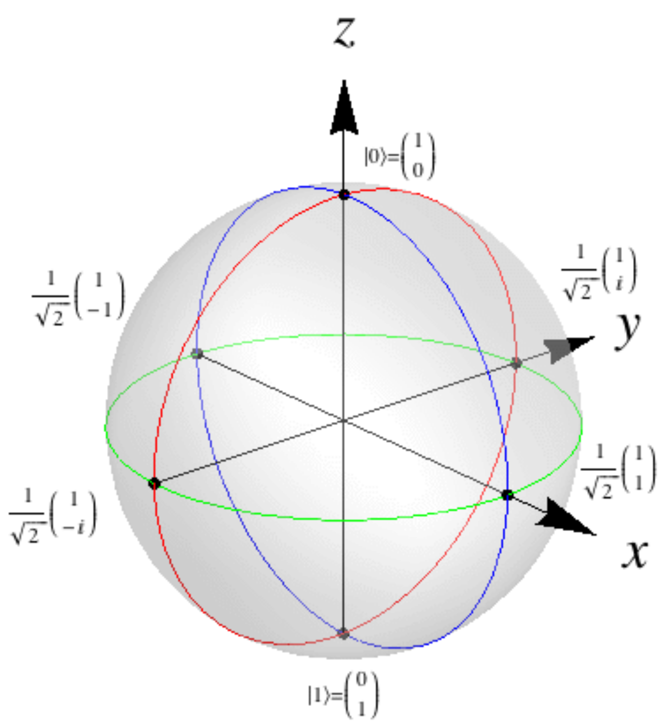
## The Bloch Sphere

A single spin-1/2 state, or "qubit", is represented as a normalized state $\begin{pmatrix} a \\ b \end{pmatrix}$ where

|a|^2+|b|^2=1. The phase of this is irrelevant, so you can always multiply **both** "a" and "b" by exp(ic) without changing the state. Note that this formalism can be used for any 2D Hilbert space state; it doesn't *have* to be a spin-1/2 particle.

For any such state, you can always find a direction that you can measure the spin and **always** get a result of $\hbar/2$. This is the same direction as the expectation value of the vector of spin-operators < **S** >. This direction can be represented as a unit vector, pointing to a location on a unit sphere, or the "Bloch sphere". For example, spin-up (a=1,b=0) corresponds to the intersection of the unit sphere with the positive z-axis. Spin-down (a=0,b=1) is the -z axis.




For an arbitrary point on this sphere, measured in usual spherical coordinates $(\theta,\phi)$, the corresponding spin-1/2 state is (see problem 4.30 in Griffiths): Eqn [4.155]:

$$\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix} \text{ or } \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{i\phi/2} \end{pmatrix}$$ (see why these two forms are really the same?)

Useful Exercise: Check that this works for the above cases in the diagram.

Notice the special notation for spin up: $|0\rangle$, and for spin down: $|1\rangle$. These are the "0"'s and "1"'s of a quantum computer. Of course, any single qubit state can be written in these terms: $\begin{pmatrix} a \\ b \end{pmatrix} = a|0\rangle + b|1\rangle$. (Don't forget a and b are complex.)

**Single Qubit Measurements:**

We already know how to do single-qubit measurements, in principle, but there is a useful shortcut when thinking about states on the Bloch sphere. For any qubit-state pointing in the **f**-direction on the Bloch sphere, suppose you measure it on the **g**-axis (for a spin-1/2 particle, you could do this by putting a magnetic field in the **g**-direction and measuring the energy.) It turns out that the probability of the outcomes only depends on the angle between **f** and **g**: call this angle $\theta$.

It is not hard to prove (done in class) that the probability of measuring the eigenvalue corresponding to the state pointing in the the +**g** direction is $\cos^2\left(\theta/2\right)$. If this was the measurement result, we already know that the qubit would "collapse" into a state whose Bloch sphere vector pointed in the +**g** direction instead of the **f** direction.

The other possibility is that it might collapse into a state whose Bloch sphere vector pointed in the **-g** direction. Obviously, the probability of this other measurement outcome would be $\sin^2\left(\theta/2\right)$. There are no other possible outcomes for such a measurement.

