Exercise 1. $\mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R}^{m+n})$. (We adopt the usual convention that $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$.)

Solution 1. $\subseteq$:

Remember $\mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^n)$ is the smallest $\sigma$-algebra containing all $\mathcal{B}(\mathbb{R}^m) \times \mathcal{B}(\mathbb{R}^n)$-measurable rectangles (i.e., sets of the form $A \times B$ where $A \in \mathcal{B}(\mathbb{R}^m)$ and $B \in \mathcal{B}(\mathbb{R}^n)$). If we show that $\mathcal{B}(\mathbb{R}^{m+n})$ contains all the $\mathcal{B}(\mathbb{R}^m) \times \mathcal{B}(\mathbb{R}^n)$-measurable rectangles, then we will get the desired containment. Consider the projection maps $\pi_1 : \mathbb{R}^{m+n} \to \mathbb{R}^m$ and $\pi_2 : \mathbb{R}^{m+n} \to \mathbb{R}^n$ defined by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. If $(x_n, y_n) \to (x, y)$, then $\lim_n \pi_1(x_n, y_n) = \lim_n x_n = x = \pi_1(x, y)$. Thus $\pi_1$ is continuous. There $\pi_1$ is $(\mathcal{B}(\mathbb{R}^m), \mathcal{B}(\mathbb{R}^m))$-measurable (adapt the proof of Theorem 4.11 of the Lecture Notes). Similarly, $\pi_2$ is $(\mathcal{B}(\mathbb{R}^{m+n}), \mathcal{B}(\mathbb{R}^n))$-measurable. Now note that

$$A \times B = (A \times \mathbb{R}^n) \cap (\mathbb{R}^m \times B) = \pi_1^{-1}(A) \cap \pi_2^{-1}(B)$$

for all $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$. Therefore, if $A \in \mathcal{B}(\mathbb{R}^m)$ and $B \in \mathcal{B}(\mathbb{R}^n)$, we have $\pi_1^{-1}(A), \pi_2^{-1}(B) \in \mathcal{B}(\mathbb{R}^{m+n})$, and hence $A \times B \in \mathcal{B}(\mathbb{R}^{m+n})$. Thus every $\mathcal{B}(\mathbb{R}^m) \times \mathcal{B}(\mathbb{R}^n)$-measurable rectangle belongs to $\mathcal{B}(\mathbb{R}^{m+n})$. Done.

$\supseteq$:

By definition, $\mathcal{B}(\mathbb{R}^{m+n})$ is the smallest $\sigma$-algebra containing all the open sets in $\mathbb{R}^{m+n}$. So if we show that $\mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^n)$ also contains all the open sets in $\mathbb{R}^{m+n}$, then we will get the desired containment. We need the following lemma. **Lemma.** Every open set in $\mathbb{R}^k$ is the union of a countable collection of rectangles of the form $R = I_1 \times \cdots \times I_k$ where $I_i$ are finite open intervals in $\mathbb{R}$.

The proof of this lemma is a simple modification of the proof of Theorem 3.9 of the Lecture Notes. Let $U$ be an arbitrary open set in $\mathbb{R}^{m+n}$. By the lemma, $U$ is a countable union of sets of the from $R = I_1 \times \cdots \times I_{m+n}$, where $I_i$ are open intervals in $\mathbb{R}$. For each such $R$, we have

$$R = (I_1 \times \cdots \times I_m) \times (I_{m+1} \times \cdots \times I_{m+n})$$

$I_1 \times \cdots \times I_m$ is an open set in $\mathbb{R}^m$ and hence a Borel set in $\mathbb{R}^m$, and $I_{m+1} \times \cdots \times I_{m+n}$ is an open set in $\mathbb{R}^n$ and hence a Borel set in $\mathbb{R}^n$; Thus $R \in \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^n)$. As $U$ is a countable union of such sets $R$, we have that $U \in \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^n)$. Done.

Exercise 2. Recall the following theorem.

**Theorem.** Let $\mathcal{S}$ be a $\sigma$-algebra on a set $X$. Let $\mathcal{T}$ be a $\sigma$-algebra on a set $Y$. If $E \in \mathcal{S} \otimes \mathcal{T}$, then $E_x \in \mathcal{T}$ for all $x \in X$ and $E^y \in \mathcal{S}$ for all $y \in Y$.

Give a counterexample that shows the converse is not true when $\mathcal{S} = \mathcal{T} = \mathcal{B}(\mathbb{R})$. That is, find a set $E \subseteq \mathbb{R} \times \mathbb{R}$ such that $E_x \in \mathcal{B}(\mathbb{R})$ for all $x \in X$ and $E^y \in \mathcal{B}(\mathbb{R})$ for all $y \in Y$ but $E \notin \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

