Exercise 1. Let $A$ be a countable subset of $\mathbb{R}$.

(a) $A$ is Borel measurable and Lebesgue measurable. In other words, $A \in \mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$.

(b) $\lambda(A) = 0$.

Exercise 2. ($\lambda^*$ is translation invariant.) For $A \subseteq \mathbb{R}$ and $t \in \mathbb{R}$, define $A + t = \{a + t : a \in A\}$. Then $\lambda^*(A + t) = \lambda^*(A)$.

Exercise 3. Prove in detail the first inequality in Lemma 25.2 of the Lecture Notes.

Exercise 4. (Improper Riemann and Lebesgue Integrals) Consider the measure space $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$. Let $a \in \mathbb{R}$. Let $f : [a, \infty) \to \mathbb{R}$.

(a) Assume $f(1_{[a,b]}$ is measurable and $f$ is Lebesgue integrable on $[a, b]$ (meaning $\int_{[a,b]} f$ is defined and finite) for every $b \geq a$. Assume that there exists an $M \in [0, \infty)$ such that $\int_{[a,b]} |f|d\lambda \leq M$ for every $b \geq a$. Then $f$ is Lebesgue integrable on $[a, \infty)$ (meaning $\int_{[a,\infty]} f$ is defined and finite) and

$$\int_{[a,\infty]} f d\lambda = \lim_{b \to \infty} \int_{[a,b]} f d\lambda.$$ 

(b) Assume $f$ is Riemann integrable on $[a, b]$ for every $b \geq a$. Assume that there exists an $M \in [0, \infty)$ such that $\int_{[a,b]} |f| \leq M$ for every $b \geq a$. Then $f$ is Lebesgue integrable on $[a, \infty)$ (meaning $\int_{[a,\infty]} f$ is defined and finite) and

$$\int_{[a,\infty]} f d\lambda = \lim_{b \to \infty} \int_{[a,b]} f d\lambda = \lim_{b \to \infty} \int_{a}^{b} f.$$ 

Exercise 5. Let $X$ be a set. Let $\mathcal{E}$ be a demi-ring on $X$. Let $\mu_0 : \mathcal{E} \to [0, \infty]$ be countably monotone and finitely additive. Let $\mu^*$ be the outer measure on $X$ generated by $\mu_0$. Define $\mathcal{E}_\delta$ to be the collection of all countable unions of sets in $\mathcal{E}$. In other words, $\mathcal{E}_\delta = \{\bigcup_{i=1}^{\infty} E_i : E_1, E_2, \ldots \in \mathcal{E}\}$. Define $\mathcal{E}_{\delta}^\ast$ to be the collection of all countable intersections of sets in $\mathcal{E}_\delta$. In other words, $\mathcal{E}_{\delta}^\ast = \{\bigcap_{i=1}^{\infty} E_i : F_1, F_2, \ldots \in \mathcal{E}_\delta\}$. Let $A \in \mathcal{P}(X)$.

(a) Assume there exist $E_1, E_2, \ldots \in \mathcal{E}$ such that $A \subseteq \bigcup_{i=1}^{\infty} E_i$. For every $\epsilon > 0$ there exists $G_\epsilon \in \mathcal{E}_\sigma$ such that $A \subseteq G_\epsilon$ and $\mu^*(A) \leq \mu^*(G_\epsilon) \leq \mu^*(A) + \epsilon$. Moreover, there exists $H \in \mathcal{E}_{\delta}^\ast$ such that $A \subseteq H$ and $\mu^*(A) = \mu^*(H)$.

(b) If $\mu^*(A) < \infty$, then the following are equivalent:

(i) $A \in M(\mu^*)$

(ii) For every $\epsilon > 0$ there exists $G_\epsilon \in \mathcal{E}_\sigma$ such that $A \subseteq G_\epsilon$ and $\mu^*(G_\epsilon \setminus A) \leq \epsilon$

(iii) There exists $H \in \mathcal{E}_{\delta}^\ast$ such that $A \subseteq H$ and $\mu^*(H \setminus A) = 0$.

(c) If $X$ is $\sigma$-finite with respect to $\mu_0$, then the following are equivalent:

(i) $A \in M(\mu^*)$

(ii) For every $\epsilon > 0$ there exists $G_\epsilon \in \mathcal{E}_\sigma$ such that $A \subseteq G_\epsilon$ and $\mu^*(G_\epsilon \setminus A) \leq \epsilon$.

(iii) There exists $H \in \mathcal{E}_{\delta}^\ast$ such that $A \subseteq H$ and $\mu^*(H \setminus A) = 0$. 

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Exercise 6. The following are equivalent:

(i) $A \in \mathcal{L}(\mathbb{R})$

(ii) For every $\epsilon > 0$ there is an open set $G_\epsilon$ such that $A \subseteq G_\epsilon$ and $\lambda^*(G_\epsilon \setminus A) \leq \epsilon$.

(iii) There is a Borel set $H$ such that $A \subseteq H$ and $\lambda^*(H \setminus A) = 0$.

(iv) For every $\epsilon > 0$ there is a closed set $C_\epsilon$ such that $C_\epsilon \subseteq A$ and $\lambda^*(A \setminus C_\epsilon) \leq \epsilon$.

(v) There is a Borel set $B$ such that $B \subseteq A$ and $\lambda^*(A \setminus B) = 0$. 