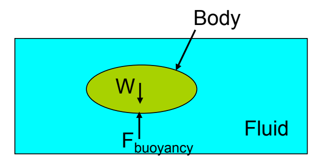
Natural Convection Heat Transfer

Net Force on a Body Completely in a Fluid

- The net force applied to a body completely submerged in a fluid is

$$F_{net} = W - F_{buoyancy}$$
$$= \rho_{body} g V_{body} - \rho_{fluid} g V_{body}$$
$$= (\rho_{body} - \rho_{fluid}) g V_{body}$$



- The body can be a bulk of hot fluid (same fluid but at a higher temperature).

$$F_{net} = (\rho_{hot\ fluid} - \rho_{fluid}) g V_{body}$$

- But $\rho_{hot\ fluid} < \rho_{fluid}$. Therefore F_{net} is negative and it means that the hot fluid will move up.

Volumetric Thermal Expansion Coefficient

- Rate of change of a unit volume of material per unit temperature change at constant pressure is called **volumetric thermal expansion coefficient**.

$$\beta = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_P$$

where v is the specific volume of the material.

- Since $v=1/\rho$ and therefore $\partial v = -\partial\rho/\rho^2$

$$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P$$

- **For an ideal gas**

$$Pv = RT \quad \Rightarrow \quad v = \frac{RT}{P} \quad \Rightarrow \quad \left(\frac{\partial v}{\partial T}\right)_P = \frac{R}{P}$$

$$\beta = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_P = \frac{1}{v}\frac{R}{P} = \frac{R}{Pv} = \frac{R}{RT} \quad \Rightarrow \quad \beta = \frac{1}{T}$$

Net Buoyancy Force and Temperature

- If V is the volume of a bulk of hot fluid, the net upward force applied to this bulk of fluid is

$$F_{net} = \Delta\rho g V$$

where $\Delta\rho = \rho_{hot\ fluid} - \rho_{fluid}$ is the density difference between the hot and cold fluids.

- A correlation between this force and temperature difference can be obtained by using an approximate expression for β.

$$\beta \approx -\frac{1}{\rho}\frac{\Delta\rho}{\Delta T} \quad \Rightarrow \quad \Delta\rho = -\rho\beta\Delta T$$

$$F_{net} = -\rho\beta\Delta T g V$$

- This shows that the larger the temperature difference between hot and cold fluids the larger the net upward force and therefore the stronger the natural convection.

Natural Convection Boundary Layer Equations

- Consider natural convection boundary layer along a vertical plate with temperature $T_s > T_\infty$.
- Since the body force in the x direction is X=-ρg, boundary layer equations are

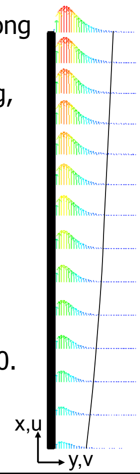
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} - g$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

- If there is no body force in y direction, ∂p/∂y=0. Therefore

$$\frac{\partial p}{\partial x} = -\rho_\infty g$$



Natural Convection Boundary Layer Equations

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} - g = \frac{\rho_\infty}{\rho}g - g = \frac{\rho_\infty - \rho}{\rho}g = \frac{-\rho\beta(T_\infty - T)}{\rho}g = g\beta(T - T_\infty)$$

- Therefore, boundary layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta(T - T_\infty) + \nu\frac{\partial^2 u}{\partial y^2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

Normalized Boundary Layer Equations

- Let's introduce normalized variables

$$x^* = \frac{x}{L} \quad y^* = \frac{y}{L} \quad u^* = \frac{u}{u_0} \quad v^* = \frac{v}{u_0} \quad T^* = \frac{T - T_\infty}{T_s - T_\infty}$$

where L is the length of the plate and u_0 is a characteristic velocity.