**Single Qubit Gates: Rotations on the Bloch Sphere**

A quantum "gate" is a transformation that you can perform on a quantum state. IMPORTANT NOTE: THIS IS NOT A MEASUREMENT. It's just a linear transformation, and can be represented by an operator: $\hat{\mathbf{Q}}|\psi\rangle = |\psi'\rangle$. In general, the gate (**Q**) takes the input state $|\psi\rangle$, and spits out the output state $|\psi'\rangle$. There is no collapse, just a transformation. (Technically, a "unitary" transformation.)

Single-qubit gates are best envisioned as rotations on the Bloch sphere. You can rotate around any axis, by any angle -- in fact, we already know how to do this to a spin-1/2 state with an appropriately aligned magnetic field. (The state precesses around the B-field direction.)

These gates/rotations are discussed in an important problem in Griffiths: Problem 4.56. The exponential notation in the earlier part of the problem is not needed; the upshot of this problem is the last equation [4.201] (although the book's equation drops the Identity matrix in my version). This tells us that the operator **R** corresponding to a rotation of an angle $\phi$ around an axis $\hat{n}$ is:

$$\mathbf{R} = \cos\left(\frac{\phi}{2}\right)\mathbf{I} + i(\hat{n}\cdot\sigma)\sin\left(\frac{\phi}{2}\right) \quad [4.201]$$

Here **I** is the 2x2 identity matrix, and $\sigma$ is the vector of Pauli matrices. (So if you wanted to rotate around the z-axis, you would put in $(\hat{n}\cdot\sigma) = \sigma_z$. Obviously **R** would also be a 2x2 matrix, so that it can operate on a qubit.

(Note: These matrices are not Hermitian! They are "unitary".)

Special/useful single-qubit gates include:

The **NOT** gate (also known as the Pauli **X**-gate); a 180° rotation around the x-axis.

The Pauli-**Z** gate: a 180° rotation around the z-axis.

The Pauli-**Y** gate: a 180° rotation around the y-axis.

The $\sqrt{NOT}$ gate; a 90° rotation around the x-axis.

Phase shift gates, $R(\phi)$; a $\phi$-angle rotation around the z-axis.

<u>Useful exercise: Build these 2x2 matrices, and check that they work as advertised!</u>

**Building Two Qubit States: Tensor Products**

In QM, when you have two single-particle Hilbert spaces, the total wavefunction lives in a larger Hilbert space that is the <u>"tensor product"</u> of those two spaces. If qubit 1 is in state $\begin{pmatrix} a \\ b \end{pmatrix}$, and qubit 2 is in state $\begin{pmatrix} c \\ d \end{pmatrix}$, then the total state of the two qubit system is: $\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$, in a 4D Hilbert space.

Now, you can also write this same equation in terms of the $|0\rangle$, $|1\rangle$ notation:

$$\left[a|0\rangle + b|1\rangle\right] \otimes \left[c|0\rangle + d|1\rangle\right] = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

Notice the new notation: The state $|00\rangle$ is just $|0\rangle \otimes |0\rangle$, etc.

You can also tensor product single-qubit operators together to make two-qubit operators. Basically, you just multiply *each* element of the first matrix by the *entire* second matrix! This clearly will make a bigger matrix:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \otimes \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} AE & AF & BE & BF \\ AG & AH & BG & BH \\ CE & CF & DE & DF \\ CG & CH & DG & DH \end{pmatrix}$$ (see the pattern??)

If the 2x2 matrix **M** is an operator on one qubit, this clearly can't operate on a 4D Hilbert space. But the operator $\mathbf{M} \otimes \mathbf{I}$ (where **I** is the 2x2 identity) represents a measurement of **M** on qubit #1; it's a 4x4 operator. The operator $\mathbf{I} \otimes \mathbf{M}$ is what you would use to measure **M** on qubit #2.

**Two-Qubits: Entangled States**

Not all states in 4D Hilbert spaces can be separated into two distinct single-particle states as in the above example. If they are in the form $\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$, the states are "separable"; otherwise they are "entangled". Entangled states are mathematically possible because you can add up superpositions of separable states, that no longer neatly split into two qubits. We'll try to make sense of them later.

The most general two-qubit state can be written:

$$A|00\rangle + B|01\rangle + C|10\rangle + D|11\rangle$$

One easy way to see if such a state is separable is if the quantity 2|AD-BC|=0. This quantity is called the "Concurrence", and is a measure of entanglement. (The maximum possible value is 2|AD-BC|=1; this occurs for a "maximally entangled state".)

