Orthogonal Functions and Fourier Series

Orthogonal Functions

- The **inner product** of two functions f_1 and f_2 on an interval $[a\,,\,b]$ is the number

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x)\,dx.$$

- Two functions f_1 and f_2 are said to be **orthogonal** on an interval $[a\,,\,b]$ if

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x)\,dx = 0.$$

- A set of real-valued functions $\{\phi_0(x),\ \phi_1(x),\ \phi_2(x),\ \ldots\}$ is said to be an **orthogonal set** on an interval $[a\,,\,b]$ if

$$(\varphi_m, \varphi_n) = \int_a^b \varphi_m(x)\varphi_n(x)\,dx = 0, \qquad m \neq n.$$

- Examples: The set $\{1,\ \cos x,\ \cos 2x,\ \ldots\}$ on the interval $[-\pi\,,\,\pi]$ is orthogonal.

Orthonormal Functions

- The **norm** or **generalized length** of a function is defined as

$$\|\varphi_n(x)\| = \sqrt{(\varphi_n, \varphi_n)} = \sqrt{\int_a^b \varphi_n^2(x)\,dx}.$$

- A set of orthogonal functions $\{\phi_0(x),\ \phi_1(x),\ \phi_2(x),\ \ldots\}$ that are normalized by their norms is called **orthonormal set**.

$$\left\{\frac{\varphi_0(x)}{\|\varphi_0(x)\|}, \frac{\varphi_1(x)}{\|\varphi_1(x)\|}, \frac{\varphi_2(x)}{\|\varphi_2(x)\|}, \cdots\right\}$$

- Examples: The set on the interval $\left\{1/\sqrt{2\pi}, \cos x/\sqrt{\pi}, \cos 2x/\sqrt{\pi}, \cdots\right\}$ $[-\pi\,,\,\pi]$ is orthogonal.

Orthogonal Series Expansion

- Suppose $\{\phi_n(x)\}$ is an infinite orthogonal set of functions on an interval $[a\,,\,b]$ and $y=f(x)$ is a function defined on this interval. Then,

$$f(x) = \sum_{n=0}^{\infty} c_n \varphi_n(x),$$

where

$$c_n = \frac{\int_a^b f(x)\varphi_n(x)\,dx}{\|\varphi_n(x)\|^2},$$

- This is called **orthogonal series expansion** of $f(x)$.

Orthogonal Functions with Weight Function

- A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is said to be **orthogonal with respect to a weight function $w(x)$** on an interval $[a , b]$ if

$$\int_a^b w(x)\varphi_m(x)\varphi_n(x)\,dx = 0, \qquad m \neq n.$$

- Suppose $\{\phi_n(x)\}$ is an infinite orthogonal set of functions on an interval $[a , b]$ and $y=f(x)$ is a function defined on this interval. Then,

$$f(x) = \sum_{n=0}^{\infty} c_n \varphi_n(x),$$

where

$$c_n = \frac{\int_a^b f(x)w(x)\varphi_n(x)\,dx}{\|\varphi_n(x)\|^2}, \qquad \|\varphi_n(x)\|^2 = \int_a^b w(x)\varphi_n^2(x)\,dx.$$

- This is called **orthogonal series expansion** of $f(x)$.

Fourier Series

- The set of trigonometric functions

$$\left\{1, \cos\frac{\pi}{p}x, \cos\frac{2\pi}{p}x, \cos\frac{3\pi}{p}x, \cdots, \sin\frac{\pi}{p}x, \sin\frac{2\pi}{p}x, \sin\frac{3\pi}{p}x, \cdots\right\}$$

is orthogonal on the interval $[-p , p]$.

- The **Fourier series** of a function f defined on the interval $(-p , p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)$$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx, \qquad a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx, \qquad b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx$$

Convergence of Fourier Series

- Let f and f' be piecewise continuous on the interval $(-p , p)$; that is f and f' be continuous except at a finite number of points in the interval and have only finite discontinuity at these points.
  - The Fourier series of f on the interval converges to $f(x)$ at a point of continuity.
  - At a point of discontinuity, the Fourier series converges to the average $[f(x^+)+f(x^-)]/2$ where $f(x^+)$ and $f(x^-)$ denote the limit of f at x from the right and from the left, respectively.