- The x-component momentum equation can be normalized as follows.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta(T - T_\infty) + \nu\frac{\partial^2 u}{\partial y^2}$$

$$u^* u_0 \frac{u_0}{L}\frac{\partial u^*}{\partial x^*} + v^* u_0 \frac{u_0}{L}\frac{\partial u^*}{\partial y^*} = g\beta(T_s - T_\infty)T^* + \nu\frac{u_0}{L^2}\frac{\partial^2 u^*}{\partial y^{*2}}$$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{g\beta(T_s - T_\infty)L}{u_0^2}T^* + \frac{\nu}{u_0 L}\frac{\partial^2 u^*}{\partial y^{*2}}$$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{g\beta(T_s - T_\infty)L^3}{\nu^2}\frac{\nu^2}{u_0^2 L^2}T^* + \frac{\nu}{u_0 L}\frac{\partial^2 u^*}{\partial y^{*2}}$$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{\mathrm{Gr}_L}{\mathrm{Re}_L^2}T^* + \frac{1}{\mathrm{Re}_L}\frac{\partial^2 u^*}{\partial y^{*2}}$$

Grashof and Rayleigh Numbers

- **Grashof number** is defined as

$$\mathrm{Gr}_\delta = \frac{g\beta(T_s - T_\infty)\delta^3}{\nu^2}$$

where δ is a characteristic length of the geometry, and T_s and T_∞ are surface and free stream temperatures.

- It can be shown that Grashof number is a measure of **ratio of buoyancy force to viscous force** acting on the fluid.
- **Rayleigh number** is the product of Grashof and Prandtl numbers.

$$\mathrm{Ra}_\delta = \mathrm{Gr}_\delta \cdot \mathrm{Pr}$$

Similarity Solutions of Boundary Layer Equations

- Boundary layer equations can be transformed into two ordinary differential equations by introducing similarity variable

$$\eta \equiv \frac{y}{x}\left(\frac{\mathrm{Gr_x}}{4}\right)^{1/4}$$

and representing the velocity components in terms of a stream function defined as

$$\psi(x,y) \equiv f(\eta)\left[4\nu\left(\frac{\mathrm{Gr_x}}{4}\right)^{1/4}\right]$$

- The two ordinary differential equations for f and T* are

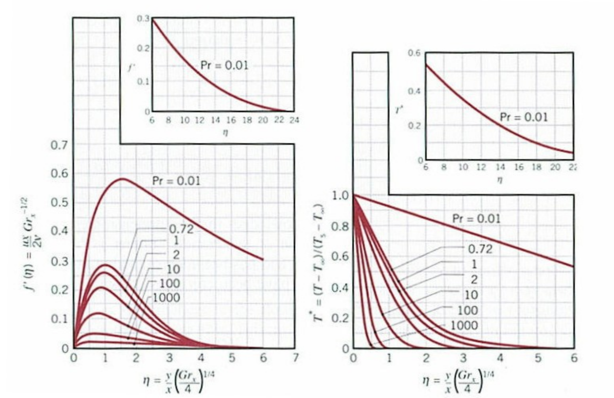
$$f''' + 3ff'' - 2(f')^2 + T^* = 0$$
$$T^{*\prime\prime} + 3\,\mathrm{Pr}\, fT^{*\prime} = 0$$

and the boundary conditions are

$$\eta = 0 \quad \Rightarrow \quad f = f' = 0 \quad T^* = 1$$
$$\eta \to \infty \quad \Rightarrow \quad f' \to 0 \qquad T^* \to 0$$

Similarity Solutions of Boundary Layer Equations

- Numerical solution of these equation are shown below.



Similarity Solutions of Boundary Layer Equations

- The local Nusselt number may be expressed as

$$\mathrm{Nu_x} = \frac{hx}{k} = \frac{[q_s''/(T_s - T_\infty)]x}{k}$$

where

$$q_s'' = -k\left.\frac{\partial T}{\partial y}\right|_{y=0} = -\frac{k}{x}(T_s - T_\infty)\left(\frac{\mathrm{Gr_x}}{4}\right)^{1/4}\left.\frac{dT^*}{d\eta}\right|_{\eta=0}$$

Therefore

$$\mathrm{Nu_x} = \frac{hx}{k} = -\left(\frac{\mathrm{Gr_x}}{4}\right)^{1/4}\left.\frac{dT^*}{d\eta}\right|_{\eta=0} = \mathrm{Gr_x}^{1/4} g(\mathrm{Pr})$$

$$g(\mathrm{Pr}) = \frac{3}{4}\left[\frac{2\,\mathrm{Pr}^2}{5(1 + 2\,\mathrm{Pr}^{1/2} + 2\,\mathrm{Pr})}\right]^{1/4}$$