**Two-Qubit Gates**

A "controlled not" or "CNOT" is an example of a gate that acts on a 2-qubit state. This operation can turn separable states into entangled states (and vice-versa) by swapping the bottom two values in this 4D Hilbert space:

$$\begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} \Rightarrow (CNOT) \Rightarrow \begin{pmatrix} e \\ f \\ h \\ g \end{pmatrix}$$

Useful Exercises: What 4x4 matrix would do this? Find a separable state that turns into a maximally-entangled state under the CNOT operation.

Obviously, two consecutive CNOTs give you back the original state.

Another two-qubit gate is the SWAP gate, which effectively swaps the two qubits.

$$\begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} \Rightarrow (SWAP) \Rightarrow \begin{pmatrix} e \\ g \\ f \\ h \end{pmatrix}$$

Useful Exercise: Check that this works as advertised for a separable state; also figure out what the 4x4 matrix might look like.

The SWAP gate cannot be used to entangle separable states; it just swaps them. But the root-SWAP, or $\sqrt{SWAP}$ gate, can do this:

$$\sqrt{\mathrm{SWAP}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(1+i) & \frac{1}{2}(1-i) & 0 \\ 0 & \frac{1}{2}(1-i) & \frac{1}{2}(1+i) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Useful Exercise: Check that two of these gives you a SWAP!

**Two Particle Measurements:**

Even if you have an entangled state, you can of course choose to measure each qubit separately. Say you choose a measurement direction for qubit 1 and a measurement direction for qubit 2. There are two possible outcomes for each direction, so there are 4 possible combined outcomes. (You would build such an operator using the Tensor Product: see above.) But finding the eigenstates of a 4x4 operator is hard; it's almost always easier to start with the two 2x2 operators separately.

The possible outcomes for qubit 1 are orthogonal eigenstates of the qubit-1 measurement operator, and so can always be represented by

$$|\psi_{1+}\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, \; |\psi_{1-}\rangle = \begin{pmatrix} b^* \\ -a^* \end{pmatrix}$$ (see why these are always orthogonal?)

You can generate the first of these from the angles of the measurement setting (see first page, "The Bloch Sphere"), where the "+" outcome has Bloch sphere vector that is aligned with the setting and the "-" outcome has a vector that is anti-aligned with the setting.

In the same way, the possible outcomes for qubit 2 are determined by a **separate** setting choice; call those:

$$|\psi_{2+}\rangle = \begin{pmatrix} c \\ d \end{pmatrix}, \; |\psi_{2-}\rangle = \begin{pmatrix} d^* \\ -c^* \end{pmatrix}.$$

So the 4 measurement eigenstates of the full 4x4 operator can be generated via a tensor product. For example, the outcome "++" (both outcomes aligned with the corresponding setting) is

$$|\psi_{1+}\rangle \otimes |\psi_{2+}\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} = |++\rangle.$$

Given the actual state in the full 4D Hilbert space, the probability of the outcome of this "++" state can be found from the generalized Born rule: $|\langle ++|\psi\rangle|^2$.

If the states are normalized, the 4 probabilities will always add to one.

**Marginal Probabilities**

We're often interested in the probabilities of a measurement of one particular particle, independent of what happens to the other one. (The quantum no-signalling theorem says that the net probabilities of one particle can't depend on the choice of measurement setting at the other particle.) So to find the probability of a + outcome of qubit#1, we can set the qubit #2 measurement setting to anything we want, say the z-axis. Then c=1, d=0. So the two possible outcomes where particle 1 is found to be "+" are:

$$|++\rangle = \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix}, \text{ and } |+-\rangle = \begin{pmatrix} 0 \\ a \\ 0 \\ b \end{pmatrix}.$$

To find the probability of the outcome "+" for qubit 1, we therefore need to calculate $|\langle ++|\psi\rangle|^2 + |\langle +-|\psi\rangle|^2$. In other words, we add up the probabilities of (+ on 1, + on 2) and (+ on 1, - on 2). The sum is just the total probability of measuring + on qubit 1.

We are about to learn an easier way to do this (and an easier way to make sense of the results), using something called a "partial trace" of a "density matrix".