Class Exercise

- Find and plot Fourier series of

$$f(x) = \begin{cases} 0, & -\pi/2 < x < 0 \\ \cos x, & 0 \le x < \pi/2 \end{cases}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi/2} \cos x\,dx = \frac{2}{\pi}$$

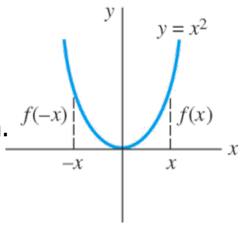
$$a_n = \frac{2}{\pi}\int_0^{\pi/2} \cos x \cos 2nx\,dx = \frac{2(-1)^{n+1}}{\pi(4n^2-1)}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi/2} \cos x \sin 2nx\,dx = \frac{4n}{\pi(4n^2-1)}$$

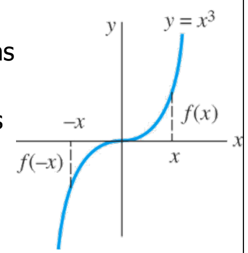
$$f(x) = \frac{1}{\pi} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^{n+1}}{\pi(4n^2-1)}\cos 2nx + \frac{4n}{\pi(4n^2-1)}\sin 2nx\right]$$

Even and Odd Functions

- A function $f(x)$ is said to be **even** if $f(-x) = f(x)$ and **odd** if $f(-x) = -f(x)$.



- The product of two even functions is even.
- The product of two odd functions is even.
- The product of an even function and an odd function is odd.
- The sum (difference) of two even functions is even.
- The sum (difference) of two odd functions is odd.
- If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$
- If f is odd, then $\int_{-a}^{a} f(x)dx = 0.$



Fourier Cosine and Sine Series

- The **Fourier series** of an even function f defined on the interval $(-p\ ,\ p)$ is a **cosine series** given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x$$

$$a_0 = \frac{2}{p}\int_0^p f(x)dx, \qquad a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx$$

- The **Fourier series** of an odd function f defined on the interval $(-p\ ,\ p)$ is a **sine series** given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x$$

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx$$

Complex Fourier Series

- **Euler's Formula:** For a real number x

$$e^{ix} = \cos x + i\sin x \qquad e^{-ix} = \cos x - i\sin x$$

- Solving for cos x and sin x gives

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

- Substituting into Fourier series of function f defined on an interval $(-p\ ,\ p)$ results in **complex Fourier series** of this function.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p}$$

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}\,dx, \qquad n = 0, \pm1, \pm2, \cdots$$

Fourier Series and Frequency Spectrum

- Fourier series of a function on the interval $(-p\ ,\ p)$ defines a periodic function with the fundamental period of $T=2p$.
- If we define $\omega=2\pi / T$ as the **fundamental angular frequency**, the Fourier series become

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega x + b_n \sin n\omega x \quad \text{and} \quad f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega x}$$

- The plot of points $(n\omega, |c_n|)$ is called **frequency spectrum** of f.

Class Exercise

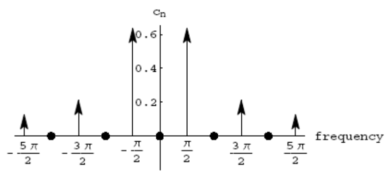
- Find complex Fourier series and frequency spectrum of

$$f(x)=\begin{cases}-1, & -2<x<0\\ 1, & 0<x<2\end{cases}$$

$$c_n=\frac{1-(-1)^n}{n\pi i},\qquad n\neq 0$$

$$c_0=0$$

$$f(x)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1-(-1)^n}{n\pi i}e^{in\pi x/2}$$