- Average Nusselt number over the entire length of the plate, L, is

$$\overline{\mathrm{Nu}}_\mathrm{L} = \frac{\bar{h}L}{k} = \frac{4}{3}\left(\frac{\mathrm{Gr_L}}{4}\right)^{1/4} g(\mathrm{Pr}) = \frac{4}{3}\mathrm{Nu_L}$$

Integral Solution of Laminar Natural Convection Boundary Layer

- The integral form of the momentum and energy equations for natural convection boundary layers are

$$\frac{d}{dx}\int_0^y u^2 dy = -\nu\left.\frac{\partial u}{\partial y}\right|_{y=0} + g\beta\int_0^y (T - T_\infty)dy$$

$$\frac{d}{dx}\int_0^y u(T - T_\infty)dy = -\alpha\left.\frac{\partial T}{\partial y}\right|_{y=0}$$

- Let's assume the velocity and temperature profiles are given by

$$\frac{u}{U} = \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^2 \quad \text{and} \quad \frac{T - T_\infty}{T_s - T_\infty} = \left(1 - \frac{y}{\delta}\right)^2$$

U is a reference velocity such as maximum velocity in the boundary layer.

Integral Solution of Laminar Natural Convection Boundary Layer

- Substituting into the integral equations gives

$$\frac{1}{105}\frac{d}{dx}(U^2\delta) = -\nu\frac{U}{\delta} + \frac{1}{3}g\beta(T_s - T_\infty)\delta$$

$$\frac{1}{30}\frac{d}{dx}(U\delta) = \frac{2\alpha}{\delta}$$

- If we assume $U=C_1x^m$ and $\delta=C_2x^n$, it can be shown that m = 1/2 and n = 1/4 and

$$C_1 = 4\left(\frac{5}{3}\right)^{1/2}\nu\left(\frac{20}{21}+\Pr\right)^{-1/2}\left(\frac{\mathrm{Gr}_x}{x^3}\right)^{1/2} \quad \text{and} \quad C_2 = 4\left(\frac{15}{16}\right)^{1/4}\left(\frac{20}{21}+\Pr\right)^{1/4}\left(\frac{\mathrm{Gr}_x}{x^3}\right)^{-1/4}\Pr^{-1/2}$$

- Boundary layer thickness and Nusselt number are

$$\frac{\delta}{x} = 3.93\left(\frac{0.952+\Pr}{\Pr^2}\right)^{1/4} Gr_x^{-1/4} \quad \text{and} \quad \mathrm{Nu}_x = 0.508\left(\frac{\Pr}{0.952+\Pr}\right)^{1/4}\mathrm{Ra}_x^{1/4}$$

Integral Solution of Turbulent Natural Convection Boundary Layer

- The integral form of the momentum and energy equations for turbulent natural convection boundary layers are

$$\frac{d}{dx}\int_0^y \bar{u}^2 dy = -\frac{\tau_s}{\rho} + g\beta\int_0^y(\bar{T}-T_\infty)dy$$

$$\frac{d}{dx}\int_0^y \bar{u}(\bar{T}-T_\infty)dy = \frac{q_s''}{\rho c_p}$$

- Let' s assume the velocity and temperature profiles are given by

$$\frac{\bar{u}}{U} = \left(\frac{y}{\delta}\right)^{1/7}\left(1-\frac{y}{\delta}\right)^4 \quad \text{and} \quad \frac{\bar{T}-T_\infty}{T_s-T_\infty} = 1-\left(\frac{y}{\delta}\right)^{1/7}$$

- Assuming $U=C_1x^m$ and $\delta=C_2x^n$, it can be shown that

$$\mathrm{Nu}_x = 0.0295\frac{\Pr^{1/15}}{(1+0.494\Pr^{2/3})^{2/5}}\mathrm{Ra}_x^{2/5}$$

Empirical Correlations for Vertical Plates

- For a vertical plate with length L the characteristic length δ=L.
- The natural convection flow is **laminar if $\mathrm{Ra}_L<10^9$** and is **turbulent if $\mathrm{Ra}_L>10^9$**.
- Average Nusselt number correlations for an isothermal vertical plate are

$$\overline{\mathrm{Nu}}_L = 0.59\mathrm{Ra}_L^{1/4} \qquad 10^4 < \mathrm{Ra}_L < 10^9$$