**Density Matrices: Pure States**

Given the complete quantum state $|\psi\rangle$, it's possible to form a "density matrix" that encodes this state, using the equation:

$$\rho = |\psi\rangle\langle\psi|$$

This crazy-looking equation is called an "outer product"; you can figure out what it means just by plugging in the bra- and ket- in vector form. For a single qubit, this gives:

$$\rho = \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a^* & b^* \end{pmatrix} = \begin{pmatrix} aa^* & ab^* \\ a^*b & bb^* \end{pmatrix}.$$

Notice the trace of this matrix is 1. And it's automatically independent of the global phase on the original state. Also, notice that $\rho^2 = |\psi\rangle\langle\psi||\psi\rangle\langle\psi| = \rho$. For this case, when $\rho^2 = \rho$, we say this is a "pure state". (meaning, it is generated from a single, complete wavefunction). We're about to encounter other density matrices for which $\rho^2 \neq \rho$; these will be "mixed states". Every state is either pure or mixed.

**Pure State of Single Qubits**

$$|\psi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}$$ corresponds to the state of a point on the Bloch Sphere (recall).

Building the density matrix from this state one finds:

$$\rho = \begin{pmatrix} \cos^2(\theta/2) & \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi} \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{+i\phi} & \sin^2(\theta/2) \end{pmatrix}$$

Then, using the half-angle formulas, this can be written in terms of the Pauli Matrices:

$$\rho = \frac{1}{2}\left(\mathbf{I} + \sin\theta\cos\phi\,\sigma_x + \sin\theta\cos\phi\,\sigma_y + \cos\theta\,\sigma_z\right)$$

But this is just the Cartesian coordinates of the original vector on the Bloch Sphere! (Think about the conversion from spherical to Cartesian; those are the 3 terms.) So a pure state can be written as

$$\rho = \frac{1}{2}\left(\mathbf{I} + \hat{n}\cdot\sigma\right)$$

Where "*n*" is the unit vector on the Bloch sphere. Later we'll see that we can generalize this even when "*n*" is not a unit vector! This is one of the easiest ways to get from an arbitrary single-qubit wavefunction to the vector on the Bloch sphere -- especially if you don't like using spherical coordinates.

**Introduction to Mixed States**

The nice thing about density matrices is that they can be combined according to ordinary probability rules. Suppose you had a machine that made the state $|\psi_1\rangle$ 30% of the time, but made the different state $|\psi_2\rangle$ 70% of the time. (This is not a superposition! Just classical ignorance.) In this case, for any given particle, you wouldn't know for sure what the state was -- and yet, your job is to make predictions all the same. This can be done by simply weighting the possible density matrices (according to their probability), and adding them up into a single density matrix: $\rho = 0.3|\psi_1\rangle\langle\psi_1| + 0.7|\psi_2\rangle\langle\psi_2|$. See how that works? You just weight the possible density matrices and add them up. It turns out that from this total density matrix you can make all the right predictions. Also, this is clearly a <u>mixed state</u>; $\rho^2 \neq \rho$. (But the trace of the mixed state density matrix is still 1.)

For any given mixed state density matrix, there are usually many ways to make it: different "ensembles". But even though those different ensembles may be made up of totally different quantum states, it turns out there is absolutely no way to experimentally distinguish the different ensembles, if they have the same density matrix! If you know the density matrix, you know all the probabilities that you can measure.

**Extracting Probabilities**

It's possible to use the rules from ordinary QM to show the expectation value of any operator Q is simply:

$\langle Q\rangle = \mathrm{Trace}(\rho Q)$

In other words, you simply multiply the density matrix times the operator matrix, and take the trace. This always works, even for mixed states! (There's also a collapse rule we may get to later.)

In principle, knowing all the expectation values gives you all the probabilities, but we'll get to exact probabilities later. (maybe!)

**Time Evolution**

The Schrodinger Equation also looks quite nice when written in terms of density matrices These matrices evolve with time, of course: $\rho(t)$. It solves the equation:

$$i\hbar\frac{d\rho}{dt} = [H,\rho]$$

Yes, that's a commutator! If H is constant (as usual, for us), the solution is:

$$\rho(t) = e^{-iHt/\hbar}\rho(t=0)e^{+iHt/\hbar}$$

We won't use this solution much, so don't worry about it too much. But see Griffiths problem 4.56 if you're wanting to understand what it means to have an operator in the exponent.