Microsoft Excel Worksheet

Sturm-Liouville Problem

- Let p, q, r, and r' be real-valued functions continuous on an interval $[a\ ,\ b]$, and let $r(x)>0$ and $p(x)>0$ for every x in the interval. Then,

$$\frac{d}{dx}\left[r(x)y'\right]+\left(q(x)+\lambda p(x)\right)y=0$$

$$A_1y(a)+B_1y'(a)=0$$

$$A_2y(b)+B_2y'(b)=0$$

is said to be a **regular Sturm-Liouville Problem.**

- Example:

$$\text{Legendre's Equation: } (1-x^2)y''-2xy'+n(n+1)y=0$$

Properties of Regular Sturm-Liouville Problem

- There exist an infinite number of real eigenvalues that can be arranged in ascending order $\lambda_1<\lambda_2<\lambda_3<\ldots<\lambda_n$ such that $\lambda_n\to\infty$ as $n\to\infty$.
- For each eigenvalue there is only one eigenfunction.
- Eigenfunctions corresponding to different eigenvalues are linearly independent.
- The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ on the interval $[a\ ,\ b]$.

Conversion to Self-Adjoint Form

- Every second-order differential equation

$$a(x)y''+b(x)y'+\left(c(x)+\lambda d(x)\right)y=0$$

can be converted to the so called self-adjoint form

$$\frac{d}{dx}\left[r(x)y'\right]+\left(q(x)+\lambda p(x)\right)y=0$$

by dividing the original equation by $a(x)$ and multiplying by

$$e^{\int(b(x)/a(x))dx}$$

- Example: Parametric Bessel Equation

$$x^2y''+xy'+(\alpha^2x^2-v^2)y=0\qquad\Rightarrow\qquad\frac{d}{dx}[xy']+\left(\alpha^2x-\frac{v^2}{x}\right)y=0$$

Bessel Functions are Orthogonal

- The parametric Bessel equation

$$x^2 y''+xy'+(\alpha^2x^2-n^2)y=0 \quad \Rightarrow \quad \frac{d}{dx}[xy']+\left(\alpha^2 x-\frac{n^2}{x}\right)y=0$$

has two solutions $J_n(\alpha x)$ and $Y_n(\alpha x)$ but only $J_n(\alpha x)$ is bounded at $x=0$.

- The set $[J_n(\alpha_i x)]$ is orthogonal with respect to the weight function $p(x)=x$ on an interval $[0\ ,\ b]$;

$$\int_0^b xJ_n(\alpha_i x)J_n(\alpha_j x)\,dx=0, \qquad \alpha_i\neq\alpha_j.$$

- α_i are given by a boundary condition at $x=b$;

$$A_2J_n(\alpha b)+B_2\alpha J_n'(\alpha b)=0.$$

Differential Recurrence Relations

$$\frac{d}{dx}\left[x^nJ_n(x)\right]=x^nJ_{n-1}(x)$$

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right]=-x^{-n}J_{n+1}(x)$$

or

$$xJ_n'(x)=nJ_n(x)-xJ_{n+1}(x)$$

$$xJ_n'(x)=xJ_{n-1}(x)-nJ_n(x)$$

Fourier-Bessel Series

- The orthogonal series expansion of a function f defined on the interval $[0\ ,\ b]$ in terms of Bessel functions,

$$f(x)=\sum_{i=1}^{\infty}c_iJ_n(\alpha_i x),$$

where

$$c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)\,dx}{\left\|J_n(\alpha_i x)\right\|^2},$$

and the square norm of the function $J_n(\alpha_i x)$ is defined by

$$\left\|J_n(\alpha_i x)\right\|^2=\int_0^b xJ_n^2(\alpha_i x)\,dx.$$

Legendre's Functions are Orthogonal

- The Legendre polynomials, which are the solutions of the Legendre's equation

$$(1-x^2)y''-2xy'+n(n+1)y=0,$$

are orthogonal with respect to the weight function $p(x)=1$ on the interval $[-1\ ,\ 1]$;

$$\int_{-1}^{1}P_m(x)P_n(x)\,dx=0, \qquad m\neq n.$$

Fourier-Legendre Series

- The orthogonal series expansion of a function f defined on the interval $[-1\ ,\ 1]$ in terms of Legendre functions,

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x),$$

where

$$c_n = \frac{\int_{-1}^{1} P_n(x) f(x)\, dx}{\| P_n(x) \|^2},$$

and the square norm of the function $P_n(x)$ is defined by

$$\| P_n(x) \|^2 = \frac{2}{2n+1}$$