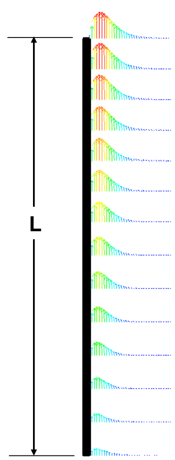
$$\overline{\mathrm{Nu}}_L = 0.1\mathrm{Ra}_L^{1/3} \qquad 10^9 < \mathrm{Ra}_L < 10^{13}$$

where

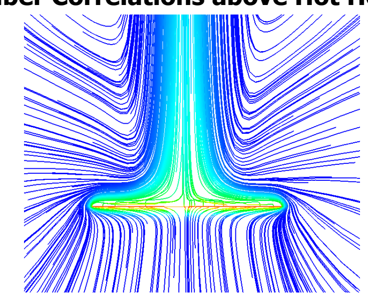
$$\overline{\mathrm{Nu}}_L = \frac{\bar{h}L}{k} \quad \text{and} \quad \mathrm{Ra}_L = \mathrm{Gr_L}\Pr = \frac{g\beta(T_s-T_\infty)L^3}{\nu\alpha}$$

- The following correlation may be used for the entire range of $\mathrm{Ra_L}$

$$\overline{\mathrm{Nu}}_\mathrm{L} = \left\{0.825 + \frac{0.387\mathrm{Ra}_L^{1/6}}{[1+(0.492/\Pr)^{9/16}]^{8/27}}\right\}^2$$



Nusselt Number Correlations above Hot Horizontal Plates

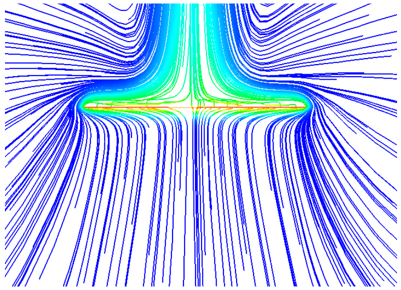


- For a horizontal plate the characteristic length is $\delta=A_s/P$ where A_s and P are surface area and perimeter of the plate.
- Average Nusselt number correlations are

$$\overline{\mathrm{Nu}}_\delta = 0.54\mathrm{Ra}_\delta^{1/4} \qquad 10^4 \le \mathrm{Ra}_\delta \le 10^7$$

$$\overline{\mathrm{Nu}}_\delta = 0.15\mathrm{Ra}_\delta^{1/3} \qquad 10^7 \le \mathrm{Ra}_\delta \le 10^{11}$$

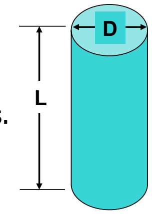
Nusselt Number Correlations under Hot Horizontal Plates



- The characteristic length is $\delta = A_s/P$ where A_s and P are surface area and perimeter of the plate.
- Average Nusselt number correlations are

$$\overline{Nu}_\delta = 0.27\,Ra_\delta^{1/4} \qquad 10^5 \le Ra_\delta \le 10^{11}$$

Nusselt Number Correlations for Cylinders



- **Vertical Cylinder**

  The characteristic length is $\delta = L$.

  If $D \ge 35L/Gr_L^{1/4}$, use vertical plate correlations.

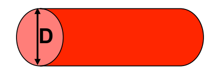
  If cylinder is thin, i.e. $D < 35L/Gr_L^{1/4}$,

$$\overline{Nu}_L = \frac{\bar{h}L}{k} = \frac{4}{3}\left[\frac{7Ra_L\,Pr}{5(20+21\,Pr)}\right]^{1/4} + \frac{4(272+315\,Pr)L}{35(64+63\,Pr)D}$$

- **Horizontal Cylinder**

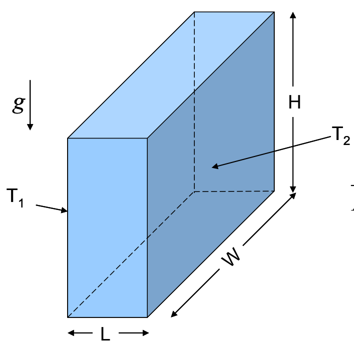
  The characteristic length is $\delta = D$.