**Gate Evolution**

From the definition of the density matrix, as well as the known action of a gate operator **R** on $|\psi\rangle$, it's easy to show that if you put a density matrix into a gate **R**, the output density matrix will be simply: $\rho_{output} = \mathbf{R}\rho\mathbf{R}^{\dagger}$. Don't forget that last matrix needs to be Hermitian-conjugated.

**Partial Traces: How to describe a smaller piece of a multi-qubit system.**

Consider a two qubit system, $A|00\rangle + B|01\rangle + C|10\rangle + D|11\rangle$. You should be able to form the 4x4 density matrix of the whole system; it will be a pure state.

But then, as before, you may ask a question about the probabilities of a measurement on just qubit #1. If you want to describe this qubit alone, independently of the other one, it would be nice to have that qubit's 2x2 density matrix. This can be extracted from the full 4x4 matrix; the procedure is to take the "partial trace" of the full 4x4 matrix.

Basically, if you want to find the $|0\rangle\langle 0|$ entry of qubit #1's density matrix, you need to add up the $|00\rangle\langle 00|$ and the $|01\rangle\langle 01|$ components of the full 4x4 matrix. In this case, you'll get AA*+BB*. (Get it? You don't care about the second qubit, so you try both options, while keeping the first qubit fixed.) If you want to find the $|0\rangle\langle 1|$ component, you add up the $\sum_n |0n\rangle\langle 1n|$ components, or AC*+BD*.

It works the other way, too To find the $|0\rangle\langle 0|$ entry of qubit #2's density matrix, you add up the $|00\rangle\langle 00|$ and the $|10\rangle\langle 10|$ components of the full 4x4 matrix (keeping the second qubit fixed, summing over all possible entries for the first one.) If you want to find the $|a\rangle\langle b|$ component for qubit #2, you add up the $\sum_n |na\rangle\langle nb|$ components.

**Pieces of Entangled Systems act like Mixed States**

If the original two-qubit state was separable, this partial trace procedure ends up with the proper two states for the two qubits. In this case, these would be pure states.

But if the original two-qubit state was entangled, this can't possibly work, because the state doesn't factor into two pure states. So what happens? You end up with two *mixed* states! Pieces of entangled systems act just like Mixed States!

When you find the partial trace corresponding to a single qubit, it turns out you can still always write it in the earlier form:

$$\rho = \frac{1}{2}\left(\mathbf{I} + \mathbf{n}\cdot\sigma\right)$$

Only now "**n**" is no longer a unit vector. (It's a unit vector when it's a pure state, just not when it's a mixed state.) But the Bloch sphere picture is still useful: the mixed state is a vector *inside* the sphere!! (Or, in the "Bloch Ball".) Knowing which way it's pointing is still useful, as it knowing its length. Maximally-entangled states wind up at the very center of the Bloch Ball, with **n**=0.

However, knowing how the two individual qubits might be measured, independently, is not the whole story. You've lost some information in the partial-trace process. What's missing is the *correlations* between the two qubits, which can only be gleaned from the full 4x4 density matrix. And this brings us to the topic of "Entanglement".

**Entanglement**

For a generic pure 2-qubit state, $A|00\rangle + B|01\rangle + C|10\rangle + D|11\rangle$ , can't always be separated into two pure single qubits. One way to see this is to generate the partial trace for each qubit, and see if it lies on the surface of the Bloch sphere or not (see previous page). Another way is to compute the "Concurrence", a measure of entanglement:

Concurrence = 2|AD-BC|.

In the homework you'll show that for each individual qubit, the magnitude of its Bloch Ball vector **n** is related to the concurrence via:

$$\mathbf{n}^2 + \left(2|AD - BC|\right)^2 = 1$$

From this it should be clear that when the concurrence is zero the state is separable (not entangled at all). When the concurrence is maximum, 1, the state is "maximally entangled", and each qubit's Bloch Ball vector is zero (at the center of the Bloch Ball). In this case, you have no information about each individual qubit; it could be measured in any direction, with equal probability.

But even in this case, not all the information has been lost! If you know the full pure state of the 2-qubit system, $A|00\rangle + B|01\rangle + C|10\rangle + D|11\rangle$ , it's true you know nothing about each individual qubit, but you know everything about the *relationship* between the two qubits. The clearest example is that of a singlet state,

$\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$. (We've seen this in chapter 4 of Griffiths; it's the spin-0 state for a 2 spin-1/2 system.)