Solution 2. Let $V$ be a subset of $\mathbb{R}$ that is not in $\mathcal{B}(\mathbb{R})$. See Cohn Theorem 1.4.9 for an example of such a set. Define $E = \{(v, v) \in \mathbb{R}^2 : v \in V\}$. For each $x \in \mathbb{R}$, we have $E_x = \{x\} \in \mathcal{B}(\mathbb{R})$ if $x \in V$ and $E_x = \emptyset \in \mathcal{B}(\mathbb{R})$ if $x \notin V$. Thus $E_x \in \mathcal{B}(\mathbb{R})$ for each $x \in \mathbb{R}$. Likewise $E^y \in \mathcal{B}(\mathbb{R})$ for each $y \in \mathbb{R}$. The map $d : \mathbb{R} \to \mathbb{R}^2$ defined by $d(x) = (x, x)$ is continuous. Indeed, if $x_n \to x$ in $\mathbb{R}$, then

$$|d(x_n) - d(x)| = |(x_n, x_n) - (x, x)| = |(x_n - x, x_n - x)| = \sqrt{|x_n - x|^2 + |x_n - x|^2} = \sqrt{2}|x_n - x| \to 0.$$
Since $d$ is continuous, $d$ is $\mathcal{B}(\mathbb{R})$-measurable. Seeking a contradiction, suppose $E \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^2)$. Then $d^{-1}(E) \in \mathcal{B}(\mathbb{R})$. But $d^{-1}(E) = V \notin \mathcal{B}(\mathbb{R})$. Contradiction. Thus $E \notin \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^2)$.

**Exercise 3.** Let $\lambda^1$ be Lebesgue measure on $\mathcal{L}(\mathbb{R})$. Let $\lambda^2 = \lambda^1 \times \lambda^1$ be Lebesgue measure on $\mathcal{L}(\mathbb{R}^2)$. Let $f : \mathbb{R} \to [0, \infty]$ be Lebesgue measurable (i.e., $\mathcal{L}(\mathbb{R})$-measurable). Define

$$U_f = \{(x, y) \in \mathbb{R}^2 : 0 \leq y < f(x)\}$$

Then $U_f$ is Lebesgue measurable (i.e., $U_f \in \mathcal{L}(\mathbb{R})$) and

$$\int f(x)d\lambda^1(x) = \lambda^2(U_f) = \int_{[0, \infty)} \lambda^1(\{x \in \mathbb{R} : 0 \leq y < f(x)\})d\lambda^1(y)$$

Hint: For the measurability of $U_f$, note that the map $(x, y) \mapsto f(x) - y$ is the composition of the map $(x, y) \mapsto (f(x), y)$ and the map $(u, v) \mapsto u - v$.

Remark: The set $U_f$ is the “region under the graph of $f$.” So $\int f(x)d\lambda^1(x) = \lambda^2(U_f)$ is the familiar statement from calculus, “the integral of a function is the area under its graph.”

**Solution 3.** Note that

$$U_f = \{(x, y) : F(x, y) < 0\} = F^{-1}(0, \infty)$$

where $F : \mathbb{R}^2 \to \mathbb{R}$ is the map defined by $(x, y) \mapsto f(x) - y$. As in the hint, note that $F$ is the composition of the map $G : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (f(x), y)$, and the map $H : \mathbb{R}^2 \to \mathbb{R}$, $(u, v) \mapsto u - v$. Let us check that $G$ is $(\mathcal{L}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}^2))$-measurable. By modifying the proof of Theorem 3.9 of the Lecture Notes, we can see that $\mathcal{B}(\mathbb{R}^2)$ is generated by the collection of open rectangles, i.e., the sets form $(a, b) \times (c, d)$. Thus it suffices to show that $G^{-1}((a, b) \times (c, d)) \in \mathcal{L}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ for every such open rectangle. We have $G^{-1}((a, b) \times (c, d)) = f^{-1}((a, b)) \times (c, d)$. Since $f$ is $\mathcal{L}(\mathbb{R})$-measurable, $f^{-1}((a, b)) \in \mathcal{L}(\mathbb{R})$. Of course, $(c, d) \in \mathcal{B}(\mathbb{R})$. Therefore

$$G^{-1}((a, b) \times (c, d)) = f^{-1}((a, b)) \times (c, d) \in \mathcal{L}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}).$$

Thus $G$ is $(\mathcal{L}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}^2))$-measurable. Since $H$ is continuous, it is $(\mathcal{B}(\mathbb{R}^2), \mathcal{B}(\mathbb{R}))$-measurable (adapt the proof of Theorem 4.11 of the Lecture Notes). Now Theorem 4.13 of the Lecture Notes implies that $F = H \circ G$ is $(\mathcal{L}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$-measurable. Now we see that

$$U_f = \{(x, y) : F(x, y) < 0\} = F^{-1}(0, \infty) \in \mathcal{L}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}).$$

Theorem 31.1(f) of the Lecture notes implies $\mathcal{L}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R}) \times \mathcal{L}(\mathbb{R}) \subseteq \mathcal{L}^2$, so $U_f$ is Lebesgue measurable, as desired.