  Average Nusselt number, if $Ra_D < 10^{12}$, is given by



$$\overline{Nu}_D = \left[0.6 + \frac{0.387\,Ra_D^{1/6}}{\left(1+(0.559/Pr)^{9/16}\right)^{8/27}}\right]^2$$

Natural Convection in Enclosures

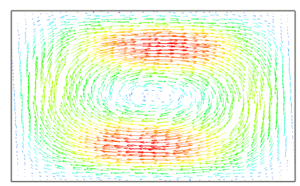


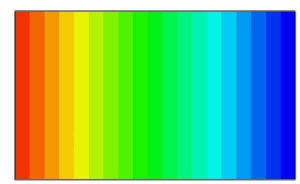
$$q''_{conv} = \bar{h}(T_1 - T_2)$$

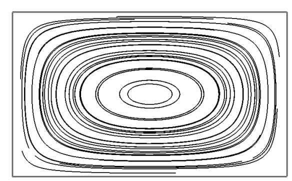
$$\overline{Nu}_L = \frac{q''_{conv}}{q''_{cond}} = \frac{\bar{h}(T_1 - T_2)}{k(T_1 - T_2)/L} = \frac{\bar{h}L}{k}$$

$$Ra_H = \frac{g\beta(T_1 - T_2)H^3}{\alpha\nu}$$

Natural Convection in Enclosures
(Conduction Limit)



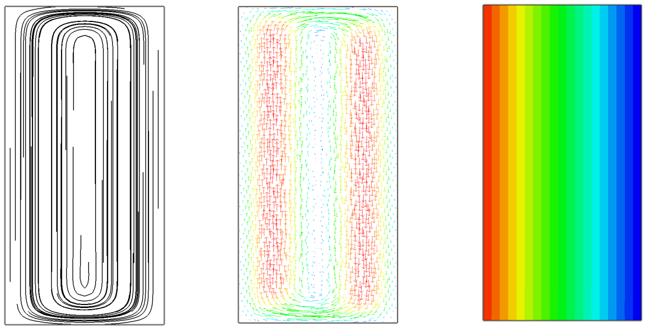




$$Ra_H \le 1$$

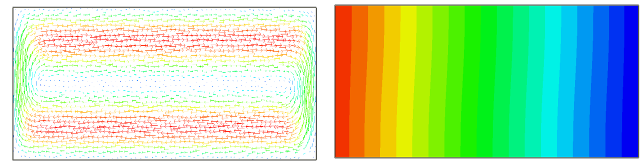
$$q''_{conv} = q''_{cond} = k(T_1 - T_2)/L$$

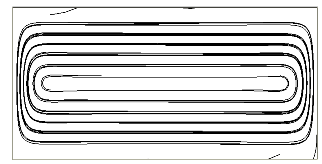
Natural Convection in Enclosures
(Tall Enclosure Limit)

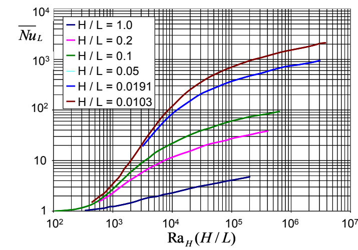


$$\frac{H}{L} > \mathrm{Ra}_H^{1/4} \qquad q''_{conv} = q''_{cond} = k(T_1 - T_2)/L$$

Natural Convection in Enclosures
(Shallow Enclosure Limit)

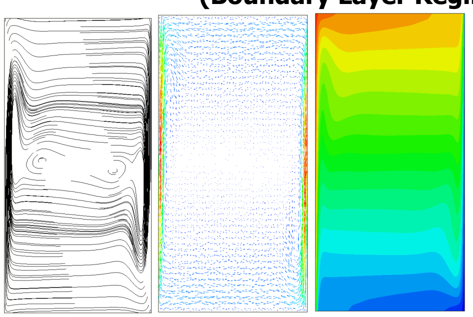






$$\frac{H}{L} < \mathrm{Ra}_H^{-1/4}$$

Natural Convection in Enclosures
(Boundary Layer Regime)



$$\mathrm{Ra}_H^{-1/4} < \frac{H}{L} < \mathrm{Ra}_H^{1/4}$$

$$\overline{Nu}_L = \frac{\bar{h}L}{k} = C_1 \frac{L}{H} \mathrm{Ra}_H^{1/4}$$