For this system, we know that a measurement of any spin direction of qubit #1 should "collapse" qubit #2 into the opposite direction. Note: this is not to say the results of two measurements on the two qubits will always be opposite, because the experimenters might choose to measure different orientations. If they are measured in the same orientation, they will always be opposite. But if they are measured (say), one with a z-measurement and one with an x-measurement, then the outcomes will be randomly correlated. (See why? A z-measurement of #1 collapses #2 into a z-state, but then an x-measurement on any z-state gives a random result.)

**No-Signaling**

The discussion in the previous paragraph made it sound like that qubit #2 is being affected by a measurement on qubit #1, in that #2 is "collapsing" into a particular state based on the outcome of the #1 measurement. That's certainly what the mathematics of QM seems to be implying: there's no "local" description of the separate two qubits, even at the fundamental level that QM supposedly provides. This raises two questions: might there be a lower-level description, below QM, that has a local description? And if not, if nature is really "non-local", does that mean distant experimenters can signal to each other, using entangled particles?

We'll tackle the last question first, where the answer is a clear "no". If Alice is the experimenter at qubit #1, and Bob is the experimenter at qubit #2, they each have a choice of what measurement to make. But no matter what Alice chooses, it doesn't change the partial trace of Bob's qubit, and vice-versa. Sure, the *outcome* of Alice's measurement seems to change Bob's qubit, but not the *choice*; and since Alice can't control her outcome, she can't signal.

For example, consider a separated singlet state. If Alice measures the spin component on any axis, 50% of the time she'll get one direction, and 50% of the time she'll get the other. This supposedly collapses Bob's state to just the opposite outcome; 50% each way, where this is just classical uncertainty for Bob at this point: ordinary ignorance. But no matter *what* direction these states are pointing, building Bob's mixed state from such a 50/50 (classical!) uncertainty mix always yields the same result:

$$50\%|0\rangle\langle 0| + 50\%|1\rangle\langle 1| = \frac{1}{2}\mathbf{I} \quad \text{(Center of the Bloch Ball)}$$

In other words, Alice hasn't done anything to change Bob's density matrix: before she measured it was a state of maximum uncertainty (in the middle of the Bloch Ball) and after she measured it's still a state of maximum uncertainty. ALICE may know Bob's state, but Bob doesn't know it until he hears the results of Alice's measurement. More importantly, he can't tell by measuring his own state, because all the probabilities are entirely determined by his own density matrix, which hasn't changed.

There exists a general "no-signaling theorem": Alice can never signal to Bob using entanglement.

**Bell Inequalities**

Just because QM gives us a non-local account of the correlations between Alice and Bob doesn't mean that QM must be correct. Perhaps there *is* a local description, and QM is just a statistical theory governing that deeper level of reality. This is certainly what Einstein thought, when presented with QM.

In fact, there seems to be an obvious way to account for having correlations between the two qubits: a common cause, or common "Hidden Instructions" sent along to Alice and Bob through the particles. (Like two opposite gloves sent through the mail, at random: it's not surprising that when Alice and Bob receive their package, they always have opposite gloves!) Now, QM does not tell us anything about what these "hidden instructions" might look like, but they might be there all the same, hiding in the particles (just as the actual state of the gloves were hiding in the packages).

Or so people thought, before John Bell showed that there is a limit on correlations that can be achieved in this manner. As detailed in class and Chapter 3 of the accompanying text, one such limit is something called the CHSH-inequality. Say Alice can measure either property a1 or a2 where the values of these properties are always either +1 or -1; numbers assigned to the 2-outcomes of a measurement like a spin orientation. (For example, removing the $\hbar/2$ from the value of a spin-component measurement, leaving only the sign of the outcome.) Bob can measure either property b1 or b2. (Also +1 or -1 in each case.)