Note that $U_f^y = \emptyset$ if $y \notin [0, \infty)$ and $U_f^y = \{x \in \mathbb{R} : 0 \leq y < f(x)\}$ if $y \in [0, \infty)$. By applying the Fubini-Tonelli theorem (Theorem 31.2) to $1_{U_f}$, we get that

$$\lambda^2(U_f) = \int 1_{U_f}d\lambda^2 = \int \int 1_{U_f^y}(x)d\lambda^1(x)d\lambda^1(y) = \int \lambda^1(U_f^y)\lambda^1(y) = \int_{[0, \infty)} \lambda^1(\{x \in \mathbb{R} : 0 \leq y < f(x)\})d\lambda^1(y)$$
This is half of the formula we are aiming for. Now we prove the other half. Note that \( U_f^y = \{ y \in \mathbb{R} : 0 \leq y < f(x) \} = [0, f(x)) \) for all \( x \in \mathbb{R} \). So if we now use the Fubini-Tonelli theorem (Theorem 31.2) to integrate first in \( y \) and then in \( x \), we get that

\[
\lambda^2(U_f) = \int 1_{U_f} d\lambda^2 = \int \int 1_{U_f}(y) d\lambda^1(y) d\lambda^1(x) = \int \lambda^1(U_f^x) d\lambda^1(x) = \int \lambda^1([0, f(x)]) d\lambda^1(x) = \int f(x) d\lambda^1(x).
\]

Exercise 4. Consider the Lebesgue measure \( \lambda \) on \( L(\mathbb{R}) \). Recall that HW6 Exercise 2 showed that

\[
\int_{(0, \infty)} \left| \frac{\sin x}{x} \right| = \infty \quad \text{and} \quad \int_{(0, \infty)} \frac{\sin x}{x} \quad \text{is not defined}.
\]

Show that

\[
\lim_{b \to \infty} \int_{(0, b)} \frac{\sin x}{x} = \frac{\pi}{2}
\]

by integrating \( e^{-xy} \sin x \) with respect to \( x \) and \( y \) in both orders and using Fubini and Tonelli to justify the equality of the integrals. Be careful when taking the limit.

Solution 4. We use \( dx \) and \( dy \) to abbreviate \( d\lambda(x) \) and \( d\lambda(y) \). As usual, \( \lambda^2 = \lambda \otimes \lambda \). Let \( (b_n) \subseteq \mathbb{R} \) be any sequence such that \( b_n \to \infty \). Without loss of generality, we assume \( b_n \geq 1 \) for all \( n \). By Tonelli’s theorem,

\[
\int |e^{-xy} \sin x 1_{(0, b_n)}(x) 1_{(0, \infty)}(y)| d\lambda^2(x, y) = \int \int |e^{-xy} \sin x 1_{(0, b_n)}(x) 1_{(0, \infty)}(y)| dy dx
\]

\[
\leq \int_0^\infty \int_0^1 e^{-xy} dx dy + \int_1^\infty \int_1^\infty e^{-xy} dx dy
\]

\[
= \int_0^1 \int_0^1 e^{-xy} dx dy + \int_1^\infty \frac{1}{y} e^{-xy} dy
\]

\[
< \infty.
\]

Therefore \( e^{-xy} \sin x 1_{(0, b_n)}(x) 1_{(0, \infty)}(y) \) is \( \lambda^2 \)-integrable. So, by Fubini’s theorem, we have

\[
\int_0^{b_n} \int_0^\infty e^{-xy} \sin x dy dx = \int_0^\infty \int_0^{b_n} e^{-xy} \sin x dx dy
\]  \hspace{1cm} (0.1)

By calculus, the integral on the left is

\[
\int_0^{b_n} \int_0^\infty e^{-xy} \sin x dy dx = \int_0^{b_n} e^{-xy} \sin x dx.
\]

Let \( g_n(y) \) denote the inner integral on the right of (0.1). That is,

\[
g_n(y) = \int_0^{b_n} e^{-xy} \sin x dx.
\]
So (0.1) becomes

$$\int_{0}^{b_n} \frac{\sin x}{x} \, dx = \int_{0}^{\infty} g_n(y) \, dy \quad (0.2)$$

By calculus,

$$g_n(y) = \frac{1}{1 + y^2} + \frac{1}{1 + y^2} e^{-b_n y} (-y \sin (b_n) - \cos (b_n))$$

Note

$$g_n(y) \mathbf{1}_{(0,\infty)}(y) \to \frac{1}{1 + y^2}$$

for all \( y \in \mathbb{R} \). Also, since \( b_n \geq 1 \), we have

$$|g_n(y) \mathbf{1}_{(0,\infty)}(y)| \leq \frac{1}{1 + y^2} (1 + ye^{-y} + e^{-y}) \mathbf{1}_{(0,\infty)}(y)$$

for all \( y \in \mathbb{R} \). So, letting \( n \to \infty \) in (0.2), the dominated convergence theorem implies that

$$\lim_n \int_{0}^{b_n} \frac{\sin x}{x} \, dx = \lim_n \int_{0}^{\infty} g_n(y) \, dy = \int_{0}^{\infty} \lim_n g_n(y) \, dy = \int_{0}^{\infty} \frac{1}{1 + y^2} \, dy = \arctan(y) \bigg|_{0}^{\infty} = \frac{\pi}{2}.$$ 

Since \((b_n)\) is arbitrary,

$$\lim_{b \to \infty} \int_{0}^{b} \frac{\sin x}{x} \, dx = \frac{\pi}{2}.$$