If the original distribution of these variables comes from some initial source with an arbitrary "joint probability distribution" P(a1,a2,b1,b2), where the 16 possible initial distributions are each assigned some "generation probability" by the source of the entangled particles, these hidden instructions could then be sent to Alice and Bob along with the particles themselves. But even then, it's possible to prove the CHSH inequality:

$$\left|\langle a_1 b_1\rangle+\langle a_1 b_2\rangle+\langle a_2 b_1\rangle-\langle a_2 b_2\rangle\right|=S\leq 2 \qquad \text{(CHSH Inequality)}$$

Here the <>'s are expectation values (weighted averages), over many identical measurements. Of course, since Alice and Bob each have to choose which property to measure, and so does Bob, they can't measure all 4 of these in a single run. Any given run only gives a piece of information towards one of these 4 terms. (Alice isn't allowed to measure both a1 and a2 at the same time, any more than she could measure two spin orientations at the same time.)

But this inequality is *violated* by Quantum Experiments! In particular, if you start with an entangled singlet state, and Alice chooses between measuring spin on the x-axis or z-axis, while Bob chooses between measuring spin on the $(\hat{x}\pm\hat{z})/\sqrt{2}$ axes, one finds that S can be a whopping $2\sqrt{2}$! (That's the maximum possible value of S, even in QM; something called "Tsirelson's Bound".)

This means that you can't account for such experiments using common-causes and hidden instructions, no matter what details you try to dream up. If there is a local account of entanglement experiments, it either has to be:

1) Based on things that don't live in ordinary spacetime (Inherently non-local)
2) Using Faster-than-Light influences (Effectively non-local)
3) Restrictive on what Alice and Bob are allowed to measure in any given run (Superdeterministic)
4) Local hidden variables that depend on Alice and Bob's future settings (Retrocausal)

For this last case, the "hidden variables" would not be "instructions"; they would in fact have different initial joint probability distributions P(a1,a2,b1,b2), depending on what Alice and Bob will eventually choose to measure. It is a key assumption of all Bell-inequalities that such a probability distribution should *not* depend on future events. If you relax this assumption, it is trivial to violate the CHSH inequality using local parameters.

**Manipulating Entangled States: Single Qubit Operations**

Given a two-qubit state, the operations you can perform on it depend on whether you have access to both qubits (in the same place). If you only have one qubit of an entangled pair (say, qubit#1), you can put that qubit through a single qubit gate/operator: some 2x2 matrix R. (This is not a measurement; just some unitary transformation.) As described above in this document, that means the operation on the full state would be:

$$\mathbf{R} \otimes \mathbf{I}$$

Here "**I**" is the 2x2 identity for qubit #2. Obviously, if you were operating on qubit #2, you would swap the R and I in the above expression.

Such a single-qubit operation is limited in several ways:

1) You can't change the partial trace of the qubit you aren't operating on. (That's the no-signaling theorem!)

2) You can't change the concurrence of the full state. (Which follows from #1, since knowledge of **n** gives you knowledge of the concurrence. )

But you *can* change the relationship between the two states, which hopefully makes sense. Say you were in a singlet state, where you knew the two spins would be measured in an opposite orientation. Then you take one qubit and rotate it by a $180^o$ rotation around the z-axis. That obviously changes the relationship between the two qubits! Now a measurement in the x- or y- direction (on both qubits) would yield the *same* result, not opposite results! In the homework you will show that such local transformations can take you from one maximally-entangled state to another.

**Manipulating Entangled States: Two-Qubit Operations**

If both qubits are together, you can put them through a 2-qubit gate, as described above, such as a CNOT or a SWAP. Such gates can in general change the concurrence, and can be used to entangle states that are (initially) separable.

Such operations can also be used to implement "joint measurements" on a pair of qubits. Suppose you want to measure whether a pair of spin-1/2 particles is in a spin-0 (singlet) state, but you only want to measure one particle at a time. The solution would be to put the two qubits into a CNOT gate, and then measure the two particles separately (in the correct basis: an x-measurement on Qubit #1 and a z-measurement on Qubit #2, as it turns out). That's because:

$$\textbf{CNOT}\ |\text{singlet}> = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In other words, the CNOT operation projects any singlet component of the original state into two separable qubits whose first qubit is in the -x direction (on the Bloch sphere) and whose second qubit is in the -z direction. A simple pair of single-qubit measurements (after a CNOT) is effectively the same as a total-spin measurement (before the CNOT).

Of course, you might find any of 4 results from this measurement (2 options for qubit #1, 2 options for qubit #2). The other 3 cases correspond to other initial components of the entangled state. Specifically, the 4 possible measurement outcomes each correspond to an initial maximally-entangled state in the so-called "Bell-Basis". There will be a homework question about this